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Ridges and Valleys on Digital Images

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Ridges and valleys on digital images are found by looking for zero crossings of the first directional derivative taken in a direction which extremizes the second directional derivative. Computation of the required directional derivative is accomplished by fitting a two-variable cubic polynomial to each neighborhood of the image. Results are shown for a face image and an airphoto scene. It indicates that the technique has a good ability to find ridges and valleys.

1. INTRODUCTION

Computer vision requires the development of an algorithm to explain (for any digital image) the cause of the spatial distribution of its gray tones. The explanation must be in terms of the shape and reflectance of the observed objects, the position of the illumination source or sources and the viewing direction of the camera. For elongated objects which have curved surfaces with a specular reflectance function, the locus of points on their surfaces having surface normals pointing in the direction of the camera generate pixels on a digital image which are ridges. Similarly, for objects which have curved surfaces with some degree of Lambertian reflectance, the locus of points on their surfaces having surface normals pointing in the direction of a point light source generate pixels on a digital image which are also ridges. Linearly narrow concavities on an object surface (such as cracks) are typically in shadow and generate pixels on a digital image which are valleys. Line and curve finding play universal roles in object analysis. Therefore, one important part of the computer vision algorithm must be ridge and valley classification of pixels. This classification task is addressed in this paper.

What is a ridge or valley in a digital image? The first intuitive notion is that a digital ridge (valley) occurs when there is a simply connected sequence of pixels having gray-tone intensity values which are significantly higher (lower) in the sequence than those neighboring the sequence. Significantly higher or lower may depend on the distribution of brightness values surrounding the sequence as well as the length of the sequence.

The facet model (Haralick [1]) can be used to help accomplish ridge and valley identification. The essence of the facet model is that any analysis made on the basis of pixel values in some neighborhood has its final authoritative interpretation relative to the underlying gray-tone intensity surface of which the neighborhood pixel values are observed noisy samples.

To use the facet model we must first translate our notion of ridge and valley to the continuous surface perspective. Here the concept of line translates in terms of directional derivatives. If we picture ourselves walking, by the shortest distance, across a ridge or valley what we would do is walk in the direction having greatest

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1/175

-13	2	7	2	-13
2	17	22	17	2
7	22	27	22	7
2	17	22	17	2
-13	2	7	2	-13

k_1 coefficient for 1

1/420

31	-5	-17	-5	31
-44	-62	-68	-62	-44
0	0	0	0	0
44	62	68	62	44
-31	5	17	5	-31

k_2 coefficient for r

1/420

31	-44	0	44	-31
-5	-62	0	62	5
-17	-68	0	68	17
-5	-62	0	62	5
31	-44	0	44	-31

k_3 coefficient for c

1/70

2	2	2	2	2
-1	-1	-1	-1	-1
-2	-2	-2	-2	-2
-1	-1	-1	-1	-1
2	2	2	2	2

k_4 coefficient for r^2

1/100

4	2	0	-2	-4
2	1	0	-1	-2
0	0	0	0	0
-2	-1	0	1	2
-4	-2	0	2	4

k_5 coefficient for rc

1/70

2	-1	-2	-1	2
2	-1	-2	-1	2
2	-1	-2	-1	2
2	-1	-2	-1	2
2	-1	-2	-1	2

k_6 coefficient for c^2

1/60

-1	-1	-1	-1	-1
2	2	2	2	2
0	0	0	0	0
-2	-2	-2	-2	-2
1	1	1	1	1

$k_7$coefficient for r^3

1/140

-4	-2	0	2	4
2	1	0	-1	-2
4	2	0	-2	-4
2	1	0	-1	-2
-4	-2	0	2	4

k_8 coefficient for r^2c

1/140

-4	2	4	2	-4
-2	1	2	1	-2
0	0	0	0	0
2	-1	-2	-1	2
4	-2	-4	-2	4

$k_9$coefficent for rc^2

1/60

-1	2	0	-2	1
-1	2	0	-2	1
-1	2	0	-2	1
-1	2	0	-2	1
-1	2	0	-2	1

k_{10} coefficient for c^3

FIG. 1. The kernels used to compute the coefficients $k_1, \ldots, k_{10}$ of the fitting bicubic $k_1 + k_2r + k_3c + k_4r^2 + k_5rc + k_6c^2 + k_7r^3k_8r^2c + k_9rc^2 + k_{10}c^3$.

magnitude of second directional derivative. The ridge peak or the valley bottom would occur when the first directional derivative has a zero crossing.

