Machine Vision and Applications (1994) 7:93–114

Machine Vision and Applications

Error propagation in machine vision

Seungku Yi, Robert M. Haralick, and Linda G. Shapiro

Intelligence Systems Laboratory, Department of Electrical Engineering, FT-10, University of Washington, Seattle, WA 98195, USA

Abstract. Machine vision systems that perform inspection tasks must be capable of making measurements. A vision system measures an image to determine a measurement of the object being viewed. The image measurement depends on several factors, including sensing, image processing, and feature extraction. We consider the error that can occur in measuring the distance between two corner points of the 2D image. We analyze the propagation of the uncertainty in edge point position to the 2D measurements made by the vision system, from 2D curve extraction, through point determination, to measurement. We extend earlier work on the relationship between random perturbation of edge point position and variance of the least squares estimate of line parameters and analyze the relationship between the variance of 2D points.

Key words: Error propagation – Measurement – Edge perturbation – Noise – Inspection – Analysis of variance

1 Introduction

Inspection tasks involve measurements. When machine vision systems perform inspection, they must be capable of making these measurements. A vision system makes a measurement on an image that determines a measurement on the object being viewed. The accuracy of the measurement on the image determines the accuracy of the final measurement. This paper is about the propagation of error from feature detection on the image through calculation of the final measurement in the inspection task.

Suppose the distance between two corner points in a 2D image is to be measured. To measure the distance between the two corner points, their positions must be determined. Since the true positions are usually not known, they must be estimated from the given data. Because a corner point is an intersection of two non-parallel lines, determining the position of a corner point involves determining the lines that pass through the corner point. A line can be determined by detecting edge points which are supposed to be on that line and fitting the line to the detected points. Since pictures have numerous sources of error, the edge points produced by a typical edge operator are not necessarily true edges. There are uncertainties in the edge point positions. Since lines are estimated by fitting noisy edge points, the parameters of the fitted lines also have uncertainties coming from the uncertainties of the edge positions. Furthermore, a corner point can be determined by finding the intersection point of two fitted lines. Therefore, the position of the corner point has uncertainty that comes from the uncertainties of the fitted line parameters. Finally, since the distance between two corner points is computed using their positions, the measured distance has an uncertainty that comes from the uncertainties of the two corner point positions.

The error propagation process is diagrammed in Fig. 1. Haralick analyzed how edge point position uncertainty is propagated to the fitted line parameter uncertainty [7]. In his analysis, the noise is assumed to come from an independent and identical distribution. In this paper, we generalize Haralick's derivation for the case that the noise comes from an independent, but non-identical distribution. The validity of our derivations is proved by experiments described in Sect. 8. We also discuss the error propagation process and develop the relationship between the variances of edge point positions and the expected variance of the measurement. Complete analyses are given for line, circle, and ellipse fitting. We begin by stating the noise model employed in our approach.

Correspondence to: L.G. Shapiro

2 Noise model

Let (x_i, y_i) be the true, but unknown coordinate, of the i-th edge point and $(\hat{x}_i, \hat{y}_i)$ be the noisy observation of (x_i, y_i). Our model for $(\hat{x}_i, \hat{y}_i)$ is

$$\hat{x}_i = x_i + \xi_i \ , \quad \hat{y}_i = y_i + \eta_i \ ,$$

where we assume that the random perturbations ξ_i and η_i are independently distributed having mean 0, variance σ_i^2, and come from a distribution which is an even function. Hence,

$$E[\xi_i] = E[\eta_i] = 0$$

$$V[\xi_i] = V[\eta_i] = \sigma_i^2$$

$$E[\xi_i \xi_j] = \begin{cases} \sigma_0^2 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

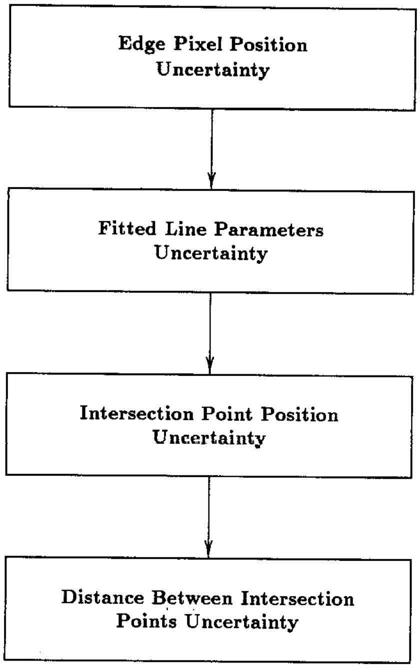

Fig. 1. Error propagation in a vision task

$$E[\eta_i\eta_j] = \begin{cases} \sigma_0^2 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$E[\eta_i\xi_j] = 0$$

These assumptions will be used throughout the paper.

3 Line fitting

3.1 Preliminaries

Consider a situation in which points (x_i, y_i), $i = 1, \ldots, I$ are assumed to lie on an unknown straight line and the problem is to determine the parameters of the line. Then, suppose (x_i, y_i) satisfies the model

$$\alpha x_i + \beta y_i + \gamma = 0 \quad i = 1, \ldots, I \tag{1}$$

where $\alpha^2 + \beta^2 = 1$. Since our noise model is not *i.i.d* and all σ_i^2 are known a priori, we define weighted mean and weighted variance instead of arithmetic mean or variance. Let the weights be $w_i = 1/\sigma_i^2$ and define

$$\mu_x = \frac{1}{W}\sum_{i=1}^{I} w_i x_i\ , \quad \mu_y = \frac{1}{W}\sum_{i=1}^{I} w_i y_i\ ,$$

$$\sigma_x^2 = \frac{1}{\tilde{W}}\sum_{i=1}^{I} w_i(x_i - \mu_x)^2\ , \quad \sigma_y^2 = \frac{1}{\tilde{W}}\sum_{i=1}^{I} w_i(y_i - \mu_y)^2\ ,$$

$$\sigma_{xy} = \frac{1}{\tilde{W}}\sum_{i=1}^{I} w_i(x_i - \mu_x)(y_i - \mu_y)\ ,$$

where

$$W = \sum_{i=1}^{I} w_i\ , \quad \tilde{W} = \frac{I-1}{I}W\ .$$

Then upon summing the model Eq. (1) multiplied by w_i over all i, we get

$$\sum_{i=1}^{I} w_i(\alpha x_i + \beta y_i + \gamma) = 0$$

so that $\gamma = -(\alpha\mu_x + \beta\mu_y)$. This permits us to rewrite the model equation as

$$\alpha(x_i - \mu_x) + \beta(y_i - \mu_y) = 0\ .$$

Multiplying the above equation by $w_i(x_i - \mu_x)$ and summing over all i

$$\sum_{i=1}^{I} w_i\left[\alpha(x_i - \mu_x)^2 + \beta(x_i - \mu_x)(y_i - \mu_y)\right] = 0\ ,$$

from which there results

$$\alpha\sigma_x^2 + \beta\sigma_{xy} = 0\ . \tag{2}$$

Now multiplying by $w_i(y_i - \mu_y)$ and summing over all i, we get

$$\sum_{i=1}^{I} w_i\left[\alpha(x_i - \mu_x)(y_i - \mu_y) + \beta(y_i - \mu_y)^2\right] = 0$$

from which there results

$$\alpha\sigma_{xy} + \beta\sigma_y^2 = 0\ . \tag{3}$$

Rewriting Eqs. (2) and (3) in matrix form, we have

$$\begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

which implies that

$$\begin{vmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{vmatrix} = 0\ .$$

Therefore, we have

$$\sigma_{xy}^2 = \sigma_x^2\sigma_y^2$$

and

$$\alpha = \frac{\sigma_{xy}}{\sqrt{\sigma_{xy}^2 + (\sigma_x^2)^2}} = \frac{\sin(\sigma_{xy})\sigma_x\sigma_Y}{\sqrt{\sigma_x^2\sigma_y^2 + (\sigma_x^2)^2}} = \frac{\sin(\sigma_{xy})\sigma_y}{\sqrt{\sigma_x^2 + \sigma_y^2}}\ .$$

Similarly,

$$\beta = \frac{-\sigma_x^2}{\sqrt{\sigma_{xy}^2 + (\sigma_x^2)^2}} = \frac{-\sigma_x}{\sqrt{\sigma_x^2 + \sigma_y^2}}\ .$$

3.2 Least squares estimates of line parameters

Up to here, the given points (x_i, y_i), $i = 1, \ldots, I$ which lie exactly on the unknown line are the points we used to compute the parameters of the line. Now we assumee that (x_i, y_i) are not given; instead, rather noisy observations $(\hat{x}_i, \hat{y}_i)$ of (x_i, y_i) are given.

