Mathematical Morphological Operations of Boundary-represented Geometric Objects

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Abstract

The resemblance between the integer number system with multiplication and division and the system of convex objects with Minkowski addition and decomposition is really striking. The resemblance also indicates a computational technique which unifies the two Minkowski operations as a single operation. To view multiplication and division as a single operation, it became necessary to extend the integer number system to the rational number system. The unification of the two Minkowski operations also requires that the ordinary convex object domain must be appended by a notion of inverse objects or *negative objects*. More interestingly, the concept of negative objects permits further unification. A nonconvex object may be viewed as a *mixture* of ordinary convex object and negative object, and thereby, makes it possible to adopt exactly the same computational technique for convex as well as nonconvex objects. The unified technique, we show, can be easily understood and implemented if the input polygons and polyhedra are represented by their *slope diagram* representations.

Key words: Mathematical morphology, Minkowski operations, negative object, slope diagram representation.

1 Introduction

1.1 Problem definition

In this paper we deal with the *computational* aspects of morphological operations of *boundary-represented* geometric objects. Two operations considered in this paper are

Minkowski addition $\oplus$ (also called *dilation*) and Minkowski decomposition $\ominus$ (also known as *erosion*). These two operations which form the kernel of all other morphological operations are defined as follows.

If A and B are two arbitrary sets of points in the real Euclidean d-dimensional space E^d, their *Minkowski addition*, $A \oplus B$, is defined as

$$A \oplus B = \{a + b \mid a \in A,\ b \in B\}, \tag{1}$$

where '+' denotes the normal vector addition of two points, and A and B are called the summands of the sum $A \oplus B$. It can also be expressed in terms of the set union and geometric translation operations. If A_p denotes the translate of a set A by a vector p, that is, $A_p = A \oplus \{p\}$, then it is easy to see that

$$A \oplus B = B \oplus A = \bigcup_{b \in B} A_b = \bigcup_{a \in A} B_a. \tag{2}$$

Minkowski decomposition $A \ominus B$ is the inverse of Minkowski addition in a "restricted" sense. It is defined as,

$$A \ominus B = \bigcap_{-b \in \check{B}} A_{-b}. \tag{3}$$

The set $\check{B} = \{-b \mid b \in B\}$ is called the *symmetrical set* of B with respect to the origin point.

We use the concept of *negative shape* in dealing with the computational problem of these two operations.

1.2 Motivation

The motivation to take up this problem and the approach adopted by us have been origined from the following facts.

- *Morphology for high-level vision.* Till this day morphological operations are mostly used for *low-level* image processing, that is, for early processing of binary or grayscale discrete images [18]. But their applications in representing or understanding $2D$ or $3D$ geometric objects for *high-level* object recognition is very limited. This is certainly an enigmatic situation by considering the fact that these operations are essentially functions of the form $f : \mathcal{G}(E^d) \to \mathcal{G}(E^d)$, where $\mathcal{G}(E^d)$ denotes the power set of the real d-dimensional Euclidean space E^d; their definitions do not discriminate whether the underlying sets are discrete images or continuous geometric objects. Moreover, a number of researchers have pointed out the relevance of these operations in high-level vision applications such as geometric modeling, spatial planning, biological form description and understanding, crystallography and textured object modeling, etc. [2, 3, 5, 11, 15]. The close resemblance between the *generalized cylinder* representation and Minkowski addition operation has also been observed.

One reason that may account for this situation is the *computational problems* associated with Minkowski operations of continuous objects. Note that any binary image can be modeled as a set consisting of finite number of discrete points. Therefore, in computing the Minkowski sum $A \oplus B$ of two binary images A and B, one can directly use Eqn.1 or Eqn.2. Similarly, Minkowski decomposition $A \ominus B$ of a binary image A can be computed by means of Eqn.3.

On the other hand, a continuous "well-formed" object (such as polygon, circle, ellipse, etc. in $2D$, or polyhedron, sphere, ellipsoid, etc. in $3D$) cannot be specified as a collection of finite number of points. In general, such an object, say A, is specified by its oriented boundary, say ∂A. This necessitates the computation of the boundary $\partial(A \oplus B)$ or $\partial(A \ominus B)$ of the products from the boundaries ∂A and ∂B of the operands. But one can immediately see that $\partial(A \oplus B) \neq \partial A \oplus \partial B$ and also $\partial(A \ominus B) \neq \partial A \ominus \partial B$. In general, $\partial(A \oplus B) = f(\partial A, \partial B)$, and also $\partial(A \ominus B) = g(\partial A, \partial B)$, where '$f$' and '$g$' denote some complicated functions whose computations turn out to be quite nontrivial.

- *Resemblance between morphology and the theory of numbers.* We observe a remarkable similarity between some number theoretic results and morphological theorems, particularly in the domain of convex objects. Let us consider the number system $(N, \cdot, /)$, where N denotes the set of natural numbers $\{0, 1, 2, \ldots\}$, and "$\cdot$" and "$/$" denote multiplication and division operations respectively. Since within N the division operation is not an exact inverse, but only a "restricted" inverse of the multiplication operation, we may define the division of a number m by another number n as $\lfloor m/n \rfloor$; the floor function notation $\lfloor x \rfloor$ means the greatest integer less than or equal to x. This number system may be compared with the geometric sytem $(\mathcal{K}, \oplus, \ominus)$, where $\mathcal{K}$ denotes the set of all compact convex subsets of E^d. We refer to Table 1, where $A, B, C \in \mathcal{K}$ and $m, n, p \in N$.

  From the table it appears, as if $A \oplus B$ in the convex domain is alike to $m \cdot n$, whereas $A \ominus B$ is alike to $\lfloor m/n \rfloor$. In this paper we do not attempt to seek why such a resemblance exists. Instead we adopt the computational guideline that may be derived from this resemblance. The value of m/n is, in general, a real number which consists of the "integer part" $\lfloor m/n \rfloor$ and the "fractional part" $m/n - \lfloor m/n \rfloor$. In the integer number domain, at the time of division we discard that fractional part and take only the integer part. We find, the morphological operations may also be treated in the similar way. The idea is to devise an operation which is the "exact" inverse of Minkowski addition operation such that the application of this operation on two convex objects will produce a *generalized* geometric object (like producing real number by division of two integer numbers) which has a physically "realizable part" (analogous to integer part) and a "non-realizable part" (analogous to fractional part). Minkowski decomposition $A \ominus B$ may then be thought of as the realizable part of the object after discarding the non-realizable part.

  The non-realizable part of a generalized object will be referred to as the *negative* part. (We avoid using terms such as "non-realizable" or "fractional" part, since

Table 1: Resemblance between morphological system and integer number system

	System $(\mathcal{K}, \oplus, \ominus)$	System $(N, \cdot, /)$
1.	$A \oplus B = B \oplus A$	$m \cdot n = n \cdot m$
2.	$A \oplus (B \oplus C) = (A \oplus B) \oplus C$	$m \cdot (n \cdot p) = (m \cdot n) \cdot p$
3.	$A \supseteq B$ implies $C \ominus A \subseteq C \ominus B$	$m > n$ implies $\lfloor p/m \rfloor \leq \lfloor p/n \rfloor$
4.	$A \subseteq B \ominus C$ iff $B \supseteq A \oplus C$	$m \leq \lfloor n/p \rfloor$ iff $n \geq m \cdot p$
5.	$A \oplus B = ((A \oplus B) \ominus B) \oplus B$	$m \cdot n = \lfloor (m \cdot n)/n \rfloor \cdot n$
6.	$A \ominus B = ((A \ominus B) \oplus B) \ominus B$	$\lfloor m/n \rfloor = \lfloor (\lfloor m/n \rfloor \cdot n)/n \rfloor$
7.	$(A \ominus B) \ominus C = A \ominus (B \oplus C)$	$\lfloor \lfloor m/n \rfloor /p \rfloor = \lfloor m/(n \cdot p) \rfloor$
8.	$A \oplus (B \ominus C) \subseteq (A \oplus B) \ominus C$	$m \cdot \lfloor n/p \rfloor \leq \lfloor (m \cdot n)/p \rfloor$
9.	$(A \ominus B) \oplus C \subseteq (A \oplus C) \ominus B$	$\lfloor m/n \rfloor \cdot p \leq \lfloor (m \cdot p)/n \rfloor$
10.	$(A \ominus B) \oplus B \subseteq A$	$\lfloor m/n \rfloor \cdot n \leq m$
11.	$(A \ominus B) \oplus B = (((A \ominus B) \oplus B) \ominus B) \oplus B$	$\lfloor m/n \rfloor \cdot n = \lfloor (\lfloor m/n \rfloor \cdot n)/n \rfloor \cdot n$
12.	If $A \oplus B = A \oplus C$, then $B = C$	If $m \cdot n = m \cdot p$, then $n = p$

they have some widely accepted literal meanings.) The realizable part, that is, any ordinary geometric object may be called *positive*.

In our subsequent discussion we show how such a computational strategy can be devised.

2 Morphological operations of convex polygons

2.1 Computation by means of support function vectors

The boundary of a convex polytope A in E^d can be precisely defined by means of *supporting function* of the polytope. (A convex polytope, the analogue of a convex polygon in E^2 and a convex polyhedron in E^3, is a bounded set which can be written as the intersection of a finite number of half-spaces in E^d.) The supporting function $H(A, u)$ of A is defined for all $u \in E^d$ by

$$H(A, u) = \sup\{< a, u > \mid a \in A\},$$

where $< a, u >$ denotes the scalar product of the vectors a and u, and "sup" stands for 'supremum' or 'least upper bound'.

Now the following result concerning Minkowski addition of convex polytopes can be easily proved [6].

Theorem 1 *If A and B are two convex polytopes in E^d, then for every $u \in E^d$,*

$$H(A \oplus B, u) = H(A, u) + H(B, u). \tag{4}$$

From Theorem 1 we can immediately infer:

- For Minkowski addition, it is the direction of u, but not its magnitude, which is of concern, provided $u \neq 0$. Thus without loss of generality we assume that u is a unit vector, that is, $u \in S^{d-1}$, where S^{d-1} denotes the unit sphere in E^d.
- Furthermore, for convex polygons addition it is not necessary to compute the supporting function $H(A \oplus B, u)$ for every $u \in S^1$ (S^1 in E^2 is nothing but a unit circle); since a convex polygon is completely specified by its oriented edges, it is sufficient to consider only those u's whose directions are the same as the outer normal directions of the edges of the summand polygons.

Let us, therefore, consider the class of convex polygons whose edges have the same outer normal vectors. In other words, any two polygons belong to this class have pairwise parallel and similarly directed edges. Note that every convex polygon in E^2 can be included in this class by means of introducing edges of zero length. Let $C(U)$ denotes this class, where U is the ordered set of outer normal vectors of the edges, that is, $U = \{u_1, \ldots, u_n\}$. In other words, if any convex polygon $A \in C(U)$, then

$$A = \{p \in E^2 \mid \, < p, u_i > \leq \, \eta_i \; (i = 1, \ldots, n)\}$$

for some $\eta_i \in R$ $(i = 1, \ldots, n)$, where R denotes the set of real numbers. Note that η_i is nothing but the value of the supporting function of A in the direction of the outer normal u_i.