Thus to determine ridges and valleys we need to use the neighborhood of a pixel to estimate a continuous surface whose direction derivatives we can compute analytically. To do this we use a functional form consisting of a cubic polynomial in the two variables row and column. For greater numerical accuracy this can be expressed as a linear combination of the tensor products of discrete orthogonal polynomials of up to degree three (Haralick [2]). These forms are often used in statistical regression problems (Draper and Smith [4]). Figure 1 illustrates the masks used to compute the coefficients of the polynomials in the natural basis set $\{1, r, c, r^2, rc, c^2, r^3, r^2c, rc^2, c^3\}$ for the 5×5 neighborhood.

Section 2 discusses the concept of directional derivatives and derives an expression for the direction which extremizes the second direction derivative. Section 3 discusses how the expressions derived in Section 2 can be applied to fitting the coefficients of the facet model.

2. DIRECTIONAL DERIVATIVES

The first directional derivative of a function f in the direction α at row, column position r, c is denoted by $f'_\alpha(r, c)$ and is defined by

$$f'_\alpha(r, c) = \lim_{d \to 0} \frac{f(r + d \sin\alpha, c + d\cos\alpha) - f(r, c)}{d}. \tag{1}$$

From this it follows that

$$f'_\alpha(r, c) = \frac{\partial f}{\partial r}(r, c)\sin\alpha + \frac{\partial f}{\partial c}(r, c)\cos\alpha \tag{2}$$

and

$$f''_\alpha(r, c) = \frac{\partial^2 f}{\partial r^2}(r, c)\sin^2\alpha + 2\frac{\partial^2 f}{\partial r\, \partial c}(r, c)\sin\alpha\cos\alpha + \frac{\partial^2 f}{\partial c^2}(r, c)\cos^2\alpha. \tag{3}$$

Rearranging the expression for f''_α we find that the second directional derivative can be expressed as a linear combination of two terms, the first term being the Laplacian of f and not depending on α and the second term depending on α:

$$f''_\alpha = \frac{1}{2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 f}{\partial c^2}\right) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial c^2} - \frac{\partial^2 f}{\partial r^2}\right)\cos^2\alpha + \frac{\partial^2 f}{\partial r\, \partial r}\sin^2\alpha \tag{4}$$

The direction α which extremizes f''_α can be determined by differentiating f''_α with respect to α, setting the derivative to zero, and solving for α:

$$\frac{\partial f''_\alpha}{\partial \alpha} = \left(\frac{\partial^2 f}{\partial r^2} - \frac{\partial^2 f}{\partial c^2}\right)\sin^2\alpha + 2\frac{\partial^2 f}{\partial r\, \partial c}. \tag{5}$$

Therefore,

$$\sin^2\alpha = \pm\left(-2\partial^2 f/\partial r\, \partial c\right)/D \qquad \text{and} \qquad \cos^2\alpha = \pm\left(\frac{\partial^2 f}{\partial r^2} - \frac{\partial^2 f}{\partial c^2}\right)/D, \tag{6}$$

where $D = \sqrt{2\left(\partial^2 f/\partial r\, \partial c\right)^2 + \left(\left(\partial^2 f/\partial r^2\right) - \left(\partial^2 f/\partial c^2\right)\right)^2}$. It is easy to see that when the plus signs are taken,

$$\frac{\partial^2 f''_\alpha}{\partial \alpha^2} > 0$$

indicating that the extremum is a relative minimum, and when the minus signs are taken,

$$\frac{\partial^2 f''_\alpha}{\partial \alpha^2} < 0$$

indicating that the extremum is a relative maximum. Also, the direction α which makes f''_α a maximum differs from the α which makes f''_α a minimum by $\pi/2$ radians.

