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A MORPHOLOGIC BRIDGING TRANSFORMATION FOR SPATIAL DOMAIN PROCESSING OF DIGITAL IMAGES

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The digital image processing community has the experience that the transformed (in the discrete domain) digital object is a subset of sample transformed (in the continuous domain) object. In this paper we have described suitable bridging transformation to fill the gap between these objects. This work enables us to implement some widely used spatial transformation, which are thus far being implemented through reverse transformations, using forward transformations.

Key Words : Mathematical Morphology; Image Transformation; Sampling; Reconstruction

1. INTRODUCTION

The digital image processing community has the experience that the transformed sampled set of points in the discrete domain is a subset of sampled corresponding object that has undergone similar transformation in the continuous domain. For example, rotation and dilation may create holes in the transformed digital objects, and magnification may make the transformed digital object disconnected, contrary to the topological characteristics of these transformations in the continuous domain. Two examples involving rotation and dilation are shown in Fig. 1. To overcome this problem sometimes reverse transformations are used (e.g., in case of rotation[5], and magnification[3], or some adhoc processing is done to reduce the difference (e.g., in case of dilation[1]). For example, rotation of an input image in the dicrete domain by an angle, say, θ is a{italic chi}eved by rotating the output image by the angle $-\theta$ and information at every pixel of the output image is acquired from the corresponding pixel (with rounded rotated coordinate) of the input image.

Our objective is to derive a suitable bridging transformation to fill the gap (at least partially) between the transformed sampled object and the sampled transformed object. Having such a bridging transformation would help us implementing above mentioned spatial transformations through known forward transformations only.

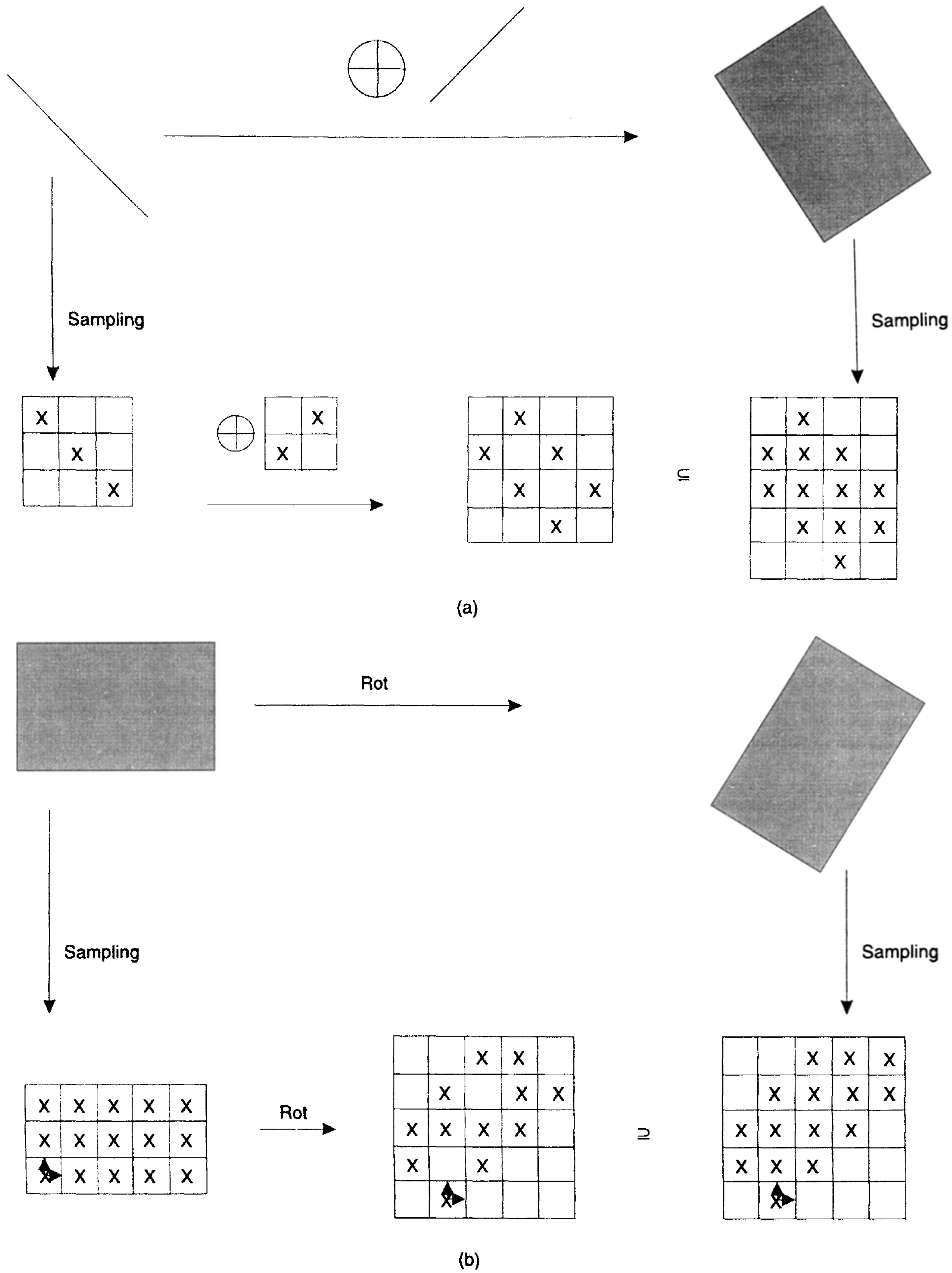


FIG. 1. Examples of transformations that illustrates that a transformed sampled set is a subset of sampled transformed set. (a) Rotation and (b) Dilation

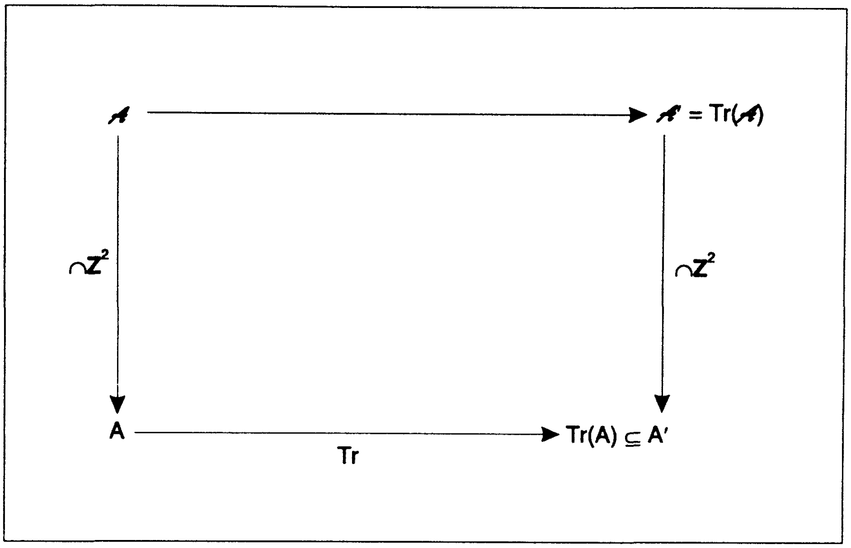


FIG. 2. Representation of corresponding objects and transformations applied on them in both the continuous and the discrete domain

Now to derive a bridging transformation we need to apply the prescribed transformation on the objects in both the continuous domain and the discrete domain, and then analyze the difference between the results of the transformation. Since we deal with or process digital objects, the only way to obtain objects in the continuous domain is by a reconstruction based on our knowledge of the sampling procedure. A comprehensive treatment of reconstruction based on our knowledge of down-sampled set of points by morphological methods is available in [2]. We make use of some of their results for reconstructing an object in the continuous domain from the corresponding object in the discrete domain.