We must estimate the parameters of the unknown line from the noisy observations $(\hat{x}_i, \hat{y}_i)$. To do this we employ the principle of minimizing the square residuals under a constraint. Using the Lagrange multiplier form, we define

$$\varepsilon^2 = \sum_{i=1}^{I} w_i(\hat{\alpha}\hat{x}_i + \hat{\beta}\hat{y}_i + \hat{\gamma})^2 - \lambda(\hat{\alpha}^2 + \hat{\beta}^2 - 1)\tilde{W} .$$

Note that ε^2 is a weighted sum and the weights are reciprocals of the variance of the random disturbances. Upon taking the partial derivative of ε^2 with respect to $\hat{\gamma}$, and setting the partial derivative to zero, we get

$$\frac{\partial \varepsilon^2}{\partial \hat{\gamma}} = 2\sum_{i=1}^{I} w_i(\hat{\alpha}\hat{x}_i + \hat{\beta}\hat{y}_i + \hat{\gamma}) = 0 .$$

Letting

$$\hat{\mu}_x = \frac{1}{W}\sum_{i=1}^{I} w_i \hat{x}_i , \quad \hat{\mu}_y = \frac{1}{W}\sum_{i=1}^{I} w_i \hat{y}_i ,$$

we obtain

$$\hat{\gamma} = -(\hat{\alpha}\hat{\mu}_x + \hat{\beta}\hat{\mu}_y) .$$

Hence,

$$\varepsilon^2 = \sum_{i=1}^{I} w_i \left(\hat{\alpha}(\hat{x}_i - \hat{\mu}_x) + \hat{\beta}(\hat{y}_i - \hat{\mu}_y)\right)^2 - \lambda(\hat{\alpha}^2 + \hat{\beta}^2 - 1)\tilde{W} .$$

Continuing to take partial derivatives of ε^2 with respect to $\hat{\alpha}$ and $\hat{\beta}$, we obtain

$$\frac{\partial \varepsilon^2}{\partial \hat{\alpha}} = \sum_{i=1}^{I} 2w_i \left(\hat{\alpha}(\hat{x}_i - \hat{\mu}_x) + \hat{\beta}(\hat{y}_i - \hat{\mu}_y)\right)(\hat{x}_i - \hat{\mu}_x) - \lambda(2\hat{\alpha})\tilde{W} = 0 ,$$

$$\frac{\partial \varepsilon^2}{\partial \hat{\beta}} = \sum_{i=1}^{I} 2w_i \left(\hat{\alpha}(\hat{x}_i - \hat{\mu}_x) + \hat{\beta}(\hat{y}_i - \hat{\mu}_y)\right)(\hat{y}_i - \hat{\mu}_y) - \lambda(2\hat{\beta})\tilde{W} = 0 ,$$

Letting

$$\hat{\sigma}_x^2 = \frac{1}{\tilde{W}}\sum_{i=1}^{I} w_i(\hat{x}_i - \hat{\mu}_x)^2 , \quad \sigma_y^2 = \frac{1}{\tilde{W}}\sum_{i=1}^{I} w_i(\hat{y}_i - \hat{\mu}_y)^2 ,$$

$$\hat{\sigma}_{xy} = \frac{1}{\tilde{W}}\sum_{i=1}^{I} w_i(\hat{x}_i - \hat{\mu}_x)(\hat{y}_i - \hat{\mu}_y)$$

substitution leads to

$$\begin{pmatrix} \hat{\sigma}_x^2 & \hat{\sigma}_{xy} \\ \hat{\sigma}_{xy} & \hat{\sigma}_y^2 \end{pmatrix} \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \lambda \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} .$$

So the sought after $\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix}$ must be an eigenvector of the sample covariance matrix. But which eigenvector? The one we want must minimize

$$\begin{aligned} &\sum_{i=1}^{I} w_i \left(\hat{\alpha}(\hat{x}_i - \hat{\mu}_x) + \hat{\beta}(\hat{x}_i - \hat{\mu}_y)\right)^2 \\ &= \tilde{W}(\hat{\alpha} \;\; \hat{\beta}) \begin{pmatrix} \hat{\sigma}_x^2 & \hat{\sigma}_{xy} \\ \hat{\sigma}_{xy} & \hat{\sigma}_y^2 \end{pmatrix} \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \\ &= \tilde{W}(\hat{\alpha} \;\; \hat{\beta})\lambda \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \tilde{W}\lambda . \end{aligned}$$

Hence, the eigenvector $\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix}$ must correspond to that eigenvalue $\hat{\lambda}$ of the sample covariance matrix having the smallest value. Any eigenvalue $\hat{\lambda}$ must satisfy

$$\left[\begin{pmatrix} \hat{\sigma}_x^2 & \hat{\sigma}_{xy} \\ \hat{\sigma}_{xy} & \hat{\sigma}_y^2 \end{pmatrix} - \hat{\lambda}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = 0$$

and this means that the determinant

$$\begin{vmatrix} \hat{\sigma}_x^2 - \hat{\lambda} & \hat{\sigma}_{xy} \\ \hat{\sigma}_{xy} & \hat{\sigma}_y^2 - \hat{\lambda} \end{vmatrix} = 0 .$$

Therefore,

$$\begin{aligned} \hat{\lambda} &= \frac{(\hat{\sigma}_x^2 + \hat{\sigma}_y^2) \pm \sqrt{(\hat{\sigma}_x^2 + \hat{\sigma}_y^2)^2 - 4(\hat{\sigma}_x^2\hat{\sigma}_y^2 - \hat{\sigma}_{xy}^2)}}{2} \\ &= \frac{(\hat{\sigma}_x^2 + \hat{\sigma}_y^2) \pm \sqrt{(\hat{\sigma}_x^2 - \hat{\sigma}_y^2)^2 + 4(\hat{\sigma}_{xy}^2)}}{2} . \end{aligned}$$

The smaller eigenvalue corresponds to the minus sign. With $\hat{\lambda}$ determined, the corresponding unit length eigenvector can be determined;

$$\begin{aligned} \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} &= \frac{1}{\sqrt{\hat{\sigma}_{xy}^2 + (\hat{\lambda} - \hat{\sigma}_x^2)^2}} \begin{pmatrix} \hat{\sigma}_{xy} \\ \hat{\lambda} - \hat{\sigma}_x^2 \end{pmatrix} \\ &= \frac{1}{\sqrt{\hat{\sigma}_{xy}^2 + (\hat{\sigma}_y^2 - \hat{\lambda})^2}} \begin{pmatrix} \hat{\sigma}_y^2 - \hat{\lambda} \\ -\hat{\sigma}_{xy} \end{pmatrix} \end{aligned}$$

3.3 Expected values and variances of least squares estimates of line parameters

The solution $\hat{\alpha}$ and $\hat{\beta}$ for the noisy situation bears a close resemblance to the solution in the noiseless case. However, the randomness of the observed data points in the noisy case leads to a randomness in the estimated parameters $\hat{\alpha}$ and $\hat{\beta}$. The question we now address is how can we determine the expected values of $\hat{\alpha}$ and $\hat{\beta}$ and the variances $\hat{\alpha}$ and $\hat{\beta}$. We consider the case for $\hat{\alpha}$;

$$\hat{\alpha} = \frac{\hat{\sigma}_{xy}}{\sqrt{\hat{\sigma}_{xy}^2 + (\hat{\lambda} - \hat{\sigma}_x^2)^2}} .$$

To find the expected value and variance of $\hat{\alpha}$ we will certainly need a way to relate the expected value and variance of $\hat{\sigma}_{xy}$ and $\hat{\lambda} - \hat{\sigma}_x^2$ to $\hat{\alpha}$. We look at the general situation.