As we know, each η_i along with u_i specify a closed half-space (in $2D$ "half-space" is generally called "half-plane"), and the intersection of all these half-spaces for $i = 1, \ldots, n$, in turn, specify all the edges of A. Therefore, once U is given, A is specified completely by the vector $(\eta_1, \ldots, \eta_n)$. We call it the *supporting function vector* of A, and denote it by $h(A)$.

From Theorem 1 we can now easily derive the following result concerning convex polygons.

Proposition 1 *If two convex polygons $A, B \in C(U)$ are represented by the supporting function vectors $(\eta_1^A, \ldots, \eta_n^A)$ and $(\eta_1^B, \ldots, \eta_n^B)$ respectively, then their Minkowski sum $A \oplus B$ is specified completely by the vector $(\eta_1^A + \eta_1^B, \ldots, \eta_n^A + \eta_n^B)$.*

Proposition 1 implies that Minkowski addition of two convex polygons can be seen as the vector addition $h(A) + h(B)$ of two points $h(A)$ and $h(B)$ in an n-dimensional space. From this observation we can now define the "exact" inverse of Minkowski addition. In computing $A \ominus B$, we shall first compute the vector $h(A) + (-h(B))$. This vector, like the vector $h(A) + h(B)$, again specifies n number of half-spaces having outer normals u_i's, but some of the half-spaces may become *redundant* in this case. Minkowski decomposition $A \ominus B$ then turns out to be discarding those redundant half-spaces, and considering the rest. Let us state this result as the following proposition.

Proposition 2 *If two convex polygons $A, B \in C(U)$ are represented by the supporting function vectors $(\eta_1^A, \ldots, \eta_n^A)$ and $(\eta_1^B, \ldots, \eta_n^B)$ respectively, then the Minkowski decomposition $A \ominus B$ can be obtained by discarding the redundant half-spaces specified by the vector $(\eta_1^A - \eta_1^B, \ldots, \eta_n^A - \eta_n^B)$.*

The novelty of our approach lies in our treatment of *not discarding* those redundant half-spaces, but retaining them as essentials.

Example. Let us clarify the idea described so far by means of an example (Fig.1).

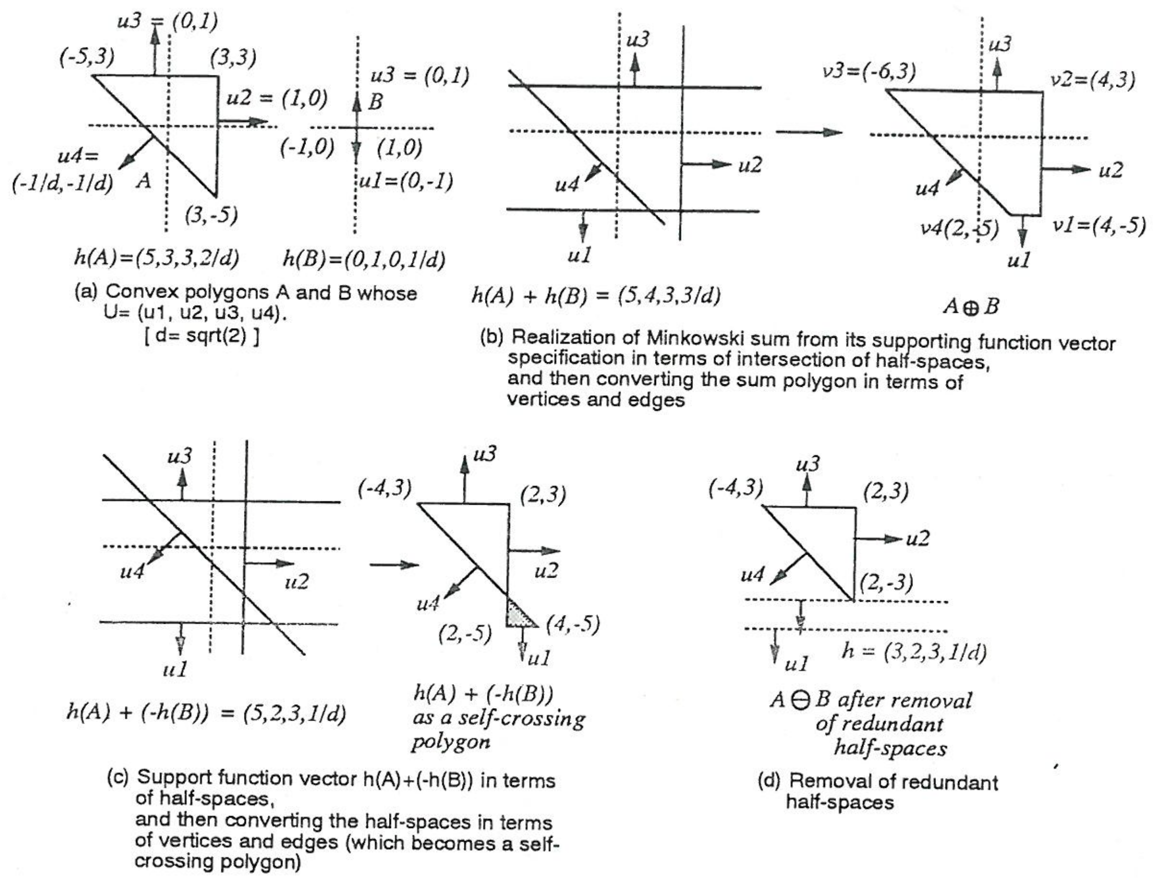


Figure 1: Minkowski addition and decomposition by means of supporting function vectors.

As is evident from the example (Fig.1b), Minkowski addition of two convex polygons is quite straightforward by means of the supporting function vector $h(A)+h(B)$. In computing the decomposition $A\ominus B$ from the supporting function vector $h(A)+(-h(B)) = (\eta_1^A-\eta_1^B,\ldots,\eta_n^A-\eta_n^B)$, note (in Fig.1c) that one of the half-planes, namely, the half-plane corresponding to the outer normal u_1, is redundant. The normal custom is to abandon this redundant half-plane altogether for further consideration. That effectively means, appropriate reduction of the corresponding value of the supporting function in the supporting function vector $h(A)+(-h(B))$. In our example, the corresponding supporting function value $\eta_1^A-\eta_1^B = 5$ has been reduced to 3 (see Fig.1d). This clearly explains why, in general, $(A\ominus B)\oplus B\subseteq A$. □

2.2 Computation by means of edges: Emergence of Boundary addition operation $\uplus$

Representing a convex polygon A by means of its supporting function vector $h(A)$ is not a very common practice. A more popular representation is the *edge length* representation where the boundary ∂A is represented by means of a starting vertex v_s^A and the length ι_i^A of the edge corresponding to every u_i, that is, $\partial A = \{\{v_s^A\}, \{\iota_1^A, \ldots, \iota_n^A\}\}$. From the $h(A)$-representation it is easy to arrive at the edge length representation by finding out the intersection points of adjacent half-spaces (see the second figures in Fig.1b or Fig.1c).

If the other summand B is also represented in the similar way, that is, $\partial B = \{\{v_s^B\}, \{\iota_1^B, \ldots, \iota_n^B\}\}$, then by a straightforward computation we get,

$$h(A) + h(B) \equiv \{\{v_s^A + v_s^B\}, \{\iota_1^A + \iota_1^B, \ldots, \iota_n^A + \iota_n^B\}\}. \tag{5}$$

Eqn.5 states that the boundary $\partial(A \oplus B)$ of the Minkowski sum of two convex polygons can be easily computed when the boundaries ∂A and ∂B are represented in their edge length forms. The computation essentially involves *addition of the lengths of the corresponding edges of the summands.* This operation may be termed as the *boundary addition* operation and is denoted by the symbol "$\uplus$". Eqn.5 can, therefore, be rewritten as,

$$\partial(A \oplus B) = \partial A \uplus \partial B = \{\{v_s^A + v_s^B\}, \{\iota_1^A + \iota_1^B, \ldots, \iota_n^A + \iota_n^B\}\}. \tag{6}$$

For computing Minkowski decomposition, we convert the support function vector $-h(B)$ to the equivalent edge length representation which turns out to be

$$-h(B) = \{\{-v_s^B\}, \{-\iota_1^B, \ldots, -\iota_n^B\}\}.$$

The polygon represented by $-h(B)$, if drawn pictorially, appears like a hole or a *negative region,* without any positive region surrounding the hole (Fig.2b). This is because, for any ordinary convex polygon (Fig.2a) all the outer normals u_i's "diverge outward", whereas the normals u_i's of $-h(B)$ appear to "converge inward"; but the *directions* of the normals remain exactly the same in both the cases. We may distinguish these two cases by introducing the concept of sense of the outer normals. The outer normals of ordinary convex objects which diverge outward may be thought of having a "positive" sense, whereas the normals of $-h(B)$ have a "negative" sense. We may consider the polygon represented by $-h(B)$ as the additive inverse of the polygon B, and may denote it by the symbol B^{-1}.

As far as the geometric shape is concerned, B^{-1} appear exactly like its symmetrical set $\check{B}$. But notice the differences among the objects B, $\check{B}$, and B^{-1}. The sense of every outer normal of both B and $\check{B}$ is the same, which is positive, while the directions of the outer normals at the corresponding faces of the two are exactly opposite. Because of the positive sense of the outer normals, we consider both B, and $\check{B}$ as positive objects. On the contrary, the directions of the outer normals at the corresponding faces of B and B^{-1} are exactly the same, but the senses are opposite. Because of the negative sense of the outer normals, B^{-1} may be considered as a negative object.

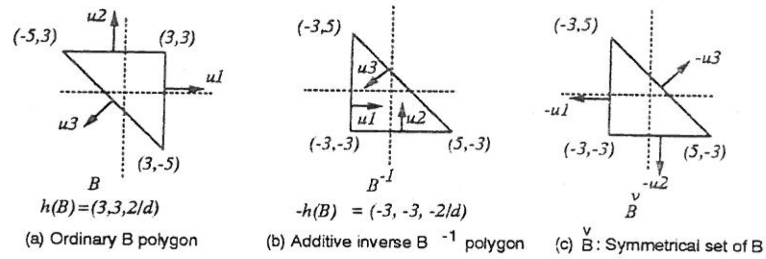


Figure 2: A geometric interpretation of inverse shape B^{-1}; $\check{B}$ is also shown for comparison.