This paper is organized as follows. Problem is defined in a more formal framework in section 2. Section 3 discusses morphological methods for reconstructing objects in the continuous domain from objects in the digital domain and presents the accuracy of such reconstructions. Bridging transformations for several known and widely used transformations are derived in section 4. Concluding remarks are cited in section 6.

2. PROBLEM DEFINITION

Let us denote objects (i.e., compact set of points) in the two-dimensional continuous domain $\boldsymbol{R}^2$ by $\mathcal{A}$, $\mathcal{B}$, C etc. Corresponding digital objects are obtained by sampling them at unit intervals in two orthogonal directions. Thus the two-dimensional discrete domain Z^2 is represented by a set of points having integer coordinates. Suppose the digital objects are denoted by A, B, C, etc. Then

$$A = \mathcal{A} \cap Z^2 \qquad \ldots (1)$$

and so on.

We have already said that the transformed sampled set of points in the discrete domain is a subset of sampled corresponding object that has undergone similar transformation in the continuous domain [see Fig. 2]. Our objective is to determine a bridging transformation $\Psi(\cdot)$ such that

$$Tr(\mathcal{A} \cap Z^2) \subseteq \Psi(Tr(\mathcal{A} \cap Z^2)) \subseteq Tr(\mathcal{A}) \cap Z^2 \qquad \text{... (2)}$$

or

$$\rho(\Psi(Tr(\mathcal{A} \cap Z^2)), Tr(\mathcal{A}) \cap Z^2) \leq \rho(Tr(\mathcal{A} \cap Z^2), Tr(\mathcal{A}) \cap Z^2), \qquad \text{... (3)}$$

where $\rho(\cdot, \cdot)$ is a set metric. The first relation (2) shows the set-bounding relationship among the objects obtained through sampling and transforming versus transforming and sampling; while the relation (3) shows their distance-bounding relationship. Since, the identity transformation can also satisfy these relations because of the presence of equality sign, more specially our objective, in terms of set-bounding relationship, should be stated as:

Determine a bridging transformation $\Psi(\cdot)$ such that

if

$$Tr(\mathcal{A} \cap Z^2) \subset Tr(\mathcal{A}) \cap Z^2,$$

then

$$Tr(\mathcal{A} \cap Z^2) \subset \Psi(\mathcal{A} \cap Z^2)) \subseteq Tr(\mathcal{A}) \cap Z^2$$

Distance-bounding relationship (3) may be re-stated in a similar way.

Now to derive the bridging transformation $\Psi(\cdot)$ we need to apply the transformation $Tr(\cdot)$ on the reconstructed object in the continuous domain as well as on the digital object in the discrete domain, and then analyze the difference between the results of the transformation.

3. Continuous Object Reconstruction

Let us denote by $\tilde{\mathcal{A}}$ the reconstructed object in the continuous domain from the discrete object A. One of the important requirements of reconstruction is that the sampled reconstructed set is the same as the original sampled set, that is,

$$\tilde{\mathcal{A}} \cap Z^2 = A. \qquad \text{... (4)}$$

It is exactly the presence of details relatively smaller than the sampling interval such as small objects, object protrusions, object intrusions, and holes that causes the sampled object and, in turn, the reconstructed object to be unrepresentative of the original object in the continuous domain, just as in signal processing, frequencies higher than the Nyquist frequency cause the sampled signal to be unrepresentative of the original signal. The morphological sampling theorem[2] tells us that if the structuring element is chosen in accordance with the sampling interval, and if the original object in the continuous domain is simplified (i.e., open and closed) with respect to this structuring element then a faithful reconstruction of the original object in the continuous domain may be done morphologically in two different ways: Either by a closing or by a dilation. The first one gives the minimal reconstruction and the second one gives the maximal. So we see the reconstruction is an approximate one. However, the error, $\rho(\tilde{\mathcal{A}}, \mathcal{A})$, in either case is no more than the radius of the disk-like structuring element used for the reconstruction.

Let $K \subseteq \boldsymbol{R}^2$ represent the structuring element for the reconstruction. Then K must satisfy the following relations :

1. $Z^2 \oplus K = R^2$

2. $x \in Z^2 \Rightarrow K_x \cap Z^2 = \{x\}$

The first relation assures that there cannot be any location in the continuous domain that cannot be covered by the reconstruction process. The second one ensures that for any sampled point area covered in the continuous domain due to reconstruction should include no other sampling point and, hence, satisfies eq. 4. The largest structuring element that satisfies these criteria is the open square in R^2 defined by $(-1, 1) \times (-1, 1)$, the Cartesian product of the open interval $(-1, 1)$ with itself.

3.1 Reconstruction with the Largest Structuring Element

We take $K = (-1, 1) \times (-1, 1)$ as the reconstruction structuring element. Since, K is symmetric, i.e., $K = \check{K}$, the reconstructed shape will be unbiased to the direction of the coordinate axes. Secondly, if $y, z \in R^2, y \in K_z \Rightarrow K_y \cap K_z \cap Z^2 \neq 0$. This characterisic along with the second criterion listed above assert that K can cover at least one and at most two sampled points in any direction: horizontal, vertical and diagonal. As we have said before, a set in the continuous domain may be reconstructed morphologically from the sampled set A either by dilating it by K or by closing it by K, such that

$$(A \bullet K) \cap Z^2 = (A \oplus K) \cap Z^2 = \mathcal{A} \cap Z^2 = A \qquad \text{... (5)}$$

or

$$\tilde{\mathcal{A}}_c^K \cap Z^2 = \tilde{\mathcal{A}}_d^K \cap Z^2 = \mathcal{A} \cap Z^2 = A, \qquad \text{... (6)}$$

where $\tilde{\mathcal{A}}_c^K$ represents the reconstructed object arising from closing the digital object by K, and $\tilde{\mathcal{A}}_d^K$ represents the reconstructed object arising from dilating the digital object by K. So in both reconstructions the sampled reconstructed sets are equal to the sampled original set. Furthermore, if the original set is both opened and closed under the structuring element K, then one can see that the original set contains its closing reconstruction and is contained in its dilation reconstruction as stated in the following proposition. Furthermore, minimal and maximal reconstructed sets differ only by a dilation of K as stated below.