3.3.1 Approximation

Suppose a function f of three variables x, y, and z is known and noisy observations $\hat{x}$, $\hat{y}$ and $\hat{z}$ are available. Furthermore, suppose that for any x, y, z the moments $E[\hat{x}-x]$, $E[(\hat{x}-x)^2]$, $E[\hat{y}-y]$, $E[(\hat{y}-y)^2]$, $E[\hat{z}-z]$, $E[(\hat{z}-z)^2]$ are known. Finally, suppose that the random variables are uncorrelated: $E[(\hat{x}-x)(\hat{y}-y)] = E[(\hat{x}-x)(\hat{z}-z)] = E[(\hat{y}-y)(\hat{z}-z)] = 0$. To determine the expected value and variance of $f(\hat{x}, \hat{y}, \hat{z})$ we can proceed as follows. Represent f as a truncated Taylor series expanded around (x, y, z);

$$f(\hat{x}, \hat{y}, \hat{z}) = f(x, y, z) + (\hat{x} - x)\frac{\partial f}{\partial \hat{x}}(x, y, z) + (\hat{y} - y)\frac{\partial f}{\partial \hat{y}}(x, y, z) + (\hat{z} - z)\frac{\partial f}{\partial \hat{z}}(x, y, z) .$$

The expected value $E[f(\hat{x}, \hat{y}, \hat{z})]$ can then be determined by taking expectations on both sides of the truncated Taylor expansion,

$$E[(f(\hat{x}, \hat{y}, \hat{z})] = f(x, y, z) + E[\hat{x} - x]\frac{\partial f}{\partial \hat{x}}(x, y, z) + E[\hat{y} - y]\frac{\partial f}{\partial \hat{y}}(x, y, z) + E[\hat{z} - z]\frac{\partial f}{\partial \hat{z}}(x, y, z) .$$

To determine $E[f(\hat{x}, \hat{y}, \hat{z}) - f(x, y, z))^2]$ we use the same expansion:

$$E\left[(f(\hat{x}, \hat{y}, \hat{z}) - f(x, y, z))^2\right] = E\left[(\hat{x} - x)^2 \left(\frac{\partial f}{\partial \hat{x}}\right)^2 + 2(\hat{x} - x)(\hat{y} - y)\frac{\partial f}{\partial \hat{x}}\frac{\partial f}{\partial \hat{y}} + 2(\hat{x} - x)(\hat{z} - z)\frac{\partial f}{\partial \hat{x}}\frac{\partial f}{\partial \hat{z}} + (\hat{y} - y)^2\left(\frac{\partial f}{\partial \hat{y}}\right)^2 + 2(\hat{y} - y)(\hat{z} - z)\frac{\partial f}{\partial \hat{y}}\frac{\partial f}{\partial \hat{z}} + (\hat{z} - z)^2\left(\frac{\partial f}{\partial \hat{z}}\right)^2\right] .$$

When the assumption $E[(\hat{x}-x)(\hat{y}-y)] = E[(\hat{x}-x)(\hat{z}-z)] = E[(\hat{y}-y)(\hat{z}-z)] = 0$ holds, we obtain the simple relation:

$$E\left[(f(\hat{x}, \hat{y}, \hat{z}) - f(x, y, z))^2\right] = E\left[(\hat{x} - x)^2\right]\left(\frac{\partial f}{\partial \hat{x}}\right)^2 + E\left[(\hat{y} - y)^2\right]\left(\frac{\partial f}{\partial \hat{y}}\right)^2 + E\left[(\hat{z} - z)^2\right]\left(\frac{\partial f}{\partial \hat{z}}\right)^2 .$$

3.3.2 Expected values and variances

We regard $\hat{\alpha}$ as a function of three variates $\hat{\sigma}_{xy}, \hat{\lambda}$, and $\hat{\sigma}_x^2$. Expanding $\hat{\alpha}$ around the point $(\sigma_{xy}, 0, \sigma_x^2)$ we obtain

$$\hat{\alpha} = \frac{\sigma_{xy}}{\sqrt{\sigma_{xy}^2 + (\sigma_x^2)^2}} + (\hat{\sigma}_{xy} - \sigma_{xy})\frac{(\sigma_x^2)^2}{(\sigma_{xy}^2 + (\sigma_x^2)^2)^{\frac{3}{2}}} + (\hat{\lambda} - 0)\frac{\sigma_{xy}\sigma_x^2}{(\sigma_{xy}^2 + (\sigma_x^2)^2)^{\frac{3}{2}}} + (\hat{\sigma}_x^2 - \sigma_x^2)\frac{-\sigma_{xy}\sigma_x^2}{(\sigma_{xy}^2 + (\sigma_x^2)^2)^{\frac{3}{2}}} .$$

Using the relationship $\sigma_{xy} = \operatorname{sign}(\sigma_{xy})\sigma_x\sigma_y$, which is true under our model and noting that

$$\alpha = \frac{\sigma_{xy}}{\sqrt{\sigma_{xy}^2 + (\sigma_x^2)^2}} \quad \text{and} \quad \beta = \frac{-\sigma_x^2}{\sqrt{\sigma_{xy}^2 + (\sigma_x^2)^2}},$$

we obtain

$$\hat{\alpha} = \alpha + \frac{1}{\sigma_x^2 + \sigma_y^2}\left[(\hat{\sigma}_{xy} - \sigma_{xy})(-\beta) + (\hat{\lambda} - \hat{\sigma}_x^2 + \sigma_x^2)\alpha\right] .$$

Then, to determine $E[\hat{\alpha}]$, we just take expectations on both sides of the equation:

$$E[\hat{\alpha}] = \alpha + \frac{1}{\sigma_x^2 + \sigma_y^2} \times \left[-E[\hat{\sigma}_{xy} - \sigma_{xy}]\beta + E[\hat{\lambda} - \hat{\sigma}_x^2 + \sigma_x^2]\alpha\right] .$$

To determine $V[\hat{\alpha}] = E\left[(\hat{\alpha} - \alpha)^2\right]$,

$$E\left[(\hat{\alpha} - \alpha)^2\right] = \frac{1}{(\sigma_x^2 + \sigma_y^2)^2} \times E\left[\left(-(\hat{\sigma}_{xy} - \sigma_{xy})\beta + (\hat{\lambda} - \hat{\sigma}_x^2 + \sigma_x^2)\alpha\right)^2\right] .$$

To complete our calculation, we need to determine $E[\hat{\sigma}_{xy} - \sigma_{xy}]$, $E[\hat{\lambda} - \hat{\sigma}_x^2 + \sigma_x^2]$, $E[(\hat{\sigma}_{xy} - \sigma_{xy})^2]$, $E[(\hat{\lambda} - \hat{\sigma}_x^2 + \sigma_x^2)^2]$, and $E[(\hat{\sigma}_{xy} - \sigma_{xy})(\hat{\lambda} - \hat{\sigma}_x^2 + \sigma_x^2)]$. These derivations, which we now employ, are given in [13]:

$$E[\hat{\sigma}_{xy} - \sigma_{xy}] = 0$$

$$E[\hat{\lambda} - \hat{\sigma}_x^2 + \sigma_x^2] = 0 .$$

$$E\left[(\hat{\sigma}_{xy} - \sigma_{xy})^2\right] = \frac{\sigma_x^2 + \sigma_y^2 + \frac{I}{W}}{\tilde{W}} .$$

$$E\left[(\hat{\lambda} - \hat{\sigma}_x^2 + \sigma_x^2)^2\right] = \frac{4\sigma_x^2(\sigma_x^2 + \sigma_y^2 + \frac{I}{W})}{\tilde{W}(\sigma_x^2 + \sigma_y^2)} .$$

$$E\left[(\hat{\sigma}_{xy} - \sigma_{xy})(\hat{\lambda} - \hat{\sigma}_x^2 + \sigma_x^2)\right] = 2\alpha\beta\frac{\sigma_x^2 + \sigma_y^2 + \frac{I}{W}}{\tilde{W}} .$$