3. RIDGE–VALLEY IDENTIFICATION

To identify a pixel as a ridge or valley, we set up a coordinate system whose origin runs through the center of the pixel. We select a neighborhood size to estimate the fitting coefficients of the polynomials. Using the fitted polynomials, we can compute all second partial derivatives at the origin, from which the two directions of the extremizing α can be computed by Eq. (6).

Having a direction α we next need to see if by traveling along a line passing through the origin in the direction α, the first directional derivative has a zero crossing sufficiently near the center of the pixel. If so, we declare the pixel to be a ridge or valley. Of course, if in one direction we find a ridge and in the other we find a valley then the pixel is a saddle point.

To express this procedure precisely and without reference to a particular basis set of polynomials tied to a neighborhood size we will rewrite the fitted bicubic surface in the canonical form

$$\begin{aligned} f(r,c) = {} & k_1 + k_2 r + k_3 c + k_4 r^2 + k_5 rc \\ & + k_6 c^2 + k_7 r^3 + k_8 r^2 c \\ & + k_9 rc^2 + k_{10} c^3. \end{aligned} \tag{7}$$

Then, $\alpha = \pm \frac{1}{2} \tan^{-1} k_5/(k_6 - k_4)$.

To walk in the direction α, we constrain r and c by

$$r = \rho \sin\alpha \qquad \text{and} \qquad c = \rho\cos\alpha.$$

Therefore, in the direction α, we have

$$f_\alpha(\rho) = A\rho^3 + B\rho^2 + C\rho + R_1, \tag{8}$$

where

$$\begin{aligned} A &= \left(k_7 \sin^3\alpha + k_8 \sin^2\alpha\cos\alpha + k_9 \cos^2\alpha\sin\alpha + k_{10}\cos^3\alpha\right), \\ B &= \left(k_4 \sin^2\alpha + k_5 \sin\alpha\cos\alpha + k_6\cos^2\alpha\right), \\ C &= k_2 \sin\alpha + k_3\cos\alpha. \end{aligned}$$

The first directional derivative in direction α is given by

$$f'_\alpha(\rho) = 3A\rho^2 + 2B\rho + C \tag{9}$$

and the second directional derivative in the direction α at ρ away from the center of the pixel is given by

$$f_{\alpha}''(\rho) = 2A\rho + B. \tag{10}$$

At those positions for ρ which make $f_{\alpha}'(\rho) = 0$, the value of $f_{\alpha}''(\rho)$ is proportional to the curvature of the surface. If $B^2 - 4AC > 0$, the largest magnitude root ρ_L of Eq. (8) can be found by

$$\rho_L = \frac{-B - \operatorname{sign}(B)\sqrt{B^2 - 4AC}}{2A}. \tag{11}$$

The smallest-magnitude root ρ_s can be found by

$$\rho_S = C/A\rho_L. \tag{12}$$

If the smallest-magnitude root of (8) is sufficiently close to zero, the center point of the pixel, we declare the pixel as a ridge or valley depending on the sign of the second directional derivative. Pixels which are both ridges and valleys can be declared as saddles.

4. PROBLEMS

There are two problems which arise in the application of the concepts in Sections 2 and 3. One arises out of the definitions for ridge and valley. The other arises from the fitting. First we discuss the problems in definition, then the problem of the fitting.