Proposition 1 — *Let $\mathcal{A}$ be an object in the continuous domain and let A be the corresponding object in the discrete domain obtained by sampling. Let $K = (-1, 1) \times (-1, 1)$ be a structuring element. If $\mathcal{A} \circ K = \mathcal{A} \bullet K = \mathcal{A}$, then following is always true.*

1. $\tilde{\mathcal{A}}_c^K \subseteq \mathcal{A} \subseteq \tilde{\mathcal{A}}_d^K$

2. $\tilde{\mathcal{A}}_c^K \oplus K = \tilde{\mathcal{A}}_d^K$,

where,

$$\tilde{\mathcal{A}}_c^K = A \bullet K \text{ and } \tilde{\mathcal{A}}_d^K = A \oplus K.$$

Proof : For 1 :

Since $A = \mathcal{A} \cap Z^2$, so $A \subseteq \mathcal{A}$.

This implies $A \bullet K \subseteq \mathcal{A} \bullet K$.

Since $\mathcal{A} \bullet K = \mathcal{A}$, $A \bullet K \subseteq \mathcal{A}$.

Thus $\tilde{\mathcal{A}}_c^K \subseteq \mathcal{A}$.

Since $\mathcal{A} \circ K = \mathcal{A}$, then there exists a set $\mathcal{A}'$ in the continuous domain such that $\mathcal{A} = \mathcal{A}' \oplus K$.

This implies $\mathcal{A}' \subseteq (\mathcal{A} \cap Z^2) \bullet K$

or $\mathcal{A}' \subseteq A \bullet K$.

Dilating both sides by $K : \mathcal{A}' \oplus K \subseteq [(A \oplus K) \ominus K] \oplus K$

or $\mathcal{A}' \oplus K \subseteq A \oplus K$

or $\mathcal{A} \subseteq A \oplus K$

Therefore, $\mathcal{A} \subseteq \tilde{\mathcal{A}}_d^K$

For 2 :

Now
$$\begin{aligned}\tilde{\mathcal{A}}_c^K \oplus K &= (A \bullet K) \oplus K \\ &= [(A \oplus K) \ominus K] \oplus K \\ &= (A \oplus K) \circ K \\ &= A \oplus K \\ &= \tilde{\mathcal{A}}_d^K\end{aligned}$$

Hence, proved.

Since $\mathcal{A}$ is open under K, to be a faithful reconstruction $\tilde{\mathcal{A}}_d^K$ should also be open under K. In the following proposition we show that the largest set in the continuous domain that is open under K and that gives the same sampled object is $\tilde{\mathcal{A}}_d^K$ itself.

***Proposition* 2** — *Let $\mathcal{A} \subseteq \boldsymbol{R}^2$ satisfy $\mathcal{A} \circ K = \mathcal{A}$, where $K = (-1, 1) \times (-1, 1)$, and let $A \subseteq \boldsymbol{Z}^2$ is the sampled set of $\mathcal{A}$, i.e. $A = \mathcal{A} \cap \boldsymbol{Z}^2$. Suppose there exists a set χ in the continuous domain such that $\chi \circ K = \chi$ and $\chi \cap \boldsymbol{Z}^2 = A$, then*

$$\chi \supseteq \tilde{\mathcal{A}}_d^K \quad \Rightarrow \quad \chi = \tilde{\mathcal{A}}_d^K .$$

PROOF : Suppose $\chi \supseteq \tilde{\mathcal{A}}_d^K$.

Then by definition of $\tilde{\mathcal{A}}_d^K$, $\chi \supseteq A \oplus K$.

We are given that $\chi \cap \boldsymbol{Z}^2 = A$.

Hence, $(\chi \cap \boldsymbol{Z}^2) \oplus K = A \oplus K$.

Again, since $\chi = \chi \circ K$ and $K = (-1, 1) \times (-1, 1)$, we have [from Proposition 1] $\chi \subseteq (\chi \cap Z^2) \oplus K$.

But $(\chi \cap Z^2) \oplus K = A \oplus K$.

This implies $\chi \subseteq A \oplus K$.

Now $\chi \subseteq A \oplus K$,

together with the supposition $\chi \supseteq A \oplus K$

implies $\chi = A \oplus K$

Hence $\chi = \tilde{\mathcal{A}}_d^K$

Similarly, since $\mathcal{A}$ is closed under K, to be a faithful reconstruction $\tilde{\mathcal{A}}_c^K$ should also be closed under K. In the following proposition we show that the smallest set in the continuous domain that is closed under K and that gives the same sampled object is $\tilde{\mathcal{A}}_c^K$ itself.

Proposition 3 — *Let $\mathcal{A} \subseteq \boldsymbol{R}^2$ satisfy $\mathcal{A} \bullet K = \mathcal{A}$, where $K = (-1, 1) \times (-1, 1)$, and $A \subseteq \boldsymbol{Z}^2$ be the sampled set of $\mathcal{A}$, i.e., $A = \mathcal{A} \cap Z^2$. Suppose there exists a set χ in the continuous domain such that $\chi \bullet K = \chi$ and $\chi \cap Z^2 = A$, then*

$$\chi \subseteq \tilde{\mathcal{A}}_c^K \quad \Rightarrow \quad \chi = \tilde{\mathcal{A}}_c^K .$$

PROOF : Suppose $\chi \subseteq \tilde{\mathcal{A}}_c^K$.

Then by definition of $\tilde{\mathcal{A}}_c^K$, $\chi \subseteq A \bullet K$.

We are given that $\chi \cap Z^2 = A$.

Hence, $(\chi \cap Z^2) \bullet K = A \bullet K$

Since $\chi \cap Z^2 \subseteq \chi$,

then $(\chi \cap Z^2) \bullet K \subseteq \chi \bullet K$

But, since $\chi \bullet K = \chi$, $(\chi \cap Z^2) \bullet K \subseteq \chi$

Again since $A \bullet K = (\chi \cap Z^2) \bullet K$ and $(\chi \cap Z^2) \bullet K \subseteq \chi$,

we obtain $A \bullet K \subseteq \chi$

Now $\chi \supseteq A \bullet K$,

together with opposition $\chi \subseteq A \bullet K$

implies $\chi = A \bullet K$

or $\chi = \tilde{\mathcal{A}}_c^K$

Hence, proved.

Thus, these reconstructed sets $\tilde{\mathcal{A}}_d^K$ and $\tilde{\mathcal{A}}_c^K$ are truely lower and upper bounding sets, respectively, of the original set $\mathcal{A}$, and they differ only by a dilation of K.

Therefore, Proposition 1 together with Proposition 2 and Proposition 3 gives set-bounding relationship between the original set and the reconstructed sets. Whereas, Proposition 1 also gives a hint about a distance-bounding relationship among them. Let us now establish a distance-bounding relationship among them more explicitly. To do so, we need the following definitions[2].