Using the notations of boundary addition, we can write

$$h(A) + (-h(B)) \equiv \partial A \uplus \partial B^{-1} = \{\{v_s^A - v_s^B\}, \{\iota_1^A - \iota_1^B, \ldots, \iota_n^A - \iota_n^B\}\}. \qquad (7)$$

Since $h(A) + (-h(B))$ may contain redundant half-spaces, the polygon represented by $\partial A \uplus \partial B^{-1}$ will be, in general, a self-crossing polygon (see the second figure in Fig.1c). Considering the senses of the outer normals, a self-crossing polygon may appear like a polygon having a positive part and a negative part (the negative part is shown shaded in Fig.1c). Since we obtain $A \ominus B$ from $h(A) + (-h(B))$ by discarding the redundant half-spaces, equivalently $A \ominus B$ is obtained from $\partial A \uplus \partial B^{-1}$ by discarding the negative part of the polygon $\partial A \uplus \partial B^{-1}$, and considering only its positive part. This can be expressed symbolically as,

$$\partial(A \ominus B) = Pos(\partial A \uplus \partial B^{-1}), \qquad (8)$$

where $Pos(X)$ denotes a unary operation which extracts the positive portion of a self-crossing object X.
(Note: In case one is interested only in the "shapes" of the objects but not upon their positions in the plane, then the starting vertices v_s^A, v_s^B etc. (appeared in Eqn.6 and Eqn.7) may be ignored.)

2.3 Computation by means of slope diagrams: Unification of Minkowski addition and decomposition

The *slope diagram* representation of a polygon is essentially the edge length representation in a pictorial form. Since all the outer normals u_i's of a convex polygon lie on a unit circle S^1, in slope diagram representation we consider a unit circle as the basis. The representation scheme goes as follows (refer to Fig.3a):

a. The outer normal direction at each edge of the polygon is represented by the corresponding point on a unit circle. It is called an *edge point.* (By "corresponding point" we mean, that point on the unit circle where the outer normal direction is the same as the outer normal direction of the edge.)

b. At each vertex of the polygon, it is possible to draw innumerably many outer normals filling an angle (supplementary to the interior angle at the vertex). This set of outer normal directions at the vertex is represented by the corresponding arc on the unit circle. It is called a *vertex arc*.

c. Apart from the direction of an outer normal, the sense of the outer normal must also be indicated. If the sense of an outer normal is negative it will be shown by thick black points or arcs, while an outer normal having positive sense will be drawn by thin lines.

d. The length of each edge is associated with its corresponding edge point like a label.

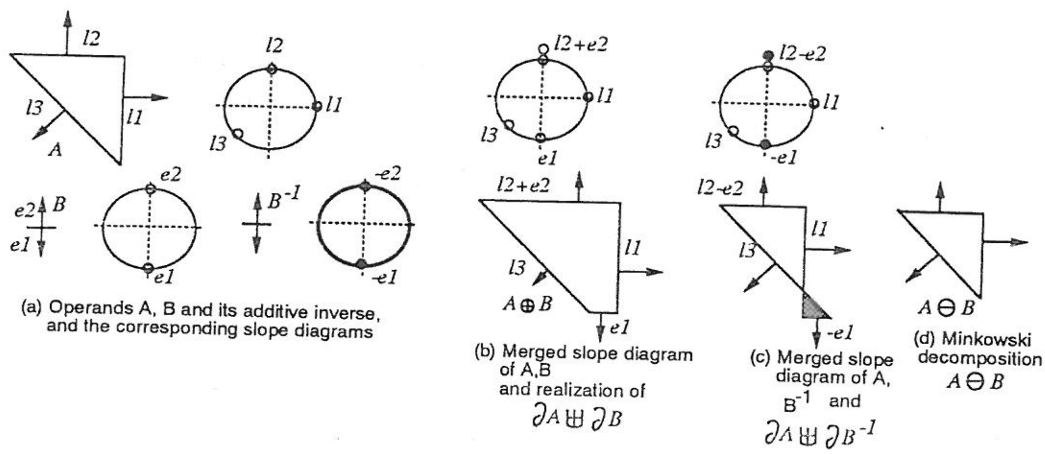


Figure 3: Minkowski addition and decomposition by means of slope diagrams; in this case, $\partial A \uplus \partial B = \partial(A \oplus B)$, but $\partial(A \ominus B) = Pos(\partial A \uplus \partial B^{-1})$.

COMPUTATION OF BOUNDARY ADDITION $\uplus$

Computation of $\partial A \uplus \partial B$ or $\partial A \uplus \partial B^{-1}$ (refer to Eqn.6 or Eqn.7; we ignore v_s^A, v_s^B etc. at present) by means of slope diagrams becomes quite starightforward:

1. *[Merging of slope diagrams]* Merge the slope diagrams of the two operands into a single one.

2. *[Realization of the boundary sum]* From the merged slope diagram, realize the polygon it represents. The term "realization" means "concatenation" of the edges (that is, joining end-point of one edge to the start-point of the next edge) in the sequence they appear in the merged slope diagram (Fig.3b or Fig.3c).

The realization/concatenation process may need some more clarification. First, refer to Eqn.6 concerning Minkowski addition. In the merged slope diagram two distinct cases may arise: (1) If two edge points of the summands occur at the same position of the unit circle, it means both the edges have the same outer normal direction. Therefore, concatenation of these two edges effectively means the addition of the lengths $\iota_i^A + \iota_i^B$ of the edges. (2) If one edge point of a summand lies on a vertex arc of the other summand, it means the edge of the second summand has zero length at that outer normal direction. Adding zero length is nothing but considering only the edge of the first summand, and that is precisely we achieve by concatenation. Next, refer to Eqn.7 concerning Minkowski decomposition. Here all other things remain as before except that some of the edge points (resulting from the B^{-1} polygon) in the merged slope diagram are black, that is, have negative sense. If the sense is negative we have to subtract the corresponding length of the edge of B from that of A. Clearly, the subtraction of a directed edge from another is nothing but reversing the direction of the former and then concatenating with the latter.

The slope diagrammatic approach clearly brings out the following facts:

- Just as in the real number arithmetic, instead of treating the division m/n as a separate operation we may consider it as the multiplication with the multiplicative inverse $\frac{1}{n}$, in the similar way the Minkowski decomposition $\partial(A \ominus B)$ operation may also be viewed as the addition with the *additive inverse* ∂B^{-1}.

- Just as in the system $(N, \cdot, /)$ we do not obtain *closure* under the division operation and we need to extend the number domain from N to the set of all positive rational numbers Q^+ for the purpose of closure, similarly to obtain closure under Minkowski decomposition it is necessary to extend the domain $\mathcal{K}$ of the ordinary convex objects; it is done here by introducing the concept of *negative shape.*

- Both Minkowski addition and decomposition can be reduced to a single operation $\uplus$; Minkowski addition involves boundary addition $\uplus$ of two positive (ordinary) objects, while Minkowski decomposition involves boundary addition of one positive and one negative object. Boundary addition, as we have shown, is essentially nothing but *addition of real numbers* (representing the lengths of the edges of the operand polygons; lengths may be positive or negative).

- Just as $\lfloor m/n \rfloor$ is obtained by discarding the the fractional part of m/n, similarly $\partial(A \ominus B)$ is obtained by discarding the negative part of $\partial A \uplus \partial B^{-1}$.

3 In the domain of convex polyhedra

One of the basic theses of this paper is: Minkowski operations of three- or higher-dimensional boundary-represented objects in E^d eventually reduce to Minkowski operations of polygons in E^2, which, in turn, boil down to addition of real numbers. At present we shall take up the case of polyhedral objects, though our approach is general enough to extend beyond 3-dimension. The approach is to find relations for convex polyhedra which will be similar to Eqn.6 and Eqn.7, and then to resort to the slope diagrammatic technique as we have done for convex polygons.

3.1 Computation by means of faces

The boundary of a convex polyhedron can be represented by means of vertices, edges, and facets. Our approach demands these concepts to be defined more precisely.

If for some $u \neq 0$, we have $H(A,u) < \infty$ (this condition ensures that A is bounded), then the hyperplane

$$L(A,u) = \{p \in E^d \mid \; < p, u >= H(A,u)\}$$

is called the supporting hyperplane of A with *outer normal* u. In E^2 the supporting hyperplane becomes the supporting line of a convex polygon A with outer normal u, while in E^3 it is the supporting plane of a convex polyhedron A.

A face of A with outer normal u, denoted by $F(A,u)$, is then defined as

$$F(A,u) = L(A,u) \cap A.$$

Thus $F(A,u)$ is precisely that set of boundary points of A where the outer normal is either u or parallel to u. Now considering all the directions of the E^d space as the directions of the outer normal u, the collection of the corresponding faces will describe the entire boundary of A. That is, the entire boundary of A can be described as $\partial A = \bigcup_{u \in S^{d-1}} F(A,u)$.

This notion of $F(A,u)$ is related to our conventional concept of boundary of an object in terms of vertices, edges, faces, etc. in the following way. If A is a convex d-dimensional object then $L(A,u)$ is a $(d-1)$-dimensional hyperplane. Therefore, $F(A,u)$ may have dimensions $0, 1, \ldots, (d-1)$. Normally, a face $F(A,u)$ of dimension r $(r = 0, 1, \ldots, d-1)$ is called a *r-face* of A. A maximal proper face of A, that is, a $(d-1)$-dimensional face is called a *facet* of A. Clearly, if A is a 2-dimensional convex polygon, then $F(A,u)$ is either a 0-face (*vertex*) or an 1-face (*edge*). Since an edge of a polygon is a maximal proper face, it may also be called a facet of the polygon. When A is a 3-dimensional convex polyhedron, in addition to being either a vertex or an edge, $F(A,u)$ may also be a 2-face (*facet*).
(Note: For $3D$ objects, instead of "facet", the term planar "face" is more frequently used.)

From Theorem 1 it is not difficult to derive the following result [6, 10].

Theorem 2 *Let A and B be two convex polytopes in E^d. Then for every $u \in S^{d-1}$,*

$$F(A \oplus B, u) = F(A,u) \oplus F(B,u).$$

Therefore, the boundary $\partial(A \oplus B)$, in terms of faces of the summands, can be expressed as,

$$\partial(A \oplus B) = \bigcup_{u \in S^{d-1}} F(A \oplus B, u) = \bigcup_{u \in S^{d-1}} (F(A,u) \oplus F(B,u)) \tag{9}$$

Eqn.9 will be our basis of computing Minkowski operations of convex polyhedra. We shall first argue that it is not necessary to compute $(F(A,u) \oplus F(B,u))$ for every $u \in S^2$ in the three-dimensional space. Just as a convex polygon is completely specified by its oriented edges, similarly a convex polyhedron is completely specified by its oriented facets (planar faces). Therefore, it is sufficient if we compute only the facets of $A \oplus B$, but not every $F(A \oplus B, u)$'s. The facets of $A \oplus B$, as is evident from Eqn.9, can be obtained by

1. *Minkowski addition of two facets:* adding a facet of A with a facet of B, that is, $facet_A \oplus facet_B$,

2. *Minkowski addition of a facet and an edge:* adding a facet of one of the two summands with an edge of the other, that is, $facet_A \oplus edge_B$ or $edge_A \oplus facet_B$,

3. *Minkowski addition of a facet and a vertex:* adding a facet of one of the two summands with a vertex of the other, that is, $facet_A \oplus vertex_B$ or $vertex_A \oplus facet_B$,

4. *Minkowski addition of two non-parallel edges:* adding non-parallel edges of A and B, that is, $edge_A \oplus edge_B$,

where the facets, edges, and vertices so added lie in supporting planes with parallel outer normals.