Hence,

$$E\left[(\hat{\alpha} - \alpha)^2\right] = \frac{1}{(\sigma_x^2 + \sigma_y^2)^2}\left\{\beta^2 E\left[(\hat{\sigma}_{xy} - \sigma_{xy})^2\right] - 2\alpha\beta E\left[(\hat{\sigma}_{xy} - \sigma_{xy})(\hat{\lambda} - \hat{\sigma}_x^2 + \sigma_x^2)\right] + \alpha^2 E\left[(\hat{\lambda} - \hat{\sigma}_x^2 + \sigma_x^2)^2\right]\right\} = \frac{1}{(\sigma_x^2 + \sigma_y^2)^2}\left\{\beta^2\frac{\sigma_x^2 + \sigma_y^2 + \frac{I}{W}}{\tilde{W}} - 4\alpha^2\beta^2\frac{\sigma_x^2 + \sigma_y^2 + \frac{I}{W}}{\tilde{W}} + \alpha^2\frac{4\sigma_x^2(\sigma_x^2 + \sigma_y^2 + \frac{I}{W})}{\tilde{W}(\sigma_x^2 + \sigma_y^2)}\right\}.$$

Therefore,

$$V(\hat{\alpha}) = E\left[(\hat{\alpha}-\alpha)^2\right] = \beta^2 \frac{\sigma_x^2 + \sigma_y^2 + \frac{I}{W}}{\tilde{W}(\sigma_x^2+\sigma_y^2)^2} \,. \tag{4}$$

Using the relation

$$\hat{\beta} = \frac{-\hat{\sigma}_{xy}}{\sqrt{\hat{\sigma}_{xy}^2 + (\hat{\sigma}_y^2 - \hat{\lambda})^2}} \,,$$

a symmetric calculation for $\hat{\beta}$ yields

$$\hat{\beta} = \beta + \frac{1}{\sigma_x^2+\sigma_y^2}\left[(\hat{\sigma}_{xy}-\sigma_{xy})(-\alpha) + (\hat{\lambda} - \hat{\sigma}_y^2 + \sigma_y^2)\beta\right]$$

from which we obtain

$$V(\hat{\beta}) = E\left[(\hat{\beta}-\beta)^2\right] = \alpha^2 \frac{\sigma_x^2 + \sigma_y^2 + \frac{I}{W}}{\tilde{W}(\sigma_x^2+\sigma_y^2)^2} \,. \tag{5}$$

Since $\alpha^2+\beta^2=1$, from Eqs. (4) and (5), we can see that

$$V(\hat{\alpha}) + V(\hat{\beta}) = T \tag{6}$$

where

$$T = \frac{\sigma_x^2 + \sigma_y^2 + \frac{I}{W}}{\tilde{W}(\sigma_x^2+\sigma_y^2)^2} \,.$$

The covariance between $\hat{\alpha}$ and $\hat{\beta}$, $\text{cov.}(\hat{\alpha},\hat{\beta}) = E[(\hat{\alpha}-\alpha)(\hat{\beta}-\beta)]$ is

$$\begin{aligned}
E[(\hat{\alpha}-\alpha)(\hat{\beta}-\beta)] &= \frac{1}{(\sigma_x^2+\sigma_y^2)^2}\alpha\beta E[(\hat{\sigma}_{xy}-\sigma_{xy})^2] \\
&- \beta^2 E[(\hat{\sigma}_{xy}-\sigma_{xy})(\hat{\lambda}-\hat{\sigma}_y^2+\sigma_y^2)] \\
&- \alpha^2 E[(\hat{\sigma}_{xy}-\sigma_{xy})(\hat{\lambda}-\hat{\sigma}_x^2+\sigma_x^2)] \\
&+ \alpha\beta E[(\hat{\lambda}-\hat{\sigma}_x^2+\sigma_x^2)(\hat{\lambda}-\hat{\sigma}_y^2+\sigma_y^2)] \\
&= \frac{-(\sigma_x^2+\sigma_y^2+\frac{I}{W})}{\tilde{W}(\sigma_x^2+\sigma_y^2)^2}\alpha\beta \,.
\end{aligned}$$

Note that

$$V(\hat{\alpha}) = \beta^2 T \,, \quad V(\hat{\beta}) = \alpha^2 T \,, \quad \text{cov.}(\hat{\alpha},\hat{\beta}) = -\alpha\beta T \,.$$

To determine the variance of $\hat{\gamma}$, we use

$$\gamma = -(\alpha\mu_x + \beta\mu_y), \quad \hat{\gamma} = -(\hat{\alpha}\hat{\mu}_x + \hat{\beta}\hat{\mu}_y) \,.$$

Therefore, we have

$$\begin{aligned}
\hat{\gamma} - \gamma &= -\hat{\alpha}(\mu_x + \bar{\xi}) - \hat{\beta}(\mu_y + \bar{\eta}) + \alpha\mu_x + \beta\mu_y \\
&= -(\hat{\alpha}-\alpha)\mu_x - (\hat{\beta}-\beta)\mu_y - (\hat{\alpha}\bar{\xi} + \hat{\beta}\eta) \,.
\end{aligned}$$

Hence the variance of $\hat{\gamma}$ is:

$$\begin{aligned}
E[(\hat{\gamma}-\gamma)^2] &= E\left[[(\hat{\alpha}-\alpha)\mu_x + (\hat{\beta}-\beta)\mu_y + (\hat{\alpha}\bar{\xi}+\hat{\beta}\eta)]^2\right] \\
&= \mu_x^2 E[(\hat{\alpha}-\alpha)^2] + \mu_y^2 E[(\hat{\beta}-\beta)^2] \\
&+ 2\mu_x\mu_y E[(\hat{\alpha}-\alpha)(\hat{\beta}-\beta)] \\
&+ E[(\hat{\alpha}\bar{\xi}+\hat{\beta}\bar{\eta})^2] + 2\mu_x E[(\hat{\alpha}-\alpha)(\hat{\alpha}\bar{\xi}+\hat{\beta}\bar{\eta})] \\
&+ 2\mu_y E[(\hat{\beta}-\beta)(\hat{\alpha}\bar{\xi}+\hat{\beta}\bar{\eta})] \,.
\end{aligned}$$

To complete our calculation, we need to determine $E[(\hat{\alpha}\bar{\xi}+\hat{\beta}\bar{\eta})^2]$, $E[(\hat{\alpha}-\alpha)(\hat{\alpha}\bar{\xi}+\hat{\beta}\bar{\eta})]$, and $E[(\hat{\beta}-\beta)(\hat{\alpha}\bar{\xi}+\hat{\beta}\bar{\eta})]$. These derivations, which we use below, are given elsewhere [13]:

$$\begin{aligned}
E[(\hat{\alpha}\bar{\xi}+\hat{\beta}\bar{\eta})^2] &= \frac{I}{W}(V(\hat{\alpha}) + V(\hat{\beta}) + 1) \,, \\
E[(\hat{\alpha}-\alpha)(\hat{\alpha}\bar{\xi}+\hat{\beta}\bar{\eta})] &= 0 \,, \\
E[(\hat{\beta}-\beta)(\hat{\alpha}\bar{\xi}+\hat{\beta}\bar{\eta})] &= 0 \,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
V(\hat{\gamma}) &= \mu_x^2 V(\hat{\alpha}) + \mu_y^2 V(\hat{\beta}) + 2\mu_x\mu_y \text{cov.}(\hat{\alpha},\hat{\beta}) \\
&+ \frac{I}{W}(V(\hat{\alpha}) + V(\hat{\beta}) + 1) \,.
\end{aligned} \tag{7}$$

Equation (7) can be rewritten as

$$V(\hat{\gamma}) = T(\mu_x^2 + \mu_y^2 - \gamma^2 + \frac{I}{W}) + \frac{I}{W}.$$

The covariance of $\hat{\alpha}$ and $\hat{\gamma}$ is

$$\begin{aligned}
\text{cov.}(\hat{\alpha},\hat{\gamma}) &= E[(\hat{\alpha}-\alpha)(\hat{\gamma}-\gamma)] \\
&= E\left[(\hat{\alpha}-\alpha)[-(\hat{\alpha}-\alpha)\mu_x - (\hat{\beta}-\beta)\mu_y - (\hat{\alpha}\bar{\xi}+\hat{\beta}\bar{\eta})]\right] \\
&= -\mu_x V(\hat{\alpha}) - \mu_y \text{cov.}(\hat{\alpha},\hat{\beta}) = -\mu_x\beta^2 T - \mu_y\alpha\beta T \\
&= \beta T(-\mu_x\beta + \mu_y\alpha) \,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\text{cov.}(\hat{\beta},\hat{\gamma}) &= -\mu_x \text{cov.}(\hat{\alpha},\hat{\beta}) - \mu_y V(\hat{\beta}) \\
&= \alpha T(\mu_x\beta - \mu_y\alpha) \,.
\end{aligned}$$