Let us define the radius of a set $\mathcal{B}$, denoted by $r(\mathcal{B})$, as the radius of its circumscribing disk whose centre is in $\mathcal{B}$. Then

$$r(\mathcal{B}) = \inf_{x \in \mathcal{B}} \sup_{y \in \mathcal{B}} \| x - y \|. \qquad \text{... (7)}$$

Since the reconstruction structuring element K is symmetric, i.e., $K = \check{K}$, where $\check{K} = \{-x \mid x \in K\}$ is the reflection of K, and $0 \in K$, then the centre of K and, consequently, the centre of its circumbscribing disk lie at the origin 0. Therefore,

$$r(K) = \sup_{x \in K} \| x \|. \qquad \text{... (8)}$$

Thus $r(K)$ is the same as the sampling interval.

For a set $\mathcal{A}$ that contains $\mathcal{B}$, a natural pseudodistance from $\mathcal{A}$ to $\mathcal{B}$ is defined by

$$\rho(\mathcal{A}, \mathcal{B}) = \sup_{x \in \mathcal{A}} \inf_{y \in \mathcal{B}} \| x - y \|. \qquad \text{... (9)}$$

It is easy to verify that -

1. $\rho(\mathcal{A}, \mathcal{B}) \geq 0$
2. $\rho(\mathcal{A}, \mathcal{B}) = 0 \Rightarrow \mathcal{A} \subseteq \mathcal{B}$
3. $\rho(\mathcal{A}, \mathcal{C}) \leq \rho(\mathcal{A}, \mathcal{B}) + \rho(\mathcal{B}, \mathcal{C}) + r(\mathcal{B})$.

Hence, $\rho(\mathcal{A}, \mathcal{B})$ is radius of the smallest disk that, when used as a structuring element to dilate $\mathcal{B}$, produces a result that contains $\mathcal{A}$. The asymmetric relation (2) is weaker than the corresponding metric requirement that $\rho(\mathcal{A}, \mathcal{B}) = 0$ if and only if $\mathcal{A} = \mathcal{B}$, and relation (3) is weaker than the metric triangular inequality. However, the pseudodistance ρ can be used as the basis for a true set metric by making it symmetric. we define a set metric (also called Hausdorf metric [4]) as

$$\rho_s(\mathcal{A}, \mathcal{B}) = \sup \{\rho(\mathcal{A}, \mathcal{B}), \rho(\mathcal{B}, \mathcal{A})\} \qquad \text{... (10)}$$

Suppose a disk of radius r with centre is at origin is denoted by *disk*(r), then $\rho_s(\mathcal{A}, \mathcal{B})$ may be interpreted as

$$\rho_s(\mathcal{A}, \mathcal{B}) = \inf \{r \mid \mathcal{A} \subseteq \mathcal{B} \oplus \text{disk(r)} \text{ and } \mathcal{B} \subseteq \mathcal{A} \oplus \text{disk(r)}\} \qquad \text{... (11)}$$

Being a metric, $\rho_s(\mathcal{A}, \mathcal{B})$ has the following properties:

Proposition 4 — Suppose $\rho_s(\mathcal{A}, \mathcal{B})$ represents distance between two sets $\mathcal{A}$ and $\mathcal{B}$ in the continuous domain, then

1. $\rho_s(\mathcal{A}, \mathcal{B}) \geq 0$
2. $\rho_s(\mathcal{A}, \mathcal{B}) = 0$ if and only if $\mathcal{A} = \mathcal{B}$

3. $\rho_s(\mathcal{A}, \mathcal{B}) = \rho_s(\mathcal{B}, \mathcal{A})$

4. $\rho_s(\mathcal{A}, \mathcal{C}) \leq \rho_s(\mathcal{A}, \mathcal{B}) + \rho_s(\mathcal{B}, \mathcal{C})$

PROOF : Proof can be found in [2].

Proposition 5 — Suppose $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are three sets in the continuous domain such that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, then

$$\rho_s(\mathcal{A}, \mathcal{B}) \leq \rho_s(\mathcal{A}, \mathcal{C}) \text{ and } \rho_s(\mathcal{B}, \mathcal{C}) \leq \rho_s(\mathcal{A}, \mathcal{C})$$

PROOF : Suppose $D_{\mathcal{A}\mathcal{B}}$ is the smallest disk such that $A \oplus D_{\mathcal{A}\mathcal{B}} \supseteq B$. $D_{\mathcal{B}\mathcal{C}}$ and $D_{\mathcal{A}\mathcal{C}}$ are defined similarly. Since $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, we can write

$$D_{\mathcal{A}\mathcal{B}} \subseteq D_{\mathcal{A}\mathcal{C}} \text{ and } D_{\mathcal{B}\mathcal{C}} \subseteq D_{\mathcal{A}\mathcal{C}}.$$

Let $r(D_{\mathcal{A}\mathcal{B}})$, $r(D_{\mathcal{B}\mathcal{C}})$ and $r(D_{\mathcal{A}\mathcal{C}})$ denote the radii of the disks $D_{\mathcal{A}\mathcal{B}}$, $D_{\mathcal{B}\mathcal{C}}$ and $D_{\mathcal{A}\mathcal{C}}$ respectively. Therefore,

$$r(D_{\mathcal{A}\mathcal{B}}) \leq r(D_{\mathcal{A}\mathcal{C}}) \text{ and } r(D_{\mathcal{B}\mathcal{C}}) \leq r(D_{\mathcal{A}\mathcal{C}}).$$

Now from the definition of Hausdorf metric, we have

$$\rho_s(\mathcal{A}, \mathcal{B}) = r(D_{\mathcal{A}\mathcal{B}}),$$

$$\rho_s(\mathcal{B}, \mathcal{C}) = r(D_{\mathcal{B}\mathcal{C}})$$

and

$$\rho_s(\mathcal{A}, \mathcal{C}) = r(D_{\mathcal{A}\mathcal{C}}).$$

Hence, proved.

Now we present a strong relationship between the set distance and the dilation of sets that is used to translate the set bounding relationship to the distance bounding relationship as suggested in Proposition 1.

Proposition 6 — $\rho_s(\mathcal{A} \oplus \mathcal{B}, \mathcal{C} \oplus \mathcal{D}) \leq \rho_s(\mathcal{A}, \mathcal{C}) + \rho_s(\mathcal{B}, \mathcal{D})$.

PROOF : Proof can be found in [2].