Although it might seem sufficient to define a ridge or valley to occur on any surface point which has a zero-crossing of the first directional derivative taken in the direction extremizing the second directional derivative, there are nonpathological simple surfaces for which every point satisfies the definition. One class of such surfaces has the radially symmetric form $f(r^2 + c^2)$. At each point (r, c) the orthogonal directions extremizing the second directional derivative either point radially towards the origin or point tangentially to the radial direction. The tangential direction has of necessity a zero-crossing of the first directional derivative since the gradient vector points radially.

We have found two criteria to partially help solve this problem. The first criterion requires that a ridge or valley have a sufficiently small gradient-to-curvature ratio, typically smaller than four. The second criterion requires that the angle the gradient

0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	1	0	0	0
1	0	0	0	0

(a)

0	0	0	1	0
0	0	1	0	0
0	1	0	0	0
1	0	0	0	0
0	0	0	0	0

(b)

0	0	1	0	0
0	1	0	0	0
1	0	0	0	0
0	0	0	0	0
0	0	0	0	0

(c)

FIG. 2. Three 5×5 neighborhoods of ideal lines which are just translates of one another.

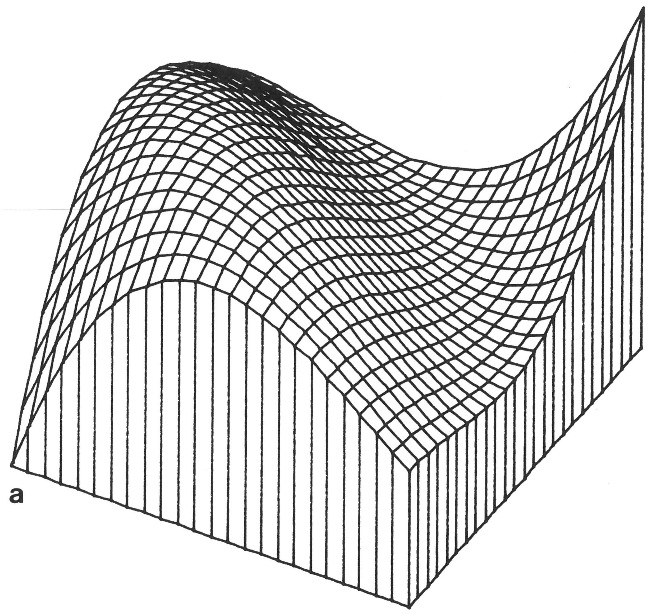

FIG. 3(a). The interpolated bicubic surface fit for the image of Fig. 2(b).

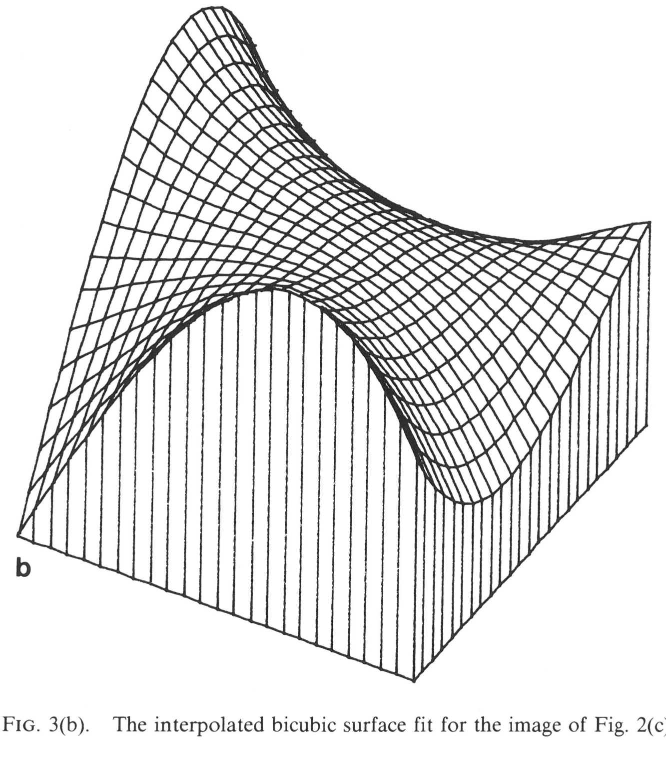

FIG. 3(b). The interpolated bicubic surface fit for the image of Fig. 2(c).