3.4 Expected value and variance of corner point position

Suppose we have two lines:

$$\begin{aligned}
\alpha_1 x + \beta_1 y + \gamma_1 &= 0 \\
\alpha_2 x + \beta_2 y + \gamma_2 &= 0
\end{aligned}$$

The intersection of the two lines is

$$(r,c) = \left(\frac{\beta_1\gamma_2 - \beta_2\gamma_1}{\alpha_1\beta_2 - \alpha_2\beta_1}, \frac{\alpha_2\gamma_1 - \alpha_1\gamma_2}{\alpha_1\beta_2 - \alpha_2\beta_1}\right) .$$

Suppose we have noisy observations for each unknown line and that $\hat{\alpha}_1, \hat{\beta}_1, \hat{\gamma}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\gamma}_2$ are the least squares estimates of the line parameters. Then the estimate of the intersection point (r,c) is

$$(\hat{r},\hat{c}) = \left(\frac{\hat{\beta}_1\hat{\gamma}_2 - \hat{\beta}_2\hat{\gamma}_1}{\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1}, \frac{\hat{\alpha}_2\hat{\gamma}_1 - \hat{\alpha}_1\hat{\gamma}_2}{\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1}\right) .$$

We want to determine how the noise propagates to the intersection point (r,c). For this purpose, we want to compute the variances of $\hat{r}$, $\hat{c}$. Note that $E[\hat{r}] = r$ and $E[\hat{c}] = c$.

We will carry out a first-order Taylor expansion of $\hat{r}$ around $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$. The derivation is given in [13]:

$$\hat{r} = r + \frac{1}{(\alpha_1\beta_2 - \alpha_2\beta_1)} \times \left[-\beta_2(\hat{\alpha}_1 r + \hat{\beta}_1 c + \hat{\gamma}_1) + \beta_1(\hat{\alpha}_2 r + \hat{\beta}_2 c + \hat{\gamma}_2)\right] ,$$
$$\hat{c} = c + \frac{1}{(\alpha_1\beta_2 - \alpha_2\beta_1)} \times \left[\alpha_2(\hat{\alpha}_1 r + \hat{\beta}_1 c + \hat{\gamma}_1) - \alpha_1(\hat{\alpha}_2 r + \hat{\beta}_2 c + \hat{\gamma}_2)\right] .$$

Therefore,

$$(\hat{r} - r)^2 = \frac{1}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2} \times \left[-\beta_2(\hat{\alpha}_1 r + \hat{\beta}_1 c + \hat{\gamma}_1) + \beta_1(\hat{\alpha}_2 r + \hat{\beta}_2 c + \hat{\gamma}_2)\right]^2 ,$$
$$(\hat{c} - c)^2 = \frac{1}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2} \times \left[\alpha_2(\hat{\alpha}_1 r + \hat{\beta}_1 c + \hat{\gamma}_1) - \alpha_1(\hat{\alpha}_2 r + \hat{\beta}_2 c + \hat{\gamma}_2)\right]^2 ,$$
$$(\hat{r} - r)(\hat{c} - c) = \frac{1}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2} \times \left[-\alpha_2\beta_2(\hat{\alpha}_1 r + \hat{\beta}_1 c + \hat{\gamma}_1)^2 - \alpha_1\beta_1(\hat{\alpha}_2 r + \hat{\beta}_2 c + \hat{\gamma}_2)^2 + (\alpha_1\beta_2 + \alpha_2\beta_1)(\hat{\alpha}_1 r + \hat{\beta}_1 c + \hat{\gamma}_1)(\hat{\alpha}_2 r + \hat{\beta}_2 c + \hat{\gamma}_2)\right] .$$

Since α_1, β_1, and γ_1 are independent of α_2, β_2, γ_2, we have

$$E[(\hat{\alpha}_1 r + \hat{\beta}_1 c + \hat{\gamma}_1)]E[(\hat{\alpha}_2 r + \hat{\beta}_2 c + \hat{\gamma}_2)] = 0 .$$

Hence,

$$V(\hat{r}) = \frac{1}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2}\left[\beta_2^2 E[(\hat{\alpha}_1 r + \hat{\beta}_1 c + \hat{\gamma}_1)^2] + \beta_1^2 E[(\hat{\alpha}_2 r + \hat{\beta}_2 c + \hat{\gamma}_2)^2]\right] ,$$
$$V(\hat{c}) = \frac{1}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2}\left[\alpha_2^2 E[(\hat{\alpha}_1 r + \hat{\beta}_1 c + \hat{\gamma}_1)^2] + \alpha_1^2 E[(\hat{\alpha}_2 r + \hat{\beta}_2 c + \hat{\gamma}_2)^2]\right] ,$$
$$\text{cov.}(\hat{r}, \hat{c}) = \frac{1}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2}\left[-\alpha_2\beta_2 E[(\hat{\alpha}_1 r + \hat{\beta}_1 c + \hat{\gamma}_1)^2] - \alpha_1\beta_1 E[(\hat{\alpha}_2 r + \hat{\beta}_2 c + \hat{\gamma}_2)^2]\right] .$$

It is shown in [13] that

$$E[(\hat{\alpha} r + \hat{\beta} c + \hat{\gamma})^2] = r^2 V(\hat{\alpha}) + c^2 V(\hat{\beta}) + V(\hat{\gamma}) + 2rc\,\text{cov.}(\hat{\alpha}, \hat{\beta}) + 2r\,\text{cov.}(\hat{\alpha}, \hat{\gamma}) + 2c\,\text{cov.}(\hat{\beta}, \hat{\gamma}) , \tag{8}$$

and the computations of $V(\hat{r})$, $V(\hat{c})$, and cov.$(\hat{r}, \hat{c})$ are straightforward.

For the case of *i.i.d.* noise, simply replacing w_i with σ^2, W with I/σ^2, and $\tilde{W}$ with $(I-1)/\sigma^2$ gives results that are identical to [7].

4 Circle fitting

4.1 Preliminaries

Consider a situation in which points $(x_i, y_i), i = 1, \ldots, I$ are assumed to lie on an unknown circle and the problem is to determine the parameters of the circle. Suppose (x_i, y_i) satisfies the model

$$2px_i + 2qy_i + r = x_i^2 + y_i^2 \quad i = 1, \ldots, I . \tag{9}$$

We define $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \sigma_{xy}$ as in the line fitting case. In addition, we define the third moments and covariances between x and y^2, and between x^2 and y as follows:

$$\mu_{x^3} = \frac{1}{\tilde{W}}\sum_{i=1}^{I} w_i(x_i - \mu_x)^3 ,$$
$$\mu_{y^3} = \frac{1}{I-1}\sum_{i=1}^{I}(y_i - \mu_y)^3 ,$$
$$\mu_{x^2y} = \frac{1}{\tilde{W}}\sum_{i=1}^{I} w_i(x_i - \mu_x)^2(y_i - \mu_y) ,$$
$$\mu_{xy^2} = \frac{1}{\tilde{W}}\sum_{i=1}^{I} w_i(x_i - \mu_x)(y_i - \mu_y)^2 .$$

Multiplying by w_i and summing the model Eq. (9) over all i, we have

$$\sum_{i=1}^{I} w_i(2px_i + 2qy_i + r) = \sum_{i=1}^{I} w_i(x_i^2 + y_i^2)$$

so that

$$r = \frac{I-1}{I}(\sigma_x^2 + \sigma_y^2) + \mu_x^2 + \mu_y^2 - 2p\mu_x - 2q\mu_y .$$

The model can be rewritten as

$$2p(x_i - \mu_x) + 2q(y_i - \mu_y) + \frac{I-1}{I}(\sigma_x^2 + \sigma_y^2) + \mu_x^2 + \mu_y^2 = x_i^2 + y_i^2 . \tag{10}$$

Multiplying Eq. (10) by $w_i(x_i - \mu_x)$ and summing over all i, we have

$$2p\sum_{i=1}^{I} w_i(x_i - \mu_x)^2 + 2q\sum_{i=1}^{I} w_i(x_i - \mu_x)(y_i - \mu_y) = \sum_{i=1}^{I} w_i(x_i - \mu_x)(x_i^2 + y_i^2) . \tag{11}$$