It immediately follows that the distance between the minimal and maximal reconstructed sets, that is between $\tilde{\mathcal{A}}_c^K$ and $\tilde{\mathcal{A}}_d^K$ respectively, which themselves differ only by a dilation by K, is no greater than the radius of K, i.e.,

$$\rho_s(\tilde{\mathcal{A}}_c^K, \tilde{\mathcal{A}}_d^K) \leq r(K). \quad \ldots (12)$$

Finally, it is not surprising that the distance between the original set and either the minimal reconstructed set or the maximal reconstructed set is no greater than $r(K)$, i.e.,

$$\rho_s(\mathcal{A}, \tilde{\mathcal{A}}_c^K) \leq r(K) \qquad \text{... (13)}$$

and

$$\rho_s(\mathcal{A}, \tilde{\mathcal{A}}_d^K) \leq r(K). \qquad \text{... (14)}$$

This implies if we can use a smaller reconstruction structuring element, then we can have reconstructed set more closer to the original set. The smallest structuring element that satisfies desired criteria is an semi-open square in $\boldsymbol{R}^2$ defined as $(-0.5, 0.5] \times (-0.5, 0.5]$.

3.2. Reconstruction with the Smallest Structuring Element

Now we take $L = (-0.5, 0.5] \times (-0.5, 0.5]$ as the reconstruction structuring element. Though L is not symmetric, i.e., $L \neq \check{L}$, the reconstructed shape will remain practically unbiased, because distance between L and its reflection is negligible. In fact, $\rho_s(L, \check{L}) = 0$. Secondly,

$$r(L) = \sup_{x \in L} \| x \|. \qquad \text{... (15)}$$

Thus $r(L)$ is the half of $r(K)$, or the sampling interval, i.e.,

$$r(L) = r(K)/2. \qquad \text{... (16)}$$

Third, if $x \in \boldsymbol{R}^2$, then $L_x \cap \boldsymbol{Z}^2 = \{\lfloor x + 0.5 \rfloor\}$ where $\lfloor a \rfloor$ is the greatest integer not exceeding a. If a is a vector, then the rule applies to its elements. This characteristic asserts that L can cover one and only one sampled point. Hence, a set in the continuous domain may be reconstructed morphologically from the sampled set A either by dilating it by L or by closing it by L, such that (rewriting Equations 5 and 6 in terms of L)

$$(A \bullet L) \cap \boldsymbol{Z}^2 = (A \oplus L) \cap \boldsymbol{Z}^2 = \mathcal{A} \cap \boldsymbol{Z}^2 = A \qquad \text{... (17)}$$

and

$$\tilde{\mathcal{A}}_c^L \cap \boldsymbol{Z}^2 = \tilde{\mathcal{A}}_d^L \cap \boldsymbol{Z}^2 = \mathcal{A} \cap \boldsymbol{Z}^2 = A. \qquad \text{... (18)}$$

It can be shown that $K \circ L = K \bullet L = K$ and $\rho_s(L, K) = r(L)$. Now we show that the error between the set reconstructed through dilation by L and the original set is no more than the half of the sampling interval. That means we can reconstruct the set with subpixel accuracy as stated in the Proposition 7.

***Proposition* 7** — *Let* $\mathcal{A} \subseteq \boldsymbol{R}^2$ *satisfy* $\mathcal{A} \circ K = \mathcal{A} \bullet K = \mathcal{A}$ *where* $K = (-1, 1) \times (-1, 1)$, *and* $A \subseteq \boldsymbol{Z}^2$ *is the sampled set of* $\mathcal{A}$, *i.e.,* $A = \mathcal{A} \cap \boldsymbol{Z}^2$. *If* $L = (-0.5, 0.5] \times (-0.5, 0.5]$, $\tilde{\mathcal{A}}_c^L = A \bullet L$ *and* $\tilde{\mathcal{A}}_c^K = A \bullet K$, *then*

1. $\tilde{\mathcal{A}}_c^L = \tilde{\mathcal{A}}_c^K$.

2. $\tilde{\mathcal{A}}_c^K \subseteq \tilde{\mathcal{A}}_d^L \subseteq \tilde{\mathcal{A}}_d^K$.

3. $\rho_s(\mathcal{A}, \tilde{\mathcal{A}}_d^L) \leq r(K)/2$.

PROOF : For 1 :

Since $L \subset K, A \bullet L \subseteq A \bullet K$

or $\tilde{\mathcal{A}}_c^L \subseteq \tilde{\mathcal{A}}_c^K$.

Now Proposition 3 states that $\tilde{\mathcal{A}}_c^K$ is the smallest set that is closed under K and that produces A when sampled.

Hence, $\tilde{\mathcal{A}}_c^L = \tilde{\mathcal{A}}_c^K$.

For 2 :

From Proposition 1 we can write $\tilde{\mathcal{A}}_d^L = \tilde{\mathcal{A}}_c^L \oplus L$

$$= \tilde{\mathcal{A}}_c^K \oplus L,$$

so

$$\tilde{\mathcal{A}}_c^K \subseteq \tilde{\mathcal{A}}_d^L$$

Again, since

$$L \subset K, \text{ so } A \oplus L \subseteq A \oplus K$$

or

$$\tilde{\mathcal{A}}_d^L \subseteq \tilde{\mathcal{A}}_d^K .$$

Hence,

$$\tilde{\mathcal{A}}_c^K \subseteq \tilde{\mathcal{A}}_d^L \subseteq \tilde{\mathcal{A}}_d^K$$

For 3 : $\rho_s(\tilde{\mathcal{A}}_c^K, \tilde{\mathcal{A}}_d^L) = \rho_s(\tilde{\mathcal{A}}_c^L, \tilde{\mathcal{A}}_d^L)$

$$= \rho_s(\tilde{\mathcal{A}}_c^L, \tilde{\mathcal{A}}_c^L \oplus L)$$

$$\leq r(L)$$

Again

$$\rho_s(\tilde{\mathcal{A}}_d^L, \tilde{\mathcal{A}}_d^K) = \rho_s(\tilde{\mathcal{A}}_c^L, + L, \tilde{\mathcal{A}}_c^K \oplus K)$$

$$= \rho_s(\tilde{\mathcal{A}}_c^L, \tilde{\mathcal{A}}_c^K) + \rho_s(L, K)$$

$$\leq r(L)$$

Now from Proposition 1 and eq. 16, we can write

$$\rho_s(\mathcal{A}, \tilde{\mathcal{A}}_d^L) \leq r(L) = r(K)/2$$

Hence, proved.

So we see that $A \oplus L$ is the closest approximation of $\mathcal{A}$, and the error is less than or equal to the half of the sampling interval. Secondly, the reconstruction by dilation by L is translation invariant unlike other reconstructions. That is

$$(A \oplus L)_x \cap Z^2 = A_{\lfloor x + 0.5 \rfloor}.$$

Therefore, for the rest of our discussion we will use this reconstruction only, i.e., dilation by L.