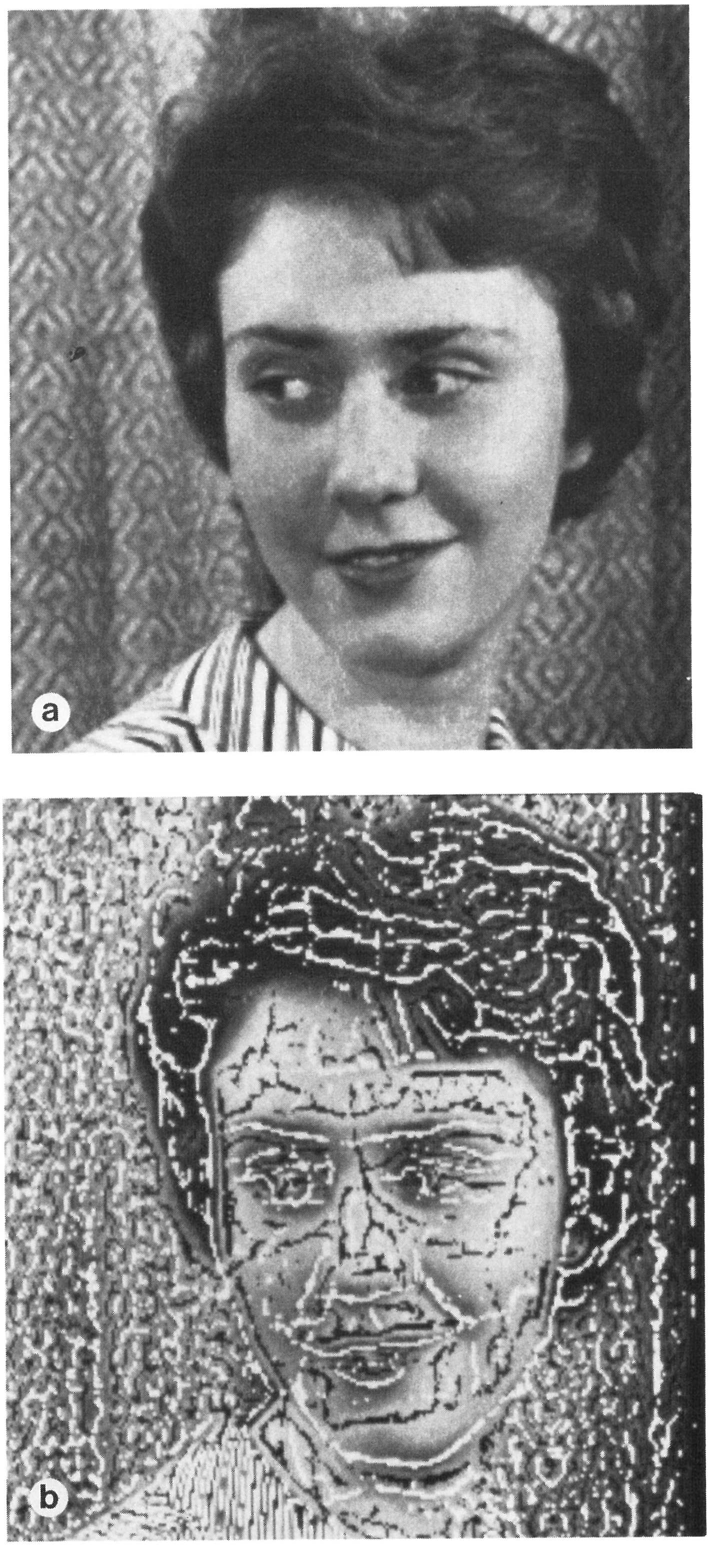

FIG. 4. The computed ridges and valleys for the girl image; (a) original girl image, (b) ridge and valley overlay of the girl image, and (c) ridges and valleys alone of the girl image.