Similarly multiplying Eq. (10) by $w_i(y_i - \mu_y)$ and summing over all i, we have

$$2p\sum_{i=1}^{I} w_i(x_i - \mu_x)(y_i - \mu_y) + 2q\sum_{i=1}^{I} w_i(y_i - \mu_y)^2 = \sum_{i=1}^{I} w_i(y_i - \mu_y)(x_i^2 + y_i^2) . \tag{12}$$

Using definitions, we can rewrite Eqs. (11) and (12);

$$2p\sigma_x^2 + 2q\sigma_{xy} = \left[\mu_{x^3} + 2\mu_x\sigma_x^2 + \mu_{xy^2} + 2\mu_y\sigma_{xy}\right] ,$$
$$2p\sigma_{xy} + 2q\sigma_y^2 = \left[\mu_{y^3} + 2\mu_y\sigma_y^2 + \mu_{x^2y} + 2\mu_x\sigma_{xy}\right] .$$

Solving the above equations for p and q, we have

$$\begin{aligned} p &= \frac{1}{2(\sigma_x^2\sigma_y^2 - \sigma_{xy}^2)}\left[\mu_{x^3}\sigma_y^2 - \mu_{y^3}\sigma_{xy} \right. \\ &\quad \left. + \mu_{xy^2}\sigma_y^2 - \mu_{x^2y}\sigma_{xy}\right] + \mu_x , \\ q &= \frac{1}{2(\sigma_x^2\sigma_y^2 - \sigma_{xy}^2)}\left[\mu_{y^3}\sigma_x^2 - \mu_{x^3}\sigma_{xy} \right. \\ &\quad \left. + \mu_{x^2y}\sigma_x^2 - \mu_{xy^2}\sigma_{xy}\right] + \mu_y , \\ r &= \frac{I-1}{I}(\sigma_x^2 + \sigma_y^2) - (\mu_x^2 + \mu_y^2) \\ &\quad - \frac{1}{\sigma_x^2\sigma_y^2 - \sigma_x^2 y}\left[\mu_{x^3}(\mu_x\sigma_y^2 - \mu_y\sigma_{xy}) \right. \\ &\quad + \mu_{y^3}(\mu_y\sigma_x^2 - \mu_x\sigma_{xy}) + \mu_{x^2y}(\mu_y\sigma_x^2 - \mu_x\sigma_{xy}) \\ &\quad \left. + \mu_{xy^2}(\mu_x\sigma_y^2 - \mu_y\sigma_{xy})\right] . \end{aligned}$$

4.2 Least squares estimates of circle parameters

Up to this point, the given points $(x_i, y_i), i = 1, \ldots, I$ (which lie exactly on the unknown circle) are the points we used to determine the parameters of the circle. Now we assume that (x_i, y_i) are not given; instead noisy observations $(\hat{x}_i, \hat{y}_i)$ of (x_i, y_i) are given. From the noisy observations $(\hat{x}_i, \hat{y}_i)$ we must estimate the parameters of the unknown circle. To do this, we employ the principle of minimizing the squared residuals. Let $\hat{p}, \hat{q}$, and $\hat{r}$ be the estimates of the unknown circle parameters. Define

$$\varepsilon^2 = \sum_{i=1}^{I} w_i(2\hat{p}\hat{x}_i + 2\hat{q}\hat{y}_i + \hat{r} - \hat{x}_i^2 - \hat{y}_i^2)^2 .$$

Note that ε^2 is again a weighted sum and the weights are reciprocals of the variances of the random disturbances. Since we want to find $\hat{p}$, $\hat{q}$, and $\hat{r}$ which minimize ε^2, we take the partial derivatives of ε^2 with respect to $\hat{p}, \hat{q}$ and $\hat{r}$ and set them equal to 0.

$$\frac{\partial\varepsilon^2}{\partial\hat{p}} = 2\sum_{i=1}^{I} 2w_i\hat{x}_i(2\hat{p}\hat{x}_i + 2\hat{q}\hat{y}_i + \hat{r} - \hat{x}_i^2 - \hat{y}_i^2) ,$$
$$\frac{\partial\varepsilon^2}{\partial\hat{q}} = 2\sum_{i=1}^{I} 2w_i\hat{y}_i(2\hat{p}\hat{x}_i + 2\hat{q}\hat{y}_i + \hat{r} - \hat{x}_i^2 - \hat{y}_i^2) ,$$
$$\frac{\partial\varepsilon^2}{\partial\hat{r}} = 2\sum_{i=1}^{I} w_i(2\hat{p}\hat{x}_i + 2\hat{q}\hat{y}_i + \hat{r} - \hat{x}_i^2 - \hat{y}_i^2).$$

Rewriting the above equations using matrix notation, we have

$$\begin{pmatrix} 4\sum_{i=1}^{I} w_i\hat{x}_i^2 & 4\sum_{i=1}^{I} w_i\hat{x}_i\hat{y}_i & 2\sum_{i=1}^{I} w_i\hat{x}_i \\ 4\sum_{i=1}^{I} w_i\hat{x}_i\hat{y}_i & 4\sum_{i=1}^{I} w_i\hat{y}_i^2 & 2\sum_{i=1}^{I} w_i\hat{y}_i \\ 2\sum_{i=1}^{I} w_i\hat{x}_i & 2\sum_{i=1}^{I} w_i\hat{y}_i & W \end{pmatrix} \begin{pmatrix} \hat{p} \\ \hat{q} \\ \hat{r} \end{pmatrix} = \begin{pmatrix} 2\sum_{i=1}^{I} w_i\hat{x}_i(\hat{x}_i^2+\hat{y}_i^2) \\ 2\sum_{i=1}^{I} w_i\hat{y}_i(\hat{x}_i^2+\hat{y}_i^2) \\ \sum_{i=1}^{I} w_i(\hat{x}_i^2+\hat{y}_i^2) \end{pmatrix} . \tag{13}$$

For notational convenience, we define

$$\boldsymbol{Q} = \begin{pmatrix} 2\sqrt{w_1}x_1 & 2\sqrt{w_1}y_1 & \sqrt{w_1} \\ 2\sqrt{w_2}x_2 & 2\sqrt{w_2}y_2 & \sqrt{w_2} \\ \vdots & \vdots & \vdots \\ 2\sqrt{w_I}x_I & 2\sqrt{w_I}y_I & \sqrt{w_I} \end{pmatrix} ,$$

$$\beta = \begin{pmatrix} p \\ q \\ r \end{pmatrix} , \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} \sqrt{w_1}(x_1^2 + y_1^2) \\ \sqrt{w_2}(x_2^2 + y_2^2) \\ \vdots \\ \sqrt{w_I}(x_I^2 + y_I^2) \end{pmatrix} .$$

Then it is easy to see that Eq. (13) has the form

$$\hat{\boldsymbol{Q}}'\hat{\boldsymbol{Q}}\hat{\beta} = \hat{\boldsymbol{Q}}'\hat{\boldsymbol{b}},$$

and this is equivalent to an overconstrained linear system

$$\hat{\boldsymbol{Q}}\hat{\beta} = \hat{\boldsymbol{b}} . \tag{14}$$

If we solve the above system for $\hat{p}, \hat{q}$ and $\hat{r}$, the center of the best fitting circle is $(\hat{p}, \hat{q})$ and its radius $\hat{R} = \sqrt{\hat{r} + \hat{p}^2 + \hat{q}^2}$. The normal solution for the system (14) is

$$\hat{\beta} = (\hat{\boldsymbol{Q}}'\hat{\boldsymbol{Q}})^{-1}\hat{\boldsymbol{Q}}'\hat{\boldsymbol{b}} .$$