Now Minkowski addition of two facets which lie in supporting planes having parallel outer normals is equivalent to Minkowski addition of those facets lying in the same plane. That, in turn, is nothing but Minkowski addition of two convex polygons in E^2. The same is true with Minkowski addition of a facet and an edge, or, addition of two edges.

In Fig.4 we depict a typical such addition.

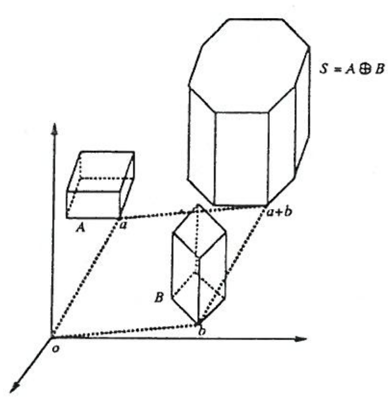
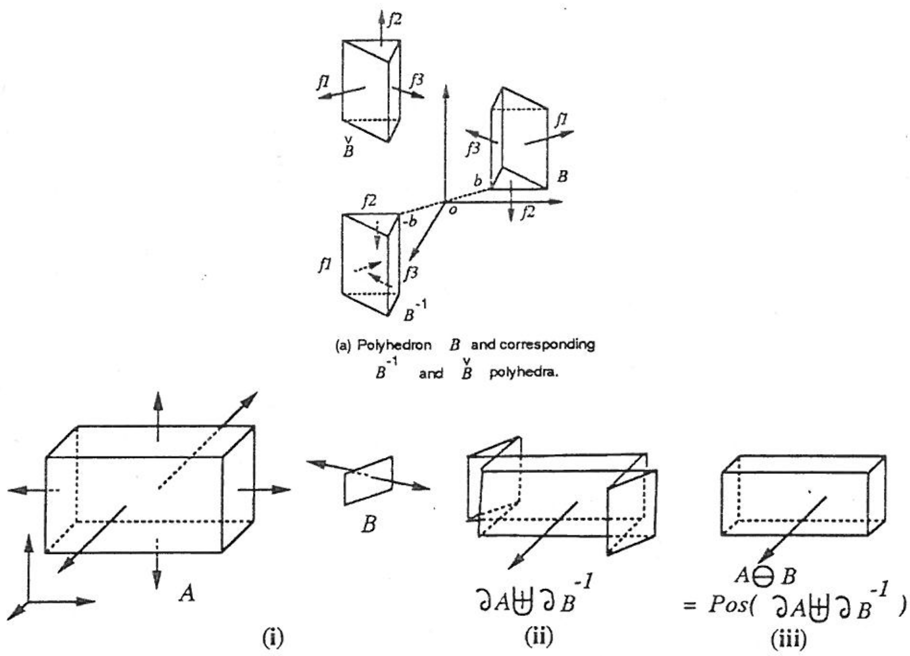


Figure 4: Computation of $\bigcup_{u \in S^{d-1}} (F(A,u) \oplus F(B,u))$ for two convex polyhedra A and B in E^3.

(In the similar way we can continue further and show that Minkowski addition of two convex polytopes in E^d reduces to Minkowski additions in $E^{d-1}, \ldots$, and eventually reduces to Minkowski additions of convex polygons in E^2. That means, it finally reduces to additions of real numbers.)

In computing Minkowski decomposition $A \ominus B$, exactly like the polygonal case, the first step is to compute the boundary addition $\partial A \uplus \partial B^{-1}$ and then to determine the positive portion $Pos(\partial A \uplus \partial B^{-1})$ of it. In terms of the faces of the operand polyhedra the first step is the computation of $\bigcup_{u \in S^{d-1}} (F(S,u) \oplus F(B^{-1},u))$. Therefore, according to our previous discussion it will reduce to $facet_A \oplus facet_{B^{-1}}$, $facet_A \oplus edge_{B^{-1}}$ or $edge_A \oplus facet_{B^{-1}}$, $facet_A \oplus vertex_{B^{-1}}$ or $vertex_A \oplus facet_{B^{-1}}$, and $edge_A \oplus edge_{B^{-1}}$ (where



(b) The input A and B are convex polyhedra.

Figure 5: (a) Geometric representation of a negative polyhedron B^{-1}; (b) Computation of $A \ominus B$ using boundary addition operation.

they lie in supporting planes with parallel outer normals). Each of these additions, as we have argued before, can first be reduced to the boundary addition of a positive and a negative convex polygons in E^2, which, in turn, reduces to subtractions of real numbers.

In Fig.5a we present an example of a negative polyhedron B^{-1}. In Fig.5b we show Minkowski decomposition of two convex polyhedra by means of boundary addition $\uplus$ and the *Pos* operation thereafter.

3.2 Slope diagram representation of convex polyhedron

The essential idea of the slope diagram representation of an object is to capture the behavior of the outer normals of the object in an explicit way. The behavior of the directions of the outer normals at various faces (that is, at the facets, edges, and vertices) of a convex polyhedron can be described as follows:

1. At each interior point of a facet of a polyhedron, there can be drawn only one outer normal.

2. At each point of an edge (different from a vertex), it is possible to draw an infinite number of normals filling a plane angle which will be supplementary to the corresponding interior angle. (In some literature this interior angle is called the *dihedral angle* corresponding to the edge. The exact definition is: the dihedral angle corresponding to an edge is the angle between the planes of its two adjacent facets).

3. At each vertex, it is possible to draw infinitely many outer normals which fill a solid angle (interpolating between the incident facet normals).

For the slope diagram of a polyhedron, we have to start with a unit sphere S^2.

a. *Facet representation.* Similar to the polygonal case, each facet can be represented by the corresponding point on the unit sphere. It is referred to as a *facet point.*

b. *Edge representation.* Each edge of the polyhedron, we claim, can be represented by the arc of the great circle joining the two facet points corresponding to the two adjacent facets of the edge. We call such an arc an *edge arc* of the polyhedron. (Note: The intersection of the surface of a sphere by a plane is called a *great circle* if the plane passes through the center of the sphere. Clearly, only one great circle can be drawn through two given points on the surface of the sphere, except when the points are the extremities of a diameter of the sphere. By the "arc of a great circle" generally we mean the shorter of the two arcs joining the two points.)

c. *Vertex representation.* According to the scheme, it easy to see that the directions of the outer normals at any vertex v of the polyhedron will be represented by a region on the unit sphere. This region is bounded by the edge arcs corresponding to the edges incident at v, and the vertices of this region are the facet points corresponding to the facets of the polyhedron incident at v. We call it a *vertex region.*

d. To denote the sense of an outer normal and the length of an edge we use the same conventions as adopted for polygons.

In Fig.6 we show the slope diagram representations of two convex polyhedra. The thick lines and black dots in the diagrams indicate negative sense of the outer normals.

(Note: In Mount and Silverman [12] and Guibas and Seidel [8] we find notions partly similar to the idea of slope diagram of convex polyhedron. But they did not take into account the notion of "sense" of a outer normal which is a crucial notion in our approach.)

Since the slope diagram of a polyhedron is a $3D$ figure, it may not be a very convenient representation to deal with for the purpose of intuitive understanding and visualization. We, therefore, transform this representation to an equivalent two-dimensional form by means of the *stereographic projection.* The stereographic projection is a projection of a sphere from one of the points s onto the plane T tangent to the sphere in the diametrically opposite point s'. The point s is called the *projection center.*

The transformation equations for the stereographic projection are given in many standard texts such as [16]. Consider a unit sphere whose center o is at the origin, and the projection center s is located on the oz axis. In this case the point s has the coordinates $(0, 0, 1)$ and the projection plane T is the plane $Z = -1$. Let the point $p(x, y, z)$ of the sphere be projected stereographically into the point $p'(x', y', -1)$ of the plane T. Then the coordinates of the point p, corresponding to the point p', are equal to

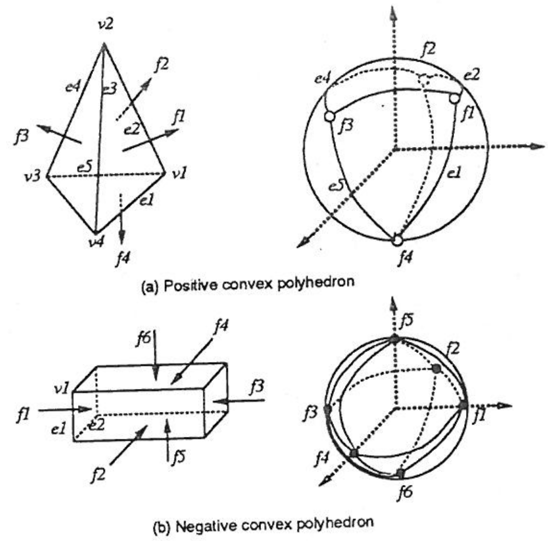


Figure 6: Two convex polyhedra and their slope diagram representations (drawings are not done according to exact measure).

$x = \frac{4x'}{x'^2+y'^2+4}$, $y = \frac{4y'}{x'^2+y'^2+4}$, $z = \frac{x'^2+y'^2-4}{x'^2+y'^2+4}$. The inverse mapping, that is, the coordinates of the point p', corresponding to the point p, are equal to $x' = \frac{2x}{1-z}$, $y' = \frac{2y}{1-z}$, $z' = -1$.

From the above equations we find that if p is the projection center s itself, that is, $p = (0,0,1)$, then we have difficulty in computing x' and y'. This difficulty is circumvented by extending the plane T by an "ideal" point which is called the *point at infinity*.

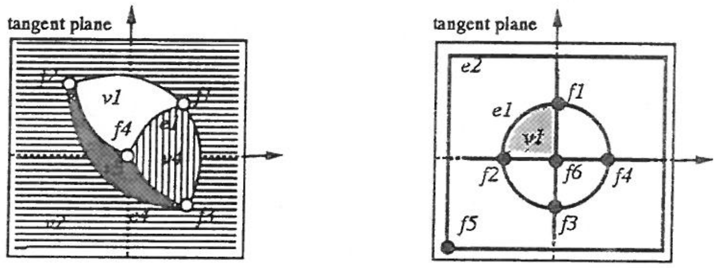
In the stereographic projection of a slope diagram, any facet point at (x_1, y_1, z_1) will be projected onto the point (x_1', y_1') of the T-plane; the point (x_1', y_1') can be determined using the above equations. Any edge arc will be projected either as a straight line or a circular arc. An edge arc joining two facet points (x_1, y_1, z_1) and (x_2, y_2, z_2) will be projected on the plane T as: $(x_1y_2 - x_2y_1)(x'^2 + y'^2) + 4(y_1z_2 - y_2z_1)x' + 4(z_1x_2 - z_2x_1)y' - 4(x_1y_2 - x_2y_1) = 0$. This is the equation of a circular arc unless $x_1y_2 = x_2y_1$ when it degenerates into a straight line segment.

In Fig.7 we present the stereographic projections of the slope diagrams shown in Fig.6.

3.3 Computation by means of slope diagrams

Assume that two convex polyhedra A and B are given in the form of their slope diagram representations. Exactly like the polygonal case, the boundary addition computation involves two steps: (a) merging of slope diagrams of the operands, and (b) realization of the boundary sum from the merged slope diagram.