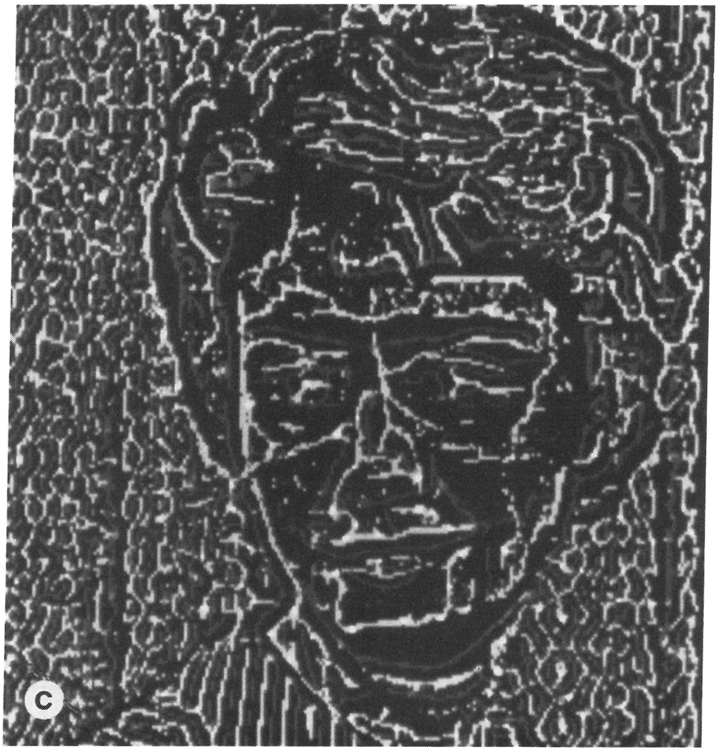

FIG. 4.—*Continued.*

vectors make with each other on each side of the ridge or valley be sufficiently large. We take "on each side of the ridge" to mean 1.5 pixels away and a sufficiently large angle to be 30 degrees.

The problem due to fitting can be understood by examining what the fit does to a simple one-pixel wide white line on a uniformly dark background. Suppose the pixels on the line have a value of zero. Figure 2a shows a 5×5 window of such a line at a 45° orientation. Computing the coefficients of the fit by taking the sums of products of the 5×5 window with each of the masks in Fig. 1 yields the cubic fit $0.2 + 0.1rc$ where $r, c = -2, -1, 0, 1, 2$. It is clear that the actual line has no character of a saddle surface but that the fit is indeed a saddle surface.

This pathologic behavior is characteristic of what happens in a cubic fit whenever the fit is to data which is comparatively simple and piecewise constant. Figures 2b and c show the images of two translations of the line in Fig. 2a. Figures 3a and b show surface plots for the respective fits. Notice that for the case of Fig. 3a the fit is reasonable while for Fig. 3b it is not.

It is clear that more work needs to be done regarding the obtaining of fits which retain the essential character of the discretized data. Questions needing answers include: What are the most appropriate basic functions? What are the most appropriate inner products with respect to which the fit is taken?

We have not found a solution to the fitting problem. The best we have been able to do is to try to disregard some of the pixels classified as ridges or valleys due to