It is straightforward to verify that

$$(\hat{\boldsymbol{Q}}'\hat{\boldsymbol{Q}})^{-1} = \begin{pmatrix} \hat{A} & \hat{B} & \hat{C} \\ \hat{B} & \hat{D} & \hat{E} \\ \hat{C} & \hat{E} & \hat{F} \end{pmatrix} ,$$

where

$$\hat{A} = \frac{\hat{\sigma}_y^2}{4\tilde{W}(\hat{\sigma}_x^2\hat{\sigma}_y^2 - \hat{\sigma}_{xy}^2)} ,$$

$$\hat{B} = \frac{-\hat{\sigma}_{xy}}{4\tilde{W}(\hat{\sigma}_x^2\hat{\sigma}_y^2 - \hat{\sigma}_{xy}^2)} ,$$

$$\hat{C} = \frac{\hat{\sigma}_{xy}\hat{\mu}_y - \hat{\sigma}_y^2\hat{\mu}_x}{2\tilde{W}(\hat{\sigma}_x^2\hat{\sigma}_y^2 - \hat{\sigma}_{xy}^2)} ,$$

$$\hat{D} = \frac{\hat{\sigma}_x^2}{4\tilde{W}(\hat{\sigma}_x^2\hat{\sigma}_y^2 - \hat{\sigma}_{xy}^2)} ,$$

$$\hat{E} = \frac{\hat{\sigma}_{xy}\hat{\mu}_x - \hat{\sigma}_x^2\hat{\mu}_y}{2\tilde{W}(\hat{\sigma}_x^2\hat{\sigma}_y^2 - \hat{\sigma}_{xy}^2)} ,$$

$$\hat{F} = \frac{1}{\tilde{W}} + \frac{\hat{\mu}_x^2\hat{\sigma}_y^2 + \hat{\mu}_y^2\hat{\sigma}_x^2 - 2\hat{\sigma}_{xy}\hat{\mu}_x\hat{\mu}_y}{\tilde{W}(\hat{\sigma}_x^2\hat{\sigma}_y^2 - \hat{\sigma}_{xy}^2)} .$$

So,

$$(\hat{Q}'\hat{Q})^{-1}\hat{Q}' = \begin{pmatrix} \sqrt{w_1}(2\hat{A}\hat{x}_1+2\hat{B}\hat{y}_1+\hat{C}) & \dots & \sqrt{w_I}(2\hat{A}\hat{x}_I+2\hat{B}\hat{y}_I+\hat{C}) \\ \sqrt{w_1}(2\hat{B}\hat{x}_1+2\hat{D}\hat{y}_1+\hat{E}) & \dots & \sqrt{w_I}(2\hat{B}\hat{x}_I+2\hat{D}\hat{y}_I+\hat{E}) \\ \sqrt{w_1}(2\hat{C}\hat{x}_1+2\hat{E}\hat{y}_1+\hat{F}) & \dots & \sqrt{w_I}(2\hat{C}\hat{x}_I+2\hat{E}\hat{y}_I+\hat{F}) \end{pmatrix} ,$$

and therefore the solution is

$$\hat{\beta} = \begin{pmatrix} \hat{p} \\ \hat{q} \\ \hat{r} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^I w_i(2\hat{A}\hat{x}_i + 2\hat{B}\hat{y}_i + \hat{C})(\hat{x}_i^2 + \hat{y}_i^2) \\ \sum_{i=1}^I w_i(2\hat{B}\hat{x}_i + 2\hat{D}\hat{y}_i + \hat{E})(\hat{x}_i^2 + \hat{y}_i^2) \\ \sum_{i=1}^I w_i(2\hat{C}\hat{x}_i + 2\hat{E}\hat{y}_i + \hat{F})(\hat{x}_i^2 + \hat{y}_i^2) \end{pmatrix} .$$

4.3 Expected values and variances of the estimated parameters

Now we want to know the expected values and variances of $\hat{p}$, $\hat{q}$ and $\hat{r}$. Suppose we have a function f of two vectors, $\Delta\boldsymbol{x}_i (i = 1, \dots, I)$ and β. Using a first order Taylor series expansion, we have

$$f(\boldsymbol{x}_i + \Delta\boldsymbol{x}_i, \beta + \Delta\beta) = f(\boldsymbol{x}_i, \beta) + \boldsymbol{J}'_{1i}\Delta\boldsymbol{x}_i + \boldsymbol{J}'_{2i}\Delta\beta,$$

where

$$\boldsymbol{J}_{1i} = \begin{pmatrix} \frac{\partial f}{\partial x_i} \\ \frac{\partial f}{\partial y_i} \end{pmatrix}\Bigg|_{\boldsymbol{x}_i,\beta} \quad \text{and} \quad \boldsymbol{J}_{2i} = \begin{pmatrix} \frac{\partial f}{\partial p} \\ \frac{\partial f}{\partial q} \\ \frac{\partial f}{\partial r} \end{pmatrix}\Bigg|_{\boldsymbol{x}_i,\beta} .$$

If we set $f(\boldsymbol{x}_i + \Delta\boldsymbol{x}_i, \beta + \Delta\beta) = f(\boldsymbol{x}_i, \beta) = 0$ for all $i = 1, \dots, I$, then we have

$$\boldsymbol{J}'_{1i}\Delta\boldsymbol{x}_i = \boldsymbol{J}'_{2i}\Delta\beta .$$

Hence

$$\Delta\beta = (\boldsymbol{J}'_2\boldsymbol{J}_2)^{-1}\boldsymbol{J}'_2 \begin{pmatrix} -\boldsymbol{J}'_{11}\Delta\boldsymbol{x}_1 \\ \vdots \\ -\boldsymbol{J}'_{1I}\Delta\boldsymbol{x}_I \end{pmatrix} ,$$

where

$$\boldsymbol{J}_2 = \begin{pmatrix} \boldsymbol{J}'_{21} \\ \vdots \\ \boldsymbol{J}'_{2I} \end{pmatrix} .$$

Also we have

$$\Delta\beta\Delta\beta' = \boldsymbol{K}\left[\begin{pmatrix} \boldsymbol{J}'_{11}\Delta\boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{J}'_{1I}\Delta\boldsymbol{x}_I \end{pmatrix}\begin{pmatrix} \boldsymbol{J}'_{11}\Delta\boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{J}'_{1I}\Delta\boldsymbol{x}_I \end{pmatrix}'\right]\boldsymbol{K}',$$

where

$$\boldsymbol{K} = (\boldsymbol{J}'_2\boldsymbol{J}_2)^{-1}\boldsymbol{J}'_2 .$$

Let $\hat{p} = p + \Delta p$, $\hat{q} = q + \Delta q$ and $\hat{r} = r + \Delta r$ and define

$$\hat{\beta} = \begin{pmatrix} \hat{p} \\ \hat{q} \\ \hat{r} \end{pmatrix}, \Delta\beta = \begin{pmatrix} \Delta p \\ \Delta q \\ \Delta r \end{pmatrix} , \quad \text{and}$$

$$\epsilon = \begin{pmatrix} \sqrt{w_1}(2\hat{p}\hat{x}_1 + 2\hat{q}\hat{y}_1 + \hat{r} - \hat{x}_1^2 - \hat{y}_1^2) \\ \vdots \\ \sqrt{w_I}(2\hat{p}\hat{x}_I + 2\hat{q}\hat{y}_I + \hat{r} - \hat{x}_I^2 - \hat{y}_I^2) \end{pmatrix} .$$

Note that $\hat{\beta} = \beta + \Delta\beta$. Let ϵ_i denote the i-th row of ϵ and let

$$f(\boldsymbol{x}_i, \beta) = \epsilon_i = \sqrt{w_i}(2px_i + 2qy_i + r - x_i^2 - y_i^2)$$

where

$$\boldsymbol{x}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} .$$

Let

$$\Delta\boldsymbol{x}_i = \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} ,$$

and note that

$$\boldsymbol{x}_i + \Delta\boldsymbol{x}_i = \begin{pmatrix} \hat{x}_i \\ \hat{y}_i \end{pmatrix} .$$

Since

$$f(\boldsymbol{x}_i, \beta) = \sqrt{w_i}(2px_i + 2qy_i + r - x_i^2 - y_i^2) = 0 ,$$

we have

$$\epsilon_i = f(\boldsymbol{x}_i + \Delta\boldsymbol{x}_i, \beta + \Delta\beta) = \boldsymbol{J}'_{1i}\Delta\boldsymbol{x}_i + \boldsymbol{J}'_{2i}\Delta\beta ,$$

where

$$J_{1i} = \sqrt{w_i}\begin{pmatrix} 2p-2x_i \\ 2q-2y_i \end{pmatrix} \text{ and } J_{2i} = \sqrt{w_i}\begin{pmatrix} 2x_i \\ 2y_i \\ 1 \end{pmatrix} .$$