If we *merge* these two slope diagrams (that is, overlay both the slope diagrams on the same unit sphere S^2), we can immediately identify the corresponding $F(A,u)$'s and $F(B,u)$'s (or, $F(B^{-1},u)$'s in case of decomposition) which have the same outer normal direction u, because they will occupy the same position on the sphere. That means,



(a) Projected slope diagram of a positive tetrahedron (the shaded regions denote vertices, the arcs denote edges, and the vertex points denote facets).

(b) Projected slope diagram of a negative parallelepiped (black dots denote facets; the facet point f5 is projected on the point at infinity).

Figure 7: Stereographic projections of the slope diagrams of the convex polyhedra depicted in Fig.6 (drawings are not done according to exact measure).

wherever there are intersections between two slope diagrams, the corresponding faces of the operands need to be added (or, subtracted).

To *realize* the boundary sum $\partial A \uplus \partial B$ or $\partial A \uplus \partial B^{-1}$ from the merged slope diagram, we have to identify only the facet points of the boundary sum. Any facet point will be created in one of the following ways: (a) intersection of a facet point of one operand with that of the other (which means, *addition of two facets*), (b) intersection of a facet point of one with an edge arc of the other (which means, *addition of a facet and an edge*), (c) intersection of a facet point of one with a vertex region of the other (which means, addition of a facet and a vertex), and (d) intersection of an edge arc of one with a non-parallel edge arc of the other (which means, addition of two non-parallel edges).

In case of Minkowski decomposition, if $\partial A \uplus \partial B^{-1}$ turns out to be a self-crossing polyhedron (as shown in Fig.5b), we have to discard the negative portion and to consider the positive portion only.

In Fig.8 we demonstrate computation by means of slope diagrams through an example of addition of a rectangular plane (A) which is parallel to the xz-plane to another rectangular plane (B) parallel to the yz-plane. The resulting sum shape $A \oplus B$ will be a rectangular parallelepiped.

4 Morphological operations of nonconvex objects

4.1 Problems with nonconvex objects

If the summands A and B are nonconvex objects, Theorem 1 or Theorem 2 do not hold any longer. We encounter three kinds of problems. We briefly discuss the problems and suggest some remedies so that Minkowski operations of convex as well as nonconvex objects can be viewed through a single algorithmic framework. (Unless otherwise stated, the operands are assumed to be *simply connected*).

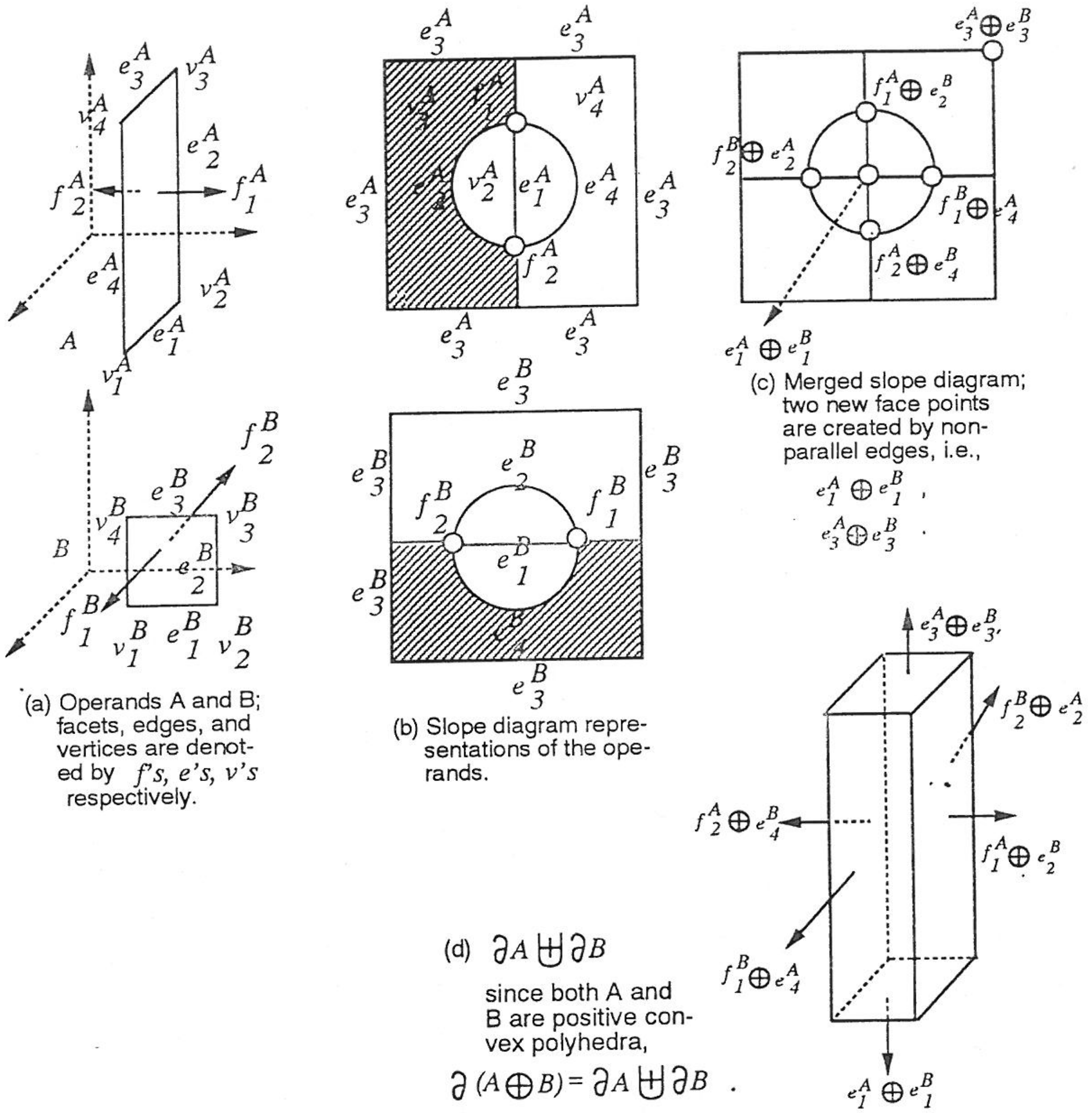


Figure 8: Minkowski addition of two polyhedra using their slope diagram representations.

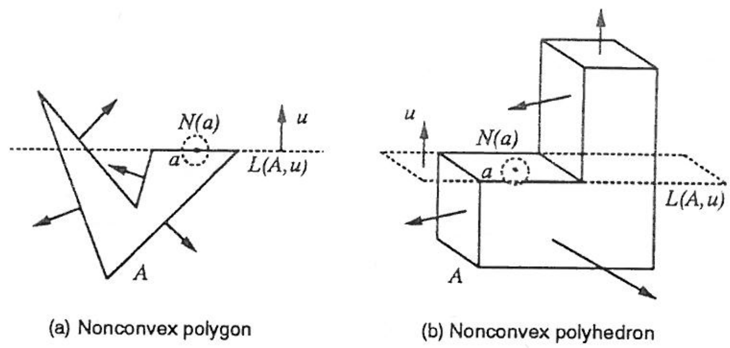


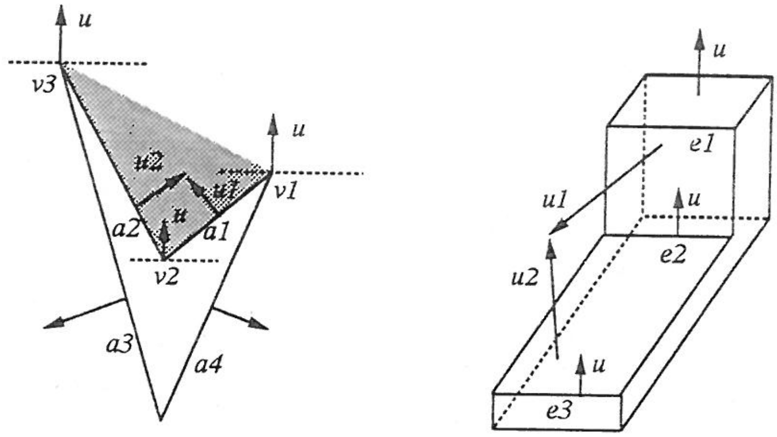
Figure 9: Local supporting hyperplane $L(A,u)$ at the neighborhood of a point a of A.

a. **Localized definition of $F(A,u)$.** A hyperplane $L(A,u)$ supporting a nonconvex object A at some point a on the boundary of ∂A, unlike convex object, may not be a supporting hyperplane of A – it may intersect the interior of A (Fig.9).

   To remedy this situation we can extend the notion of supporting hyperplane to *relative/local supporting hyperplane*. The idea is to consider, not the whole of the object A, but only a *neighborhood* of the point a. The neighborhood of a, a very small circular disk (or spherical ball in case of $3D$ object), is commonly denoted by $N(a)$. We call $L(A,u)$ a local supporting hyperplane of A if it is a supporting hyperplane of the set $A \cap N(a)$. That means, $L(A,u)$ is a supporting hyperplane only locally near the point $a \in \partial A$, but not globally for all the points of ∂A. Assuming $L(A,u)$ to be local supporting hyperplane, we may extend the definition of a face $F(A,u)$ accordingly, that is, $F(A,u) = L(A,u) \cap (A \cap N(a))$.

   The price we pay for this localized definition of $F(A,u)$ is this: if any of the summands A or B is nonconvex, it can no longer be assumed, like the convex case, that all the points in the collection $\bigcup_{u \in S^{d-1}} (F(A,u) \oplus F(B,u))$ will lie on the global boundary $\partial(A \oplus B)$ of the sum; some of the points in the collection may happen to be interior points of $A \oplus B$. Symbolically, $\partial(A \oplus B) \subseteq \bigcup_{u \in S^{d-1}} (F(A,u) \oplus F(B,u))$.

b. **Anomalous behavior of the outer normals at the nonconvex faces.** Some of the faces of a nonconvex object are convex faces, while the rest are *nonconvex faces*. In Fig.10a the vertex v_2, or the edges a_1, a_2 of the polygon are nonconvex faces, while v_1, v_3, or edges a_3, a_4 are convex; similarly, the edge e_2 of the polyhedron is a nonconvex face. (The term "nonconvex face" is used here in the following sense. Consider a simple (nonconvex) polygon. A vertex of it is called a nonconvex vertex (v_2 in the example figure) if the internal angle at the vertex is more than 180 degrees; otherwise it is convex. The edges of the polygon that are incident to a nonconvex vertex (a_1 and a_2 in Fig.10a) are called nonconvex edges, or, nonconvex faces, in general. Similarly, for a polyhedron an edge is nonconvex if the internal angle is more than 180 degrees, and the facets incident to a nonconvex edge are the nonconvex faces.)



(a) Nonconvex polygon (b) Nonconvex polyhedron

Figure 10: Anomalous behavior of the outer normals at the nonconvex faces.

The basic problem is, *the nature of outer normal at a nonconvex face is different from that at a convex face.* As we have already described in Sec.2.2, all the faces of an ordinary convex object are convex, and the outer normals at any two adjacent convex faces appear to diverge outwards from a point inside the object. In contrast to that, the outer normals at any two adjacent nonconvex faces converge to a point outside the object.