4. DETERMINING THE BRIDGING TRANSFORMATION

Given a sampled set we proceed as follows. First, we apply the prescribed transformation $Tr(\cdot)$ on both the sampled set and the reconstructed set. Then we apply a corrective operation τ on the transformed reconstructed set to obtain compatible results at the end of two paths as shown in Fig. 3. Finally, comparing the changes that take place in the objects (both sampled and reconstructed ones) along two paths, we decide on the desired bridging transformation $\Psi(\cdot)$. However, analytical derivation of $\Psi(\cdot)$ is not possible because the reconstructed set is only an approximation of the original set. Here we determine bridging transformation $\Psi(\cdot)$ for several widely used and known transformations. If we can show [consider Figs. 2 and 3] that $\tilde{\mathcal{A}}''$ is a subset of $\mathcal{A}'$ and that $\tilde{\mathcal{A}}''$ is a superset of $Tr(A)$, then our criteria presented in eqs. 2 and 3 are satisfied. We argue that the corrective operation denoted by τ in the first path and in turn the desired bridging transformation $\Psi(\cdot)$ in the second path depend on the prescribed transformation $Tr(\cdot)$ [see Fig. 3]. In the following discussion we consider only reconstruction by dilation by L. Therefore, unless otherwise stated $\tilde{\mathcal{A}}$ represents $\tilde{\mathcal{A}}_d^L$.

4.1. DILATION

Suppose $x \equiv (r_1, c_1)$ and $y \equiv (r_2, c_2)$ are two points in Z^2. Now if we dilate x by y, we get another point $z \equiv (r_1 + r_2, c_1 + c_2)$ in Z^2. Formally, we can write

$$Tr(x) = x \oplus y = z \qquad \text{... (19)}$$

Now if we like to perform the same operation on the reconstructed sets, we proceed as follows. First dilate x and y by L to get L_x and L_y, respectively which are equivalent to the reconstructed sets, i.e., $L_x = x \oplus L$ and $L_y \oplus L$. Then we dilate L_x by L_y, i.e.,

$$Tr(L_x) = L_x \oplus L_y \qquad \text{... (20)}$$

Sampling of $L_x \oplus L_y$ produces a set of four points:

$$\{(r_1 + r_2, c_1 + c_2), (r_1 + r_2 + 1, c_1 + c_2), (r_1 + r_2, c_1 + c_2 + 1), (r_1 + r_2 + 1, c_1 + c_2 + 1)\}$$

So we have to apply some corrective operation before sampling to get a result closes to what we get when transformation $Tr(\cdot)$ is applied on the sampled set [see eq. 19]. One such corrective operation may be erosion by $2L$, where nL may be defined as [3]

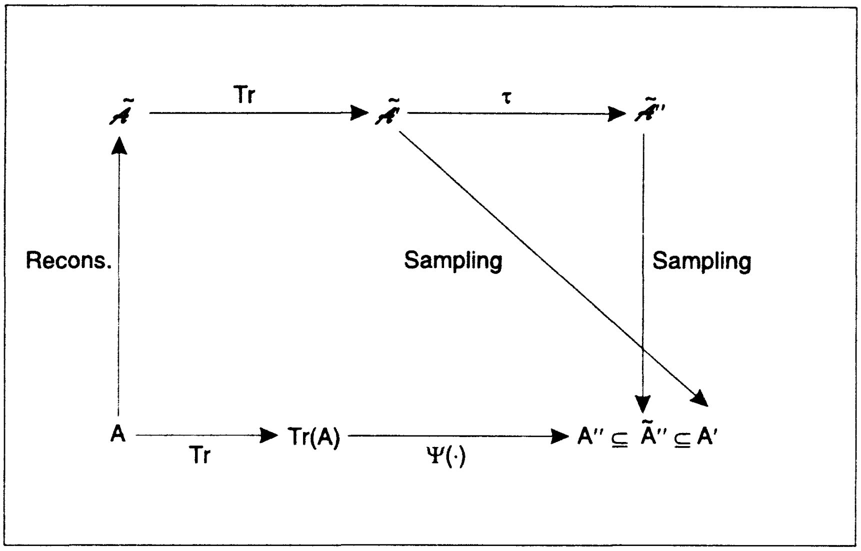


FIG. 3. Showing two paths: The first one consists of reconstruction followed by applying transformation on the reconstructed set folowed by corrective operation followed by sampling, and the second one consists of applying transformation on the sampled set followed by applying the bridging transformation for comparing results.

$$nL = L \oplus L \oplus \ldots \oplus L,\ n > 0 \qquad \ldots (21)$$

Thus the corrective operation $\tau[\equiv \ominus 2L]$ is applied to produce a point same as $z \equiv (r_1 + r_2, c_1 + c_2)$ as follows

$$[(L_x \oplus L_y) \ominus 2L] \cap Z^2 = \{(r_1 + r_2 + c_1 + c_2)\} \qquad \ldots (22)$$

Now consider two sampled sets A and B, and perform a dilation between them. The complete transformation along the first path of Fig. 3 is given by $[\{A \oplus L) \oplus (B \oplus L)\} \ominus 2L] \cap Z^2 \equiv [\{A \oplus B) \oplus L \oplus L\} \ominus 2L] \cap Z^2 \equiv [\{(A \oplus B) \oplus 2L\} \ominus 2L] \cap Z^2 \equiv [(A \oplus B) \bullet 2L] \cap Z^2 \equiv (A \oplus B) \bullet S$ where $S = 2L \cap Z^2$, that means S is a 2×2 structuring element in the discrete domain. Now $(A \oplus B) \bullet S$ may be considered as the transformation along the second path of Fig. 3. Hence, the bridging transformation $\Psi(\cdot)$ we are looking for appears to be closing by S, i.e.,

$$\Psi(Tr(A)) = Tr(A) \bullet S \qquad \ldots (23)$$

where, $Tr(A)$ is equivalent to dilation of A by B, and S represents a 2×2 structuring element in discrete domain.

Comparing Figs. 2 and 3 alongwith equations (19-23) we may say that $\tilde{\mathcal{A}}$ is approximation of $\mathcal{A}'$ and $(A \oplus B) \bullet S \subseteq [(A \oplus B) \bullet L] \cap Z^2$, i.e.,

$$A'' \subseteq \tilde{\mathcal{A}}''$$

$Tr(A) \bullet S = A''$ This implies $Tr(A) \subseteq A''$ (since closing is extensive). Now proposition 7 says that $\mathcal{A}' \supseteq \tilde{\mathcal{A}}' \ominus L$, so $\mathcal{A}' \supseteq (\tilde{\mathcal{A}}' \ominus L) \ominus L \supseteq \tilde{\mathcal{A}}' \ominus 2L = \tilde{\mathcal{A}}''$.