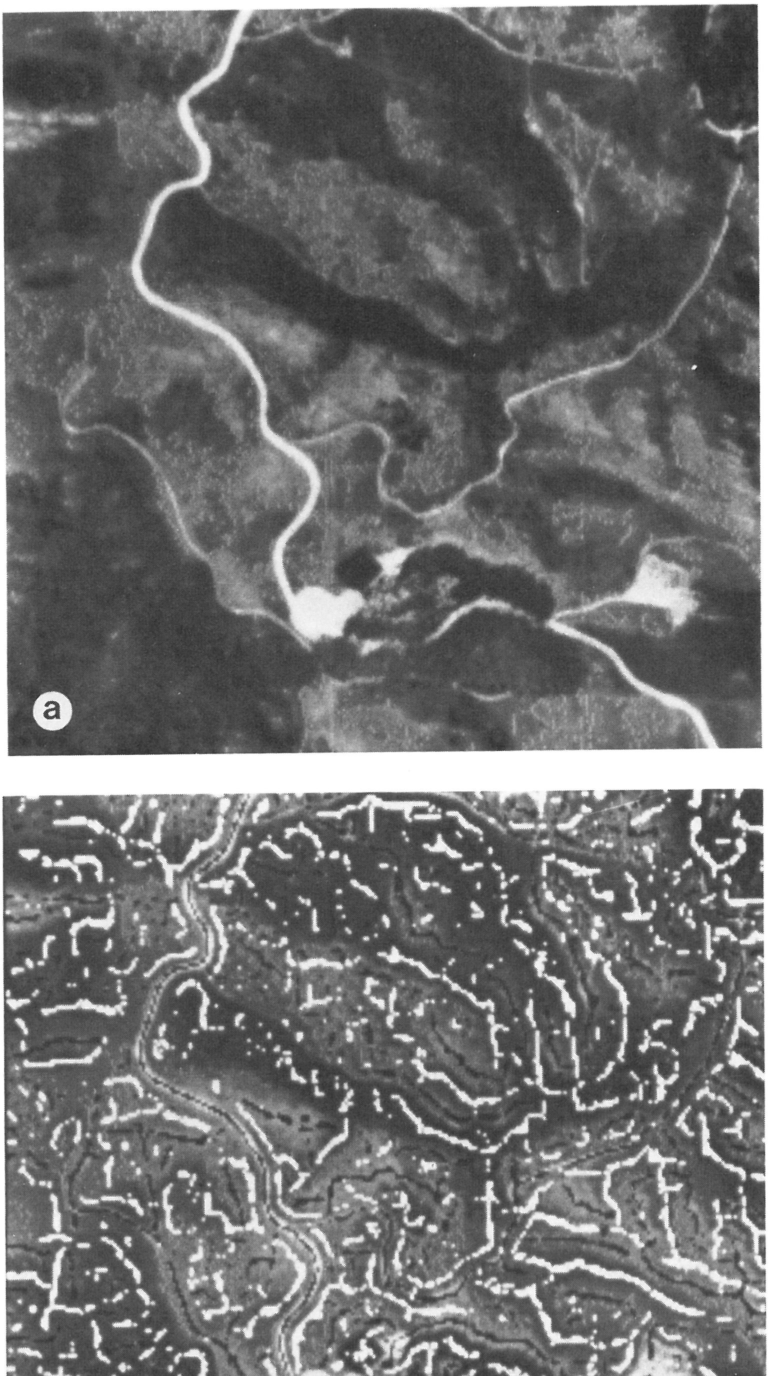

FIG. 5. The computed ridges and valleys for the airphoto scene; (a) original airphoto scene image, (b) ridge and valley overlay of the airphoto scene, and (c) ridges and valleys alone of the airphoto scene.