We remedy this problem in the following way. Consider the region complement of the nonconvex part of the object A (part of this complement region for the polygon is shown shaded in Fig.10a). This complement region is like a hole or a negative region, and the nonconvex faces of A constitute a part of the boundary of this hole. (Intuitively, *a nonconvex object can be regarded as a combination of positive and negative convex objects*).

If we would have added (in the Minkowski addition sense) the positive object B to this hole, the resulting faces would have been determined by subtracting each face of the hole from the corresponding face of B. In other words, a nonconvex face $F_2(A, u)$ of A and the corresponding convex face $F(B, u)$ of B are of opposite types; if one is considered as positive, the other one would be considered as negative. Since we are adding to B – not the hole – but the positive part of A, it necessitates that the corresponding face of B has to be subtracted from the nonconvex face of A.

Going by the same logic, in computing $A \ominus B$ which reduces to the boundary addition of the positive object A and the negative object B^{-1}, a convex face of B^{-1} is subtracted from the corresponding convex face of A, whereas it has to be added with the corresponding nonconvex face of A.

c. **Need to maintain explicit topological information of the operands.** Here by "topological information" we mean how the various faces of the object are connected. In case of a convex polytope, the topological information need not be maintained explicitly, since it can be easily derived from the outer normal directions of the faces. For a given outer normal direction u, a convex polytope has one and only one face; moreover the faces of a convex polytope are connected in such a way that their outer normal directions are automatically arranged in sorted angular order (though the sorting orders for three- and higher-dimensional polytopes are more complicated). Because of this reason, we have already seen in case of a convex polygon, the topological connections are automatically established if the consecutive edge points and vertex arcs are appropriately marked on the unit circle of its slope diagram (see Fig.3 where we can say, by inspecting the slope diagram of a polygon, that an edge l_1 is connected to the edge l_2, etc.).

However, that does not happen with nonconvex objects. Consider the nonconvex polygon shown in Fig.10a. For some outer normal direction, say u, the polygon has three vertices v_1, v_2, and v_3, and the vertices are all disconnected. The remedy is to maintain explicitly, in case of a nonconvex object, the information how the various faces of the object are connected. By means of slope diagram representation, we shall shortly show, this can be easily achieved.

4.2 Boundary addition of nonconvex polygons by means of slope diagrams

The slope diagram representation of a nonconvex polygon is slightly more complicated than that of a convex polygon. For the convex edges and vertices, the representation is exactly the same as explained in Sec.2.3. For the nonconvex portion, the following additional considerations must be made:

- To maintain the topological connectivity of the edges, we have to observe a forward and backward motion along the unit circle corresponding to a nonconvex vertex (Fig.11b or Fig.12b).
- The *sense* of the outer normals at a nonconvex vertex is opposite to that at a convex vertex. Therefore, the vertex arc corresponding to the nonconvex vertex must be depicted by thick black lines to indicate its negative sense.

Computation of boundary addition $\uplus$ by means of slope diagrams remains exactly the same, that is, (a) merging of the slope diagrams of the operands, and (b) realization of the boundary sum from the merged slope diagram. However, the second step is slightly more involved in case any of the operands is nonconvex.

First, consider the boundary addition $\partial A \uplus \partial B$. As we have noted, in a nonconvex portion of a slope diagram the path along the unit circle is traversed "three times" – twice in the positive sense and once in the negative sense (Fig.11b or Fig.12b). Therefore, at the time of realization of the merged slope diagram if there is any edge point of the other summand lying within this portion, it must be considered three times in the appropriate manner. The term "appropriate manner" means, in the negative arc portion (depicted by thick black lines) the edge has to be subtracted, while in the positive arc portions it has to be added in the usual way (Fig.11c). Clearly, the subtraction of a directed edge

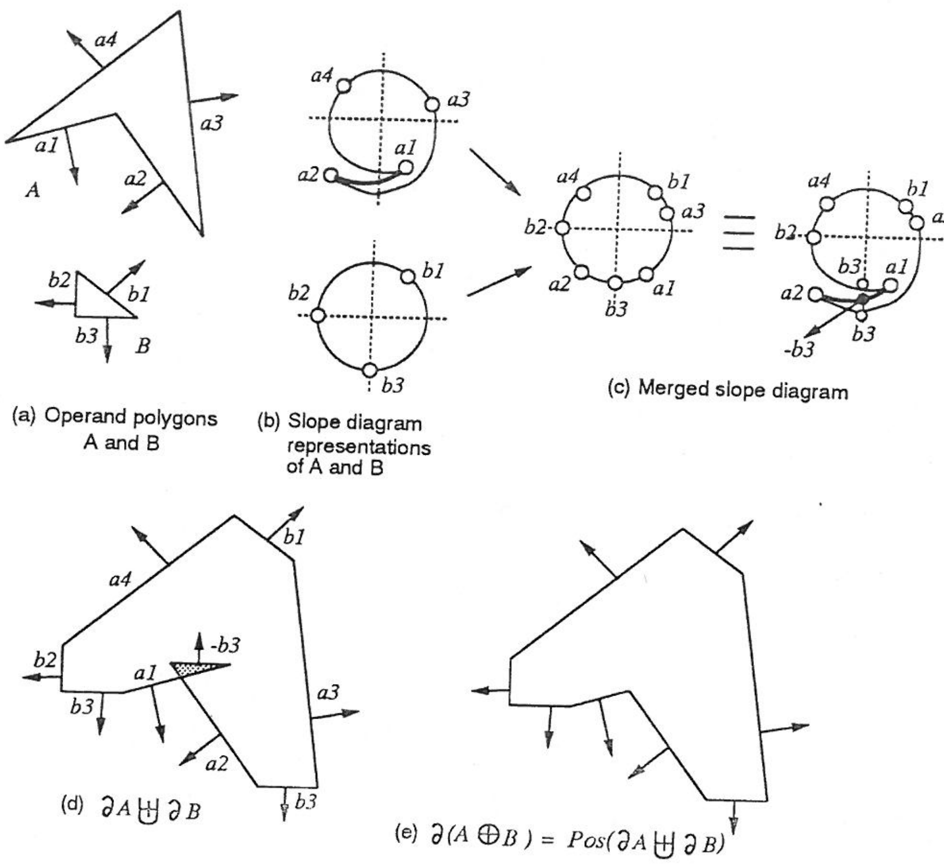


Figure 11: Minkowski addition of nonconvex polygons by means of slope diagrams.

from another is nothing but reversing the direction of the former and then adding it with the latter (Fig.11d).

One important point to be noted here. Since a nonconvex polygon is treated here as a combination of positive object and negative object, the boundary addition $\partial A \uplus \partial B$ of nonconvex polygons, unlike the convex case, does not directly produce the boundary $\partial(A \oplus B)$ of the Minkowski sum. In general, as shown in Fig.11d, $\partial A \uplus \partial B$ may be an oriented self-crossing polygon. The boundary of the sum can be obtained by determining the *positive region* enclosed by the resulting self-crossing polygon (Fig.11e). We may symbolically express the complete Minkowski addition as,

$$\partial(A \oplus B) = Pos(\partial A \uplus \partial B), \tag{10}$$

for general polygons – convex or nonconvex.

The computation of $\partial A \uplus \partial B^{-1}$ is in no way different, except that senses of the edge points and vertex arcs of B^{-1} are exactly opposite to those of B. $\partial A \uplus \partial B^{-1}$ will be, in general, a self-crossing polygon, and $\partial(A \ominus B) = Pos(\partial A \uplus \partial B^{-1})$.

In Figure 12 we show an example of Minkowski decomposition.

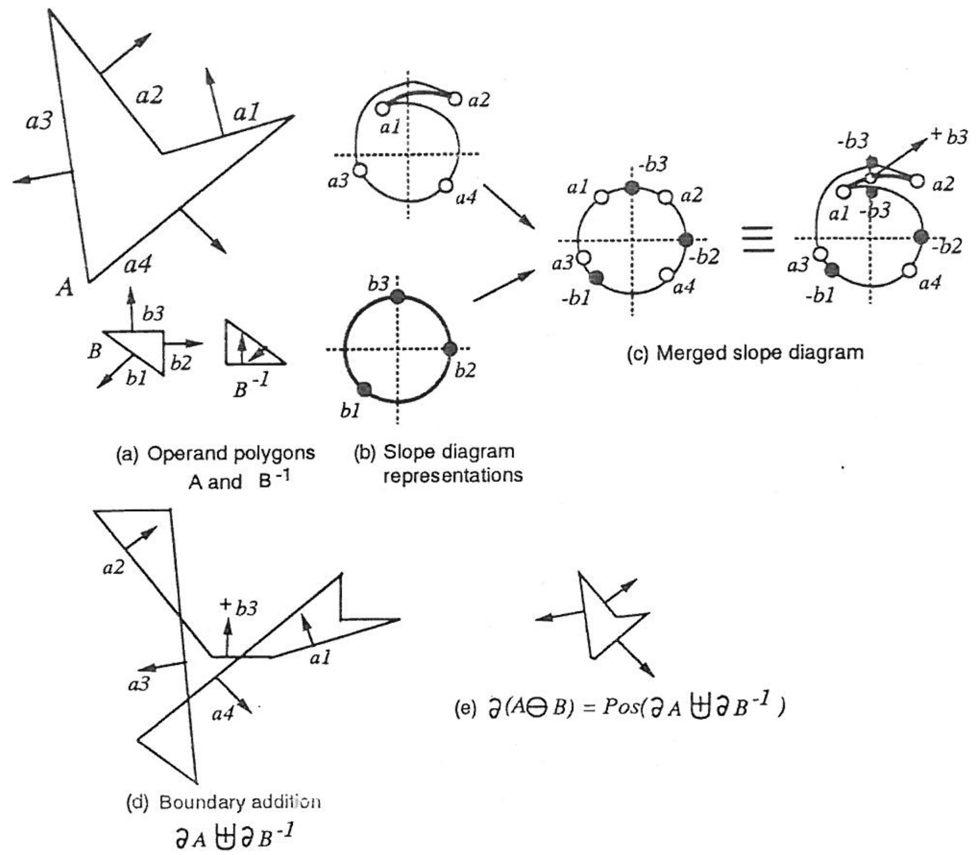


Figure 12: Minkowski decomposition of nonconvex polygons by means of slope diagrams.

4.3 Nonconvex polyhedra and the slope diagrammatic approach

The slope diagram of a nonconvex polyhedron, like a convex one, can be captured on a unit sphere by indicating the facet points, edge arcs, and vertex regions. A typical such example is shown in Fig.13b. Note that the edge arc connecting the facet points f_2 and f_3 are drawn by thick black lines to indicate its negative sense. The projected slope diagram is also shown in Fig.13c.

The boundary addition $\uplus$ of nonconvex polyhedra also remains exactly identical, that is, merging of the slope diagrams of the operands, and then realization of the boundary sum from the merged slope diagram. In case of nonconvex polyhedra too, like the nonconvex polygonal case, the boundary sum will be a self-crossing polyhedron, in general, even if both the summands are positive polyhedra. In Fig.14 we present an example of Minkowski addition by means of the boundary addition and the *Pos* operation.