Therefore, $Tr(A) \subseteq A'' \subseteq \tilde{\mathcal{A}}'' \subseteq \mathcal{A}'$

This implies equations 2 and 3 are satisfied by the suggested bridging transformations $\Psi(\cdot)$

4.2 Erosion

Suppose, again, $x \equiv (r_1, c_1)$ and $y \equiv (r_2, c_2)$ are two points in $\mathbf{Z}^2$. Now if we erode x by y, we get another point $z \equiv (r_1 - r_2, c_1 - c_2)$ also in $\mathbf{Z}^2$. Formally, we can write

$$Tr(x) = x \ominus y = z \qquad \text{... (24)}$$

Now if we like to perform the same operation on the reconstructed sets, we proceed as before. That is, we first dilate x and y by L to get L_x and L_y respectively, which are equivalent to the reconstructed sets, and then erode L_x by L_y. This eventually produces the same point $z \in \mathbf{Z}^2$. Hence, the desired bridging transformation $\Psi(\cdot)$ in this case is a unity transformation. Equations 2 and 3 are also satisfied by this $\Psi(\cdot)$

4.3 Translation

We have said earlier that the reconstruction by dilation by L is a translation invariant operation. That means, if a set is reconstructed from $A \subseteq \mathbf{Z}^2$ by dilation by L and if the reconstructed set is translated by an amount t in the continuous domain, then sampling of the translated reconstructed set gives the original sampled set A translated by an amount which is nearest integer of t, i.e.,

$$(A \oplus L)_t \cap \mathbf{Z}^2 = A_{\lfloor t + 0.5 \rfloor} \qquad \text{... (25)}$$

Hence, here also the desired bridging transformation $\Psi(\cdot)$ is a unity transformation. Equations 2 and 3 are also satisfied by this $\Psi(\cdot)$

4.4 Rotation

Let us consider rotation of a set of points by an angle θ in anti-clock direction. Then a point $x \equiv (r, c) \in \mathbf{Z}^2$ moves to a new location $z \in \mathbf{Z}^2$ whose coordinate is obtained by first transforming coordinate of x and then rounding it to nearest integer, i.e., $z \equiv (\lfloor r \cos\theta - c \sin\theta + 0.5 \rfloor, \lfloor r \sin\theta \; c \cos\theta + 0.5 \rfloor)$. Thus, for one point x we get one and only one point z. On the other hand, if we rotate the reconstructed set L_x due to the point x through the same angle, and then sample the rotated set, depending on the values of r, c and θ we may get :

1) no point; or

2) exactly one point; or

3) two points that are adjacent horizontally or vertically.

First situation suggests dilation of transformed sampled set by a 1×2 or a 2×1 structuring element, while the third situation suggests erosion of transformed sampled set by a 1×2 and by a 2×1 structuring element independently. And the second situation suggests a unity transformation as the bridging transformation. Following these arguments and considering Fig. 3, the desired bridging transformation $\Psi(\cdot)$ is given by

$$\Psi(Tr(A)) = [Tr(A) \bullet S_h] \cup [Tr(A) \bullet S_\upsilon], \qquad \text{... (26)}$$

where, TR (A) is equivalent to rotation of A by θ, and S_h and S_υ represent 1×2 and 2×1 structuring elements, respectively in the discrete domain.

4.5 MAGNIFICATION

Suppose $x \equiv (r, c)$ is a member of a set of points A in $\mathbf{Z}^2$. Now if we magnify that by a scaling factor of m, then corresponding to x we get one and only one point $z \equiv (mr, mc)$ in $\mathbf{Z}^2$. Whereas if we perform the same operation on the reconstructed set L_x we get a magnified set mL_x produces a set of $m \times m$ points:

$$\{(r, c), (r+1, c), ..., (r+m-1, c), (r, c+1),$$

$$... (r, c+m-1), ..., (r+m-1, c+m-1)\}$$

This suggests that corrective operation may be an erosion by mL. Hence, the corrective operation $\tau[\equiv \Theta\, mL]$ is applied to produce a point the same as $z \equiv (mr, mc)$ as follows

$$(L_x \,\Theta\, mL) \cap \mathbf{Z}^2 = \{(mr,\ mc)\}. \qquad \text{... (28)}$$

Now consider a set A, and magnify it by a scale factor m. Thus the complete transformation along the first path of Fig. 3 is given by

$$[\{m(A \oplus L)\}\ \Theta\, mL] \cap \mathbf{Z}^2 \equiv [\{mA \oplus mL\}\ \Theta\, mL] \cap \mathbf{Z}^2 \equiv [mA \bullet mL] \cap \mathbf{Z}^2 \equiv mA \bullet S,$$

where $S = mL \cap Z^2$ is a $m \times m$ structuring element in $\mathbf{Z}^2$. Now $mA \bullet S$ may be considered as the transformation along the second path of Fig. 3. Hence, the bridging transformation $\Psi(\cdot)$ we are looking for appears to be closing by S, i.e.,

$$\Psi(Tr(A)) = Tr(A) \bullet S, \qquad \text{... (29)}$$

where $Tr(A)$ is equivalent to magnification of A by m, and S represents a $m \times m$ structuring element in discrete domain. Note that if m is non-integer then S represents a $\lceil m \rceil \times \lceil m \rceil$ structuring element, where $\lceil a \rceil$ represents smallest integer integer not less than a. Finally, it can be shown that equations 2 and 3 satisfied by the suggested bridging transformation $\Psi(\cdot)$

5. EXAMPLES

Here we present two examples that show how the proposed bridging transform fills the gap between the transformed sampled image and the sampled transformed image. Two different transformations are considered here: Rotation and magnification. The image contains a *key* against an almost uniform background.

Result of rotation is shown in Fig. 4. The object, the key, is imaged and digitized by a CCD camera. During imaging optical axis of the camera is held perpendicular to the object plane (a table in this case). Corresponding digital image is shown in Fig. 4(a). Thus it represents a sampled