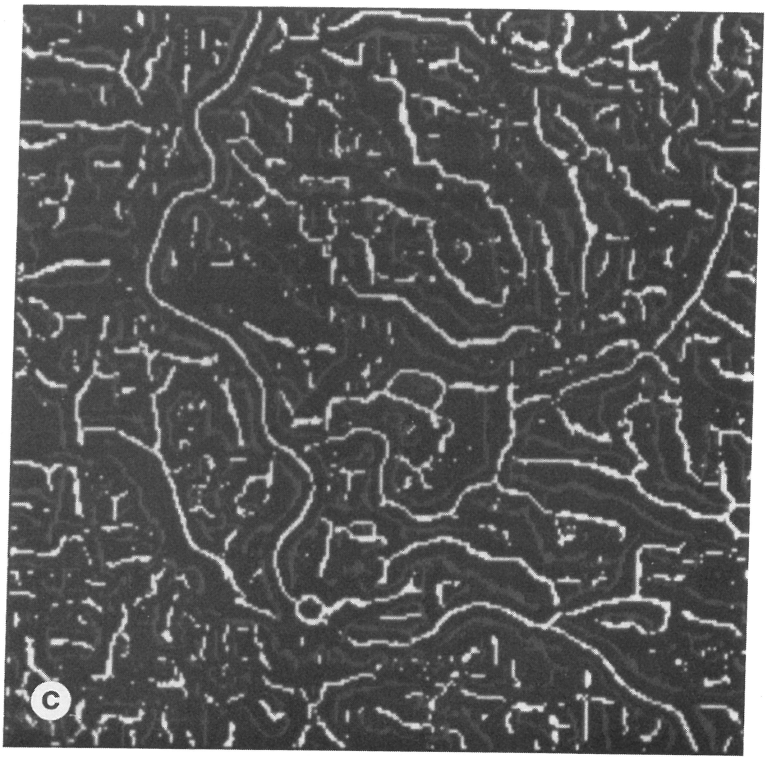

FIG. 5.—*Continued.*

artifacts of the fit by using a criterion of depth of valley or height of ridge. If the depth is too small, then the classification of the pixel is disregarded. Depth D is easily defined for the one-dimensional cubic polynomial of Eq. (8) as the difference in the value of its stationary points $D = |f_\alpha(\rho_L) - f_\alpha(\rho_s)|$.

5. RESULTS

To illustrate the power of the ridge–valley classification scheme, we show results on two images of entirely different character. Figure 4a shows the image of a girl which has been used in experiments by other researchers. Figure 4b shows the ridges and valleys overlayed. Ridges appear as black and valleys appear as white. Figure 4c shows the ridges and valleys by themselves, the ridges being brighter and the valleys being darker. Notice how the highlights are ridges and the shadow lines are valleys. To obtain these results a 5×5 neighborhood was used for fitting, and the depth threshold was set to one. The interval in which a zero-crossing has to occur is $(-0.85, +0.85)$. The gradient magnitude-to-curvature ratio had to be less than four, and the angle between the gradient vectors at a distance of 1.5 pixels on each side of the ridge or valley had to be greater than 30°.

The second image of a road scene is shown in Fig. 5a. Figure 5b shows the ridges and valleys overlayed and Fig. 5c shows the ridges and valleys alone. Notice the relative connectedness of the ridges and valleys and how they seem to alternate. It is clear examining the overlay that ridges and valleys are placed where they should be and that in some sense the placement of the ridges and valleys determines the essential character of the image. The results of Fig. 5 were obtained by using a 9×9 neighborhood for fitting. The depth threshold was set to ten, and the interval in which the zero-crossing had to occur was $(-1., +1.)$. The gradient magnitude-to-curvature ratio had to be less than four and the angle between the gradient vectors at a distance of 1.5 pixels on each side of the ridge or valley had to be greater than 30°.

6. RELATED LITERATURE

The closest approach to the classification scheme discussed here is the one presented by Paton [3] and Hsu *et al.* [5]. Paton uses a quadratic surface fit and defines a ridge or valley to exist at any pixel whose surface fit has a significant quadratic component most of whose energy is directed in one direction. Paton uses the continuous least squares fit formulation in setting up the surface fit equations. We use the discrete least squares fit formulation. Because Paton's surface is only quadratic, the ridge–valley definitions can only apply to the center point of a pixel. Because our surface fit is cubic, we are able to classify a pixel as a ridge or valley if there is a ridge or valley anywhere in the area of the pixel.

Hsu *et al.* [5] also use a quadratic approximation, but (as we do) use a discrete least squares formulation. Their paper also does more than labeling. They link ridges and valleys together in a web network and use this representation to approximate the image.

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