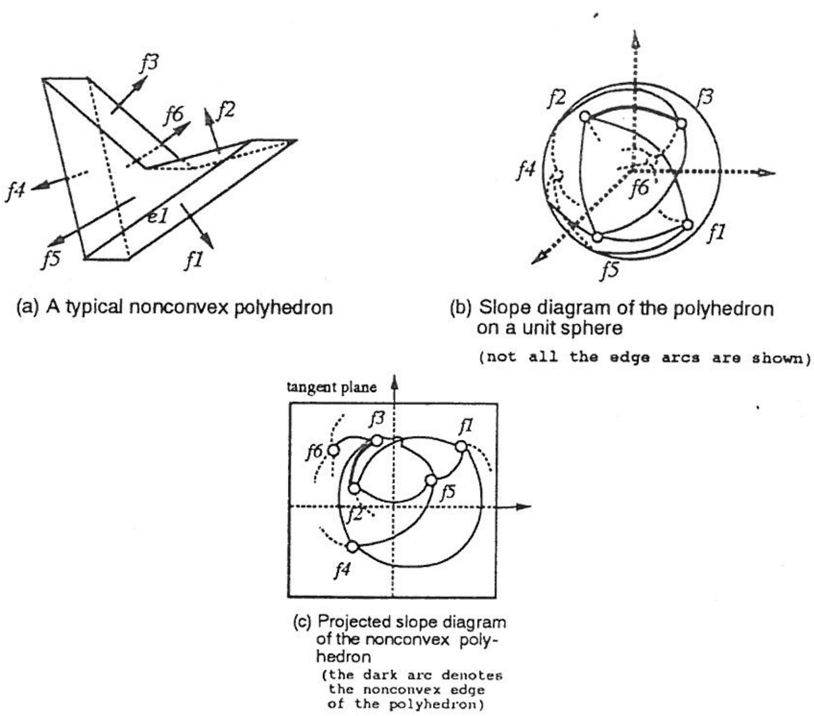


Figure 13: A nonconvex polyhedron and its slope diagram representation; the projected slope diagram is also shown (drawings are not done according to exact measure).

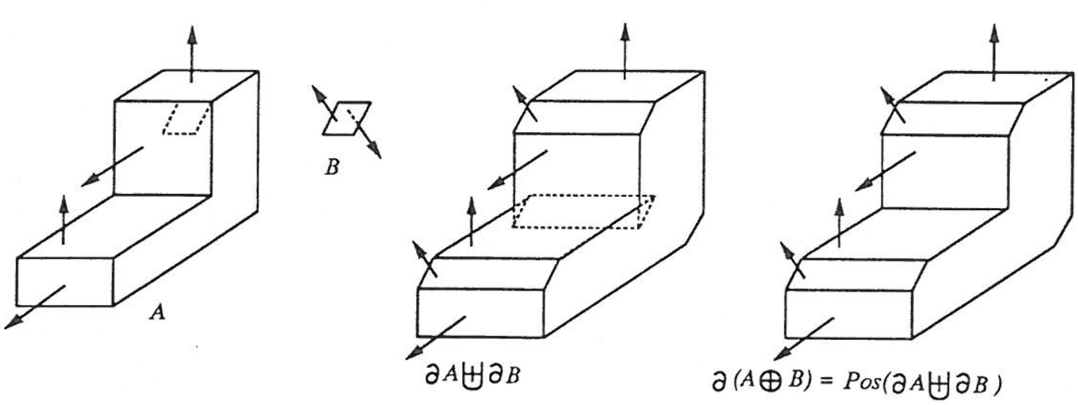


Figure 14: Minkowski addition: boundary addition of two positive polyhedra, and the behavior of the sum at the nonconvex portion.

5 The summary of our approach: A unified algorithm

5.1 Unified algorithm

We can summarize our discussion by saying that we have arrived at a *unified approach* to compute Minkowski operations of boundary represented geometric objects. We find that:

- Both Minkowski addition and decomposition are essentially the same operation; while Minkowski addition is basically a boundary addition operation of two positive objects, Minkowski decomposition is the boundary addition of one positive and one negative objects;
- Minkowski operations of both convex and nonconvex objects are essentially the same; the only extra consideration needed is to treat the nonconvex faces as the faces of a negative object;
- Minkowski operations of both $2D$ and $3D$ objects (in fact, general d-dimensional objects) are exactly alike; they eventually reduce to addition/subtraction of real numbers.

Below we present our *unified algorithm* to compute Minkowski operations of polygonal and polyhedral objects.

Unified algorithm to compute Minkowski operations
(Input operands are polygons or polyhedra which are specified in their boundary represented forms.)

A. COMPUTATION OF BOUNDARY ADDITION OPERATION $\uplus$

1. *[Formation of slope diagrams]* Represent the operands in their slope diagram forms. (In case of computing $A \oplus B$ the operands are ∂A and ∂B, while the operands are ∂A and ∂B^{-1} in computing $A \ominus B$.)
2. *[Merging of slope diagrams]* Merge the slope diagrams of the two operands into a single one.
3. *[Realization of the boundary sum]* From the merged slope diagram, realize the polygon or polyhedron it represents. This polygon or polyhedron is the boundary sum $\partial A \uplus \partial B$ or $\partial A \uplus \partial B^{-1}$, which is, in general, a self-crossing geometric object. Let us call it S_{boun}.

B. COMPUTATION OF MINKOWSKI OPERATIONS

4. *[Determination of $Pos(S_{boun})$]* Compute the boundary of the positive portion of S_{boun}. This will be equal to $\partial(A \oplus B)$ or $\partial(A \ominus B)$.

5.2 Complexity analysis of the unified algorithm

Consider the complexity of the algorithm for $2D$ polygons. Let n_1 and n_2 be the number of edges of the input polygons. Step 1 of the algorithm can then be carried out in $O(n_1 + n_2)$, that is, in $O(n)$ time, where $n = n_1 + n_2 =$ *total size of the input.* Let k be the number of edges of the boundary sum S_{boun}. Step 2 and Step 3 then require $O(k)$ time. Thus the computation of S_{boun} totally takes $O(n + k)$ time. Note that the size of k may vary from $n_1 + n_2$ (in case of Minkowski addition of two convex polygons) to $O(n_1 n_2)$ (in the most general case). In computing $Pos(S_{boun})$, that is, in Step 4, the basic algorithmic step is to "determine all the intersections among k given straight line segments in the plane". This can be done by using the method of Bently and Ottman [14] whose time complexity is $O((k+m)\log k)$, where m denotes the number of intersections among the k line segments. There is another improved, but slightly difficult algorithm proposed by Chazelle [14] whose time complexity is $O(m+(k\log^2 k/\log\log k))$. Thus the overall time complexity of the unified algorithm will be $O(n+k+m+(k\log^2 k/\log\log k))$. Note that, in general, $m \leq \binom{k}{2} = O(k^2)$.

In case of $3D$ polyhedra, let n_1 and n_2 be the number of edges of the two input polyhedra. Since the number of vertices, edges, and facets of a polyhedron without any hole follow the Euler's formula,

$$\textit{no. of vertices} - \textit{no. of edges} + \textit{no. of facets} = 2,$$

we can say that the numbers of vertices and facets of a polyhedron are related to its number of edges by a small constant factor. In other words, we say that the sizes of the input polyhedra are of the orders $O(n_1)$ and $O(n_2)$ respectively. Therefore, Step 1 of the unified algorithm can be carried out in linear time, that is, in $O(n)$ time, where $n = n_1 + n_2$. If k denotes the total number of edges in S_{boun}, then Step 2 and Step 3 take $O(k)$ time. Therefore, the computation of S_{boun} totally takes $O(n + k)$ time. The major computational cost in the $3D$ case arises in Step 4. The crudest approach to the computation of $Pos(S_{boun})$ consists of determining the intersections of each facet of S_{boun} with every other facet. It is easy to verify that such an approach may take as much as $O(k^2)$ time. Clearly this step dominates the initial step of determining S_{boun}, and the overall time complexity of the algorithm becomes $O(k^2)$, that is, $O(n^4)$ in the worst case.

6 Simplification of the unified algorithm depending on the type of inputs

At this point we must clearly state that the unified algorithm presented above is a general framework in which the underlying methodology of computation of Minkowski operations is expressed in an algorithmic form; but we do not stress that exactly the same algorithm should be used in every case. In other words, if the input set is endowed with more structure, for example, if both the operands are convex, etc., it is necessary to modify the algorithm and the data structures appropriately in order to increase the computational efficiency. We take up here a few special cases to demonstrate how the structure of the input set can be exploited fully to make the algorithm more efficient. We also show how, in certain cases, the unified algorithm automatically reduces to a simpler algorithm, since some of the computational steps may no longer be required.

TWO-DIMENSIONAL CASES

1. Minkowski addition of two convex polygons

In Sec.2.3 we have given an algorithm for addition when both the input polygons A and B are convex. Step 4 of the unified algorithm is not required, since $\partial(A \oplus B) = Pos(\partial A \uplus \partial B)$ in this case. Therefore, the complexity of the algorithm becomes $O(n+k)$. Since, in this case, $k = n_1 + n_2 = n$, the complexity is linear, that is, $O(n)$. The reader may also refer to an algorithm by Schwartz [17].

2. Minkowski decomposition of two convex polygons

In Sec.2.3 we have also discussed how to determine $A \ominus B$, when both A and B are convex. The same algorithm is discussed in detail in Ghosh [3, 4]. It is easy to see that the computation of the boundary sum $\partial A \uplus \partial B^{-1}$ will take $O(n)$ time, since $k = n_1 + n_2 = n$ in this case too. However, Step 4 may not be avoided since the boundary sum, in general, is a self-crossing polygon. But it is quite easy to obtain a much faster algorithm to execute Step 4, since the n line segments in the boundary sum are not arbitrary line segments in the plane; these segments are, in fact, translated edges of the convex polygons A and B. In [3] an $O(n_1)$ algorithm for Step 4 is given, where n_1 denotes the number of edges of A. Thus the overall time complexity again reduces to $O(n)$. We also refer the reader to Guibas *et al* [7].

3. Minkowski decomposition $A \ominus B$ where A is a convex polygon

Consider the decomposition $A \ominus B$, where A is a convex polygon, but B is a general polygon, not necessarily convex. From the set theoretic result we know [3], if A is convex set, then

$$A \ominus B = A \ominus conv(B),$$

where $conv(B)$ denotes the convex hull of B. Therefore, to obtain an efficient algorithm, it is advisable to determine the convex hull of B first, and then to use the ***convex-convex decomposition*** algorithm as discussed above. Let n_1 and n_2 be the number of edges of A and B respectively. To determine $conv(B)$ we may use any standard convex hull construction algorithm, such as Graham's scan algorithm [14] that runs in $O(n_2 \log n_2)$ time. As shown above, $A \ominus conv(B)$ takes $O(n)$ time, where n denotes the total size of the input polygons. Therefore, $A \ominus B$ can be determined in $O(n + n_2 \log n_2)$ time.