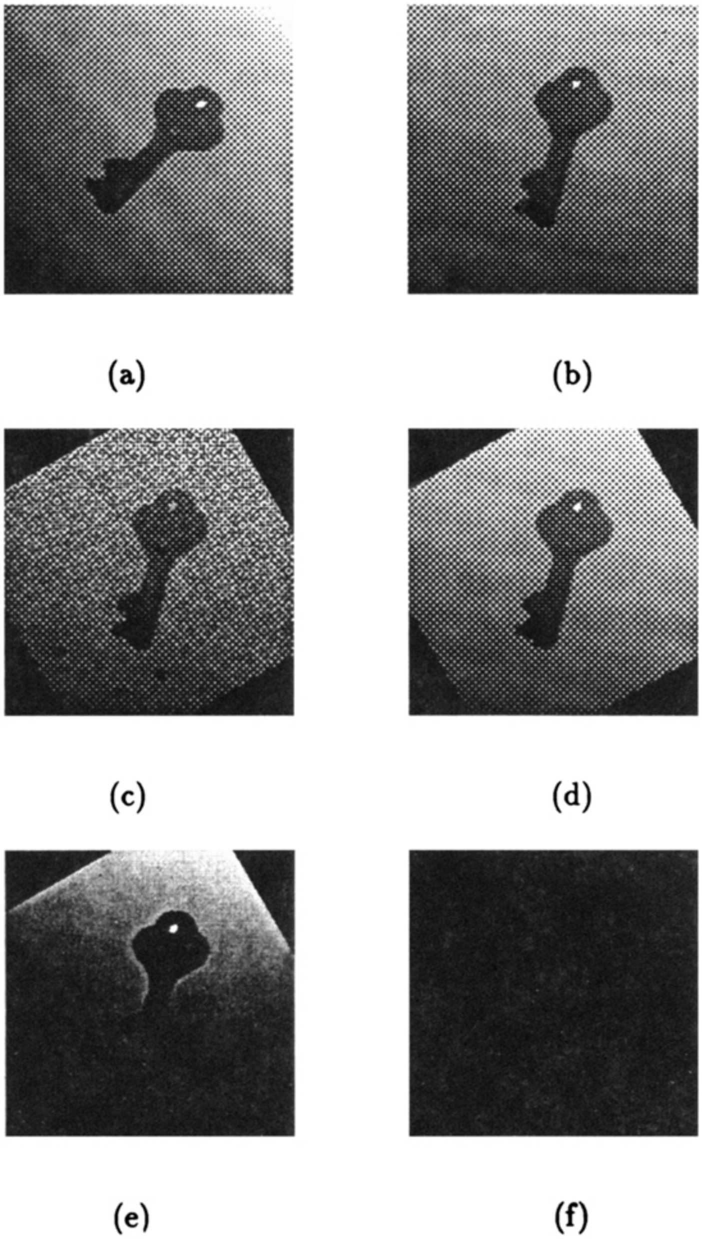
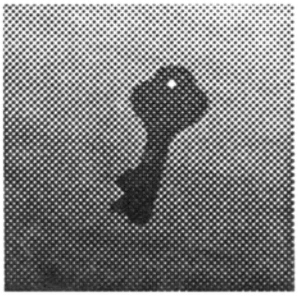
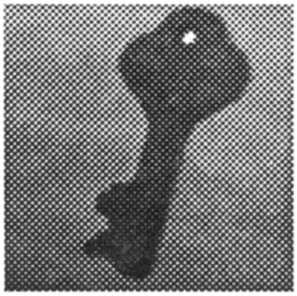


FIG. 4. Illustrates usefulness of bridging transform. (a) Image of original object, (b) Image of sampled transformed object. Here transformation is rotation by 30°. (c) Transformed sampled object where transformation is same as before. (d) Image after applying bridging transformation on (c). (e) Image obtained by rotating (a) using conventional (i.e., inverse transformation) method. (f) Illustrates difference between (d) and (e)

or digitized object. Now the camera is rotated by 30° about its optical axis and another image is taken as shown in Fig. 4(*b*). So this image represents the object that is first transformed in continuous domain and then sampled. The said transformation is, therefore, a rotation by 30° in two-dimensional object plane. We apply similar transformation on Fig. 4(*a*) which is a sampled object, and obtain Fig. 4(*c*) that represents the transformed sampled object. Finally, we apply bridging transformation



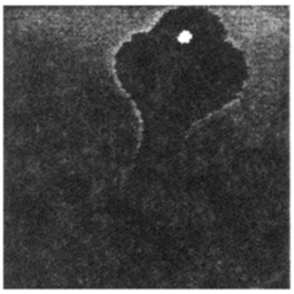
(a)



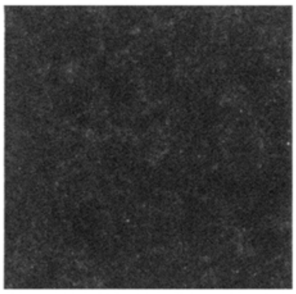
(b)

(c)

(d)



(e)



(f)

FIG. 5. Illustrates usefulness of bridging transform. (a) Image of original object. This is same as Fig. 4(b). (b) Image of sampled transformed object. Here transformation is magnification by a factor of 2 in both the directions. (c) Transformed sampled object where transformation is same as before. (d) Image after applying bridging transformation on (c). (e) Image obtained by magnifying (a) using conventional (i.e., inverse transformation) method. (f) Illustrates difference between (d) and (e)

on Fig. 4(*c*). The resultant image is Fig. 4(*d*). A simple comparison shows that figure (*b*) is much closer to (*d*) than (*c*). That means bridging transformation fills the gap between (*b*) and (*c*).

In case of magnification, we use the same sampled object as shown in Fig. 5(*a*) which is same as Fig. 4(*b*). Now the camera lens is zoomed to a{italic chi}eve magnification by a factor of

2 in both the directions. The digital image of magnified object is shown in Fig. 5(*b*). We magnify the relevant portion of the image of sampled object (i.e., Fig. 5(*a*)) by the same factor to obtain an image as shown in Fig. 5(*c*). Now, we apply appropriate bridging transformation on Fig. 5(*c*). The resultant image is Fig. 5(*d*). A simple comparison shows that figure (*b*) is much closer to (*d*) than (*c*). That means bridging transformation fills the gap between (*b*) and (*c*).

Fig. 4(*a*) is also transformed (i.e., rotated) by conventional method, that is, by applying suitable inverse transformation in the discrete domain. The result is shown in Fig. 4(*e*). An image depicting difference between Fig. 4(*d*) and Fig. 4(*e*) is shown in Fig. 4(*f*). We also obtain Figs. 5(*e*) and (*f*) in a similar way starting from Fig. 5(*a*). Visual comparison between (*d*) and (*e*) [of both Fig. 4 and Fig. 5], as well as quantitative analysis of (*f*) reveals that results obtained through forward transformation followed by bridging transform are as good as that of conventional method. However, our intention is not to establish superiority of our method over conventional ones. What we claim is information lost due to applying (forward) transformation in the discrete domain can be recovered, at least partially, by bridging transform. Thus it is extremely useful for the transformations (e.g., dilation or other nonlinear transformations) for which inverse transformations do not exist.

6. CONCLUSION

In this work we have derived suitable bridging transformations for some widely used transformations, like dilation, erosion, translation, rotation and magnification, to fill the gap (at least partially) between the transformed sampled set and the sampled transformed set. This bridging transformations enables us to implement some of these said transformations (e.g., rotation and magnification) using forward transformations only. Note that these transformations are thus far being implemented using reverse transformations.

Determination of the bridging transformations requires reconstruction of the original object in the continuous domain. The reconstruction methodology suggested in this work can approximate the original set to subpixel accuracy. However, the entire analysis is based on the assumption that the original object in the continuous domain is both open and close under the structuring element $K = (-1, 1) \times (-1, 1)$. That means the suggested bridging transformations are suitable only to those discrete objects that are both open and close under a 2×2 structuring element.

Finally, it should be pointed out that, in this paper, our intention is not to propose any algorithm for geometric transformation in discrete domain that is superior to conventional ones. In stead what we intend to show is that bridging transformation can recover, at least partially, information lost due to applying forward transformation in the discrete domain. Thus it is extremely useful for the nonlinear transformations (e.g., dilation) for which inverse transformations do not exist. However, wherever such inverse transformations are available, it can be shown that results obtained through forward transformation followed by bridging transform are as good as that of conventional method. Experimental results also support above observations and propositions.

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