4. Minkowski operations of two planar regions whose boundaries are smooth curves

First, consider the addition $A \oplus B$. We assume that the boundary curves of both A and B can be represented by smooth analytic functions. Let us denote them, in the

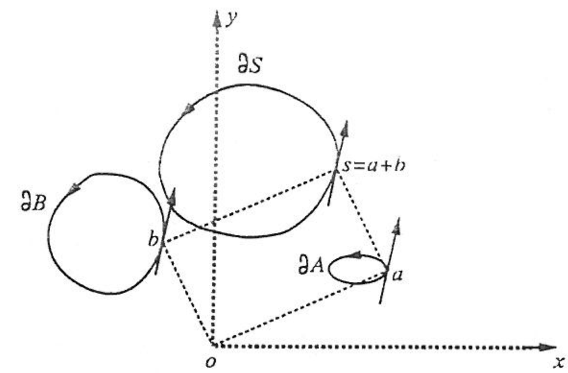


Figure 15: Minkowski addition of two regions bounded by smooth curves.

parametric form, as follows:

$$\partial A = [A_x(t_1), A_y(t_1)],$$
$$\partial B = [B_x(t_2), B_y(t_2)],$$

where t_1, t_2 are two scalar quantities and are closed intervals on t_1-axis and t_2-axis respectively; $A_x(t_1)$ represents a polynomial function of t_1, etc. We assume that ∂A and ∂B are oriented in the sense corresponding to an increase in the parameters t_1 and t_2 respectively.

The corresponding faces $F(A, u)$ and $F(B, u)$, in this case, means the points of ∂A and ∂B respectively where the tangent lines are parallel and similarly oriented (Fig.15). Therefore, the determination of $\partial A \uplus \partial B$ reduces to the following sequence of operations: (a) For every point $a \in \partial A$ find the direction of the tangent line, and then determine the corresponding point, say $b \in \partial B$, where the tangent line is parallel and similarly oriented; then (b) vectorially add a and b, that is, $a + b$. The method is clearly depicted in Fig.15.

In certain circumstances it may be possible to determine $\partial A \uplus \partial B$ completely analytically. Let $a = [A_x(t_1'), A_y(t_1')]$ for some $t_1 = t_1'$. To obtain the corresponding point $b \in \partial B$, we have to solve the following equation for t_2,

$$\left[\frac{\delta B_y(t_2)/\delta t_2}{\delta B_x(t_2)/\delta t_2}\right] = \left[\frac{\delta A_y(t_1)/\delta t_1}{\delta A_x(t_1)/\delta t_1}\right]_{t_1=t_1'} \tag{11}$$

Here $\delta X(t)/\delta t$ denotes the differentiation of the function $X(t)$ with respect to t.

Eqn.11 yields the solution for t_2 in terms of t_1'. Let $t_2 = h(t_1')$, where $h(t_1')$ denotes some polynomial in t_1'. Then the corresponding point b can be expressed as $b = [B_x(h(t_1')), B_y(h(t_1'))]$. Therefore, the analytic expression for the boundary sum will be,

$$\partial A \uplus \partial B = [A_x(t_1) + B_x(h(t_1)), A_y(t_1) + B_y(h(t_1))] \tag{12}$$

Whether $\partial A \uplus \partial B$ will be a self-crossing curve or not clearly depends on the nature of ∂A and ∂B. It is, therefore, not possible, in general, to comment on the overall complexity of the algorithm which involves determination of $Pos(\partial A \uplus \partial B)$ if the boundary sum is self-crossing. However, in special cases where both A and B are convex, we may get constant time algorithms.

For more details in this regard, we refer to Ghosh [1, 3]. We also refer to Horn and Weldon [9] where there is a mention of sum of two *extended circular images*. An extended circular image of a shape is somewhat analogous to the slope diagram representation of the shape.

THREE-DIMENSIONAL CASES

5. Minkowski addition of two convex polyhedra

From the unified algorithm it is not difficult to derive the following result.

Theorem 3 *Let A and B be two convex polyhedra. Let $v_i{}^A$ $(i = 1, \ldots, n_1)$ be the (position vectors of the) vertices of A, and $v_j{}^B$ $(j = 1, \ldots, n_2)$ be the vertices of B. Then the sum*

$$A \oplus B = conv\{v_i{}^A + v_j{}^B \mid i = 1, \ldots, n_1,\ j = 1, \ldots, n_2\},$$

where $conv(X)$ represents the convex hull of a set X.

This result can be used to devise a very simple two-step algorithm to add a convex polyhedron to another convex polyhedron. The first step is to vectorially add every vertex of A with every vertex of B. The total number of points thus generated will be $n_1 n_2$. So this step takes $O(n_1 n_2)$ time. The second step is to determine the convex hull of these $n_1 n_2$ points in the $3D$ space. Using some standard algorithm, say Preparata-Hong algorithm [13], this can be accomplished in $O(n_1 n_2 \log n_1 n_2)$ time, which clearly dominates the computation of the first step.

For the addition of convex polyhedra the reader may also refer to Guibas and Seidel [8].

6. Minkowski addition of a space curve by a spherical ball

We have already shown that if the boundaries of $2D$ operands can be expressed as smooth analytic functions, it may be possible to obtain the boundary of the product object purely analytically. Here we take up an example to demonstrate this fact for $3D$ operands as well.

Let A be a space curve whose parametric equation is given by,

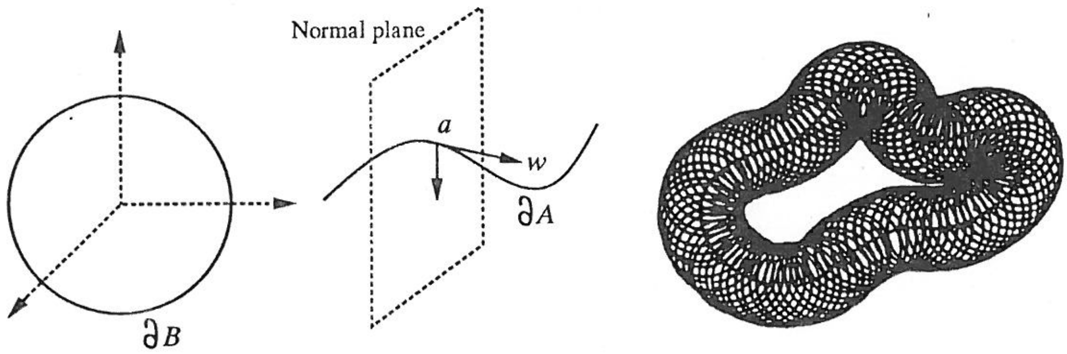
$$\partial A = [A_x(t), A_y(t), 0)],$$

and B be a sphere of radius r whose equation of the boundary is,

$$\partial B = [x, y, z], \text{ where, } x^2 + y^2 + z^2 = r^2.$$

The center of the sphere is assumed to be at $(0, 0, 0)$.

Let a be a point on the space curve, say $a = \partial A(t = t')$. We have to find out the corresponding point(s) $F(B, u)$ on ∂B. Since A is a space curve, the outer normals at a form a plane, called the *normal plane* at that point (Fig.16b). The equation of the normal plane can be obtained in the following way.



(a) Sphere of radius r.

(b) A space curve and its "normal plane" at the point a.

(c) A typical S_{boun} for Minkowski addition of a closed space curve and a sphere

Figure 16: Minkowski addition of space curve by spherical ball.

The unit tangent vector w of ∂A at a is given by,

$$w = \left[\frac{(\delta A/\delta t)}{|(\delta A/\delta t)|}\right]_{t=t'} = \left[\frac{\dot{A}_x(t)}{l}, \frac{\dot{A}_y(t)}{l}, 0\right]_{t=t'},$$

where $\dot{A}_x(t) = \frac{\delta A_x(t)}{\delta t}$, etc., and $l = \sqrt{(\dot{A}_x(t))^2 + (\dot{A}_y(t))^2}$. (Hereafter we shall write A_x, etc., instead of $A_x(t)$).

One point of the normal plane is the the point a, and the unit tangent vector w is normal to that plane. Therefore, the equation of the normal plane at a is given by,

$$\frac{\dot{A}_x}{l}(x - A_x) + \frac{\dot{A}_y}{l}(y - A_y) + 0.(z - 0) = 0.$$

Or,

$$\dot{A}_x(x - A_x) + \dot{A}_y(y - A_y) = 0,$$

where $t = t'$.

The corresponding points $F(B, w)$ will be the intersection points of ∂B with a plane which is parallel to the normal plane and passes through the center of the sphere.

The equation of the plane parallel to the normal plane and passes through the point $(0, 0, 0)$ is,

$$\dot{A}_x.x + \dot{A}_y.y = 0, \quad \text{where } t = t'.$$

Substituting the value of y (in terms of x) from the above equation to the equation of the sphere, we obtain

$$\frac{x^2}{\left(\sqrt{\frac{r^2}{\left(1+\dot{A}_x^{\,2}/\dot{A}_y^{\,2}\right)}}\right)^2} + \frac{z^2}{r^2} = 1.$$

Similarly, substituting the value of x, we get

$$\frac{y^2}{\left(\sqrt{\frac{r^2}{\left(1+\dot{A_y}^2/\dot{A_x}^2\right)}}\right)^2} + \frac{z^2}{r^2} = 1.$$

That means, the intersection points, that is, $F(B, u)$ can be expressed as,

$$x = F_x \cos\theta, y = F_y \cos\theta, z = r \sin\theta,$$

where θ varies from 0 to 2π radians, and

$$F_x = \left(\sqrt{\frac{r^2}{\left(1+\dot{A_x}^2/\dot{A_y}^2\right)}}\right) = F_x(t = t'),$$

$$F_y = \left(\sqrt{\frac{r^2}{\left(1+\dot{A_y}^2/\dot{A_x}^2\right)}}\right) = F_y(t = t').$$

Therefore, $\partial A \uplus \partial B$ will be given by,

$$x(t,\theta) = F_x(t)\cos\theta + A_x(t),\ y(t,\theta) = F_y(t)\cos\theta + A_y(t),\ z(t,\theta) = r\sin\theta.$$

One typical $\partial A \uplus \partial B$ is shown in Fig.16c.

7 A brief summing up

Let us mention the significant concepts introduced and developed in this paper:

- *Resemblance* between the geometric system $(\mathcal{K}, \oplus, \ominus)$ and the number system $(N, \cdot, /)$
- Concept of inverse or negative shape, and thereby, a more *generalized* notion of geometric objects
- *Unification* of Minkowski addition and Minkowski decomposition as a single operation of boundary addition $\uplus$
- Boundary addition operation as *additions* of real numbers
- Nonconvex object as a *mixture* of positive and negative objects
- *Slope diagram* representations of $2D$ and $3D$ operands
- A unified *computational framework* for carrying out morphological operations of boundary-represented objects by means of slope diagrams.

Acknowledgments

The first author like to thank Dr.S.P.Mudur and Dr.S.Ramani for supporting this work at N.C.S.T., Bombay, and to thank Mr.T.M.Vijayaraman for suggesting many improvements in the manuscript.

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