Since

$$\begin{pmatrix} J'_{21} \\ \vdots \\ J'_{2I} \end{pmatrix} = \begin{pmatrix} 2\sqrt{w_1}x_1 & 2\sqrt{w_1}y_1 & \sqrt{w_1} \\ 2\sqrt{w_2}x_2 & 2\sqrt{w_2}y_2 & \sqrt{w_2} \\ \vdots & \vdots & \vdots \\ 2\sqrt{w_I}x_I & 2\sqrt{w_I}y_I & \sqrt{w_I} \end{pmatrix} = Q,$$

if we let $\epsilon = \mathbf{0}$, then we have

$$Q\Delta\beta = -\begin{pmatrix} J'_{11}\Delta x_i \\ \vdots \\ J'_{1I}\Delta x_i \end{pmatrix} .$$

Hence

$$\Delta\beta = (Q'Q)^{-1}Q'\begin{pmatrix} -J'_{11}\Delta x_1 \\ \vdots \\ -J'_{1I}\Delta x_I \end{pmatrix} . \tag{15}$$

Since $E[\Delta x_i] = 0$ for all $i = 1,\ldots,I$, after taking expectations on both sides of Eq. (15), there results;

$$E[\Delta\beta] = \begin{pmatrix} E[\Delta p] \\ E[\Delta q] \\ E[\Delta r] \end{pmatrix} = (Q'Q)^{-1}Q'\begin{pmatrix} -J'_{11}E[\Delta x_1] \\ \vdots \\ -J'_{1I}E[\Delta x_I] \end{pmatrix} = \mathbf{0} ,$$

which means $E[\hat{\beta}] = \beta$. Letting $K = (Q'Q)^{-1}Q'$, we have

$$\Delta\beta\Delta\beta' = K\left[MM'\right]K', \tag{16}$$

where

$$M = \begin{pmatrix} J'_{11}\Delta x_1 \\ \vdots \\ J'_{1I}\Delta x_I \end{pmatrix}\begin{pmatrix} J'_{11}\Delta x_1 \\ \vdots \\ J'_{1I}\Delta x_I \end{pmatrix}' = \begin{pmatrix} J'_{11}\Delta x_1\Delta x'_1 J_{11} & \ldots & J'_{11}\Delta x_1\Delta x'_I J_{1I} \\ \vdots & \ddots & \vdots \\ J'_{1I}\Delta x_I\Delta x'_1 J_{11} & \ldots & J'_{1I}\Delta x_I\Delta x'_I J_{1I} \end{pmatrix} .$$

For the variance of $\Delta\beta$, we take expectations on both sides of Eq. (16). Since

$$E[\Delta x_i\Delta x'_i] = \begin{pmatrix} \sigma_i^2 & 0 \\ 0 & \sigma_i^2 \end{pmatrix}$$

and

$$E[\Delta x_i\Delta x'_j] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{when} \quad i \neq j ,$$

we have

$$E[M] = \begin{pmatrix} J'_{11}E[\Delta x_1\Delta x'_1]J_{11} & J'_{11}E[\Delta x_1\Delta x'_2]J_{12} & \ldots & J'_{11}E[\Delta x_1\Delta x'_I]J_{1I} \\ J'_{12}E[\Delta x_2\Delta x'_1]J_{11} & J'_{12}E[\Delta x_2\Delta x'_2]J_{12} & \ldots & J'_{12}E[\Delta x_2\Delta x'_I]J_{1I} \\ \vdots & \vdots & \ddots & \vdots \\ J'_{1I}E[\Delta x_I\Delta x'_1]J_{11} & J'_{1I}E[\Delta x_I\Delta x'_2]J_{12} & \ldots & J'_{1I}E[\Delta x_I\Delta x'_I]J_{1I} \end{pmatrix}$$

$$= 4\begin{pmatrix} (p-x_1)^2+(q-y_1)^2 & 0 & \ldots & 0 \\ 0 & (p-x_2)^2+(q-y_2)^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (p-x_I)^2+(q-y_I)^2 \end{pmatrix}$$

$$= 4(r+p^2+q^2)I_I = 4R^2 I_I ,$$

where I_I is an I by I identity matrix. We know that

$$K = (Q'Q)^{-1}Q' = \begin{pmatrix} \sqrt{w_1}(2Ax_1+2By_1+C) & \ldots & \sqrt{w_I}(2Ax_I+2By_I+C) \\ \sqrt{w_1}(2Bx_1+2Dy_1+E) & \ldots & \sqrt{w_I}(2Bx_I+2Dy_I+E) \\ \sqrt{w_1}(2Cx_1+2Ey_1+F) & \ldots & \sqrt{w_I}(2Cx_I+2Ey_I+F) \end{pmatrix} .$$

Therefore, we have

$$E[\Delta\beta\Delta\beta'] = 4(r+p^2+q^2)KK' . \tag{17}$$

and

$$KK' = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{pmatrix} ,$$

where

$$K_{11} = \sum_{i=1}^{I} w_i(2Ax_i + 2By_i + C)^2 ,$$

$$K_{12} = \sum_{i=1}^{I} w_i(2Ax_i + 2By_i + C)(2Bx_i + 2Dy_i + E) ,$$

$$K_{13} = \sum_{i=1}^{I} w_i(2Ax_i + 2By_i + C)(2Cx_i + 2Ey_i + F) ,$$

$$K_{22} = \sum_{i=1}^{I} w_i(2Bx_i + 2Dy_i + E)^2 ,$$

$$K_{23} = \sum_{i=1}^{I} w_i(2Bx_i + 2Dy_i + E)(2Cx_i + 2Ey_i + F) ,$$

$$K_{33} = \sum_{i=1}^{I} w_i(2Cx_i + 2Ey_i + F)^2 .$$

Since the center of the circle is at (p, q), we already know the variance of these variables. For the case of *i.i.d.* noise, we simply replace w_i with σ^2, W with I/σ^2, and $\tilde{W}$ with $(I-1)/\sigma^2$.

5 Ellipse fitting

5.1 Least squares estimates of ellipse parameters

Consider a situation in which points $(x_i, y_i), i = 1, \ldots, I$ are assumed to lie on an unknown ellipse and the problem is to determine the parameters of the ellipse. Suppose (x_i, y_i) satisfies the model

$$Ax_i^2 + Bx_iy_i + Cy_i^2 + Dx_i + Ey_i + F = 0 , \quad i = 1, \ldots, I .$$

We assume that (x_i, y_i) are not given; instead noisy observations $(\hat{x}_i, \hat{y}_i)$ of (x_i, y_i) are given. From the noisy observations $(\hat{x}_i, \hat{y}_i)$, we must estimate the parameters of the unknown ellipse. To do this we employ the principle of minimizing the squared residuals. We are assuming the noise comes from an independent and identical distribution with mean 0 and variance σ^2. The general case, noise from non-identical distribution, will be mentioned at the end of this section. Let $\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}$, and $\hat{F}$ be the estimates of the unknown ellipse parameters. We want to minimize the squared residual errors

$$\varepsilon^2 = \sum_{i=1}^{I} (\hat{A}\hat{x}_i^2 + \hat{B}\hat{x}_i\hat{y}_i + \hat{C}\hat{y}_i^2 + \hat{D}\hat{x}_i + \hat{E}\hat{y}_i + \hat{F})^2 .$$

There have been many methods proposed to find least squares estimates of the parameters of the above model [1, 2, 4, 5, 10, 11], but their solutions are not invariant under translation and rotation since D, E and F are involved in their constraints which are functions of the origin of coordinates. Bookstein proposed a general method whose solution is invariant under the Euclidean group [3]. For any conic, the forms $A + C$ and $B^2 - 4AC$ are invariant under the Euclidean group. Since the only positive-definite invariant that can be formed from these quantities is $(A+C)^2 + (B^2 - 4AC)/2 = A^2 + B^2/2 + C^2$, Bookstein suggested $A^2 + B^2/2 + C^2 = 2$ as a constraint. Also if the center of the conic is at (p, q), there are two more linear constraints:

$$2Ap + Bq + D = 0 , \quad \text{and} \quad Bp + 2Cq + E = 0 .$$

We will employ Bookstein's method for ellipse fitting.
From now on, we will write x instead of $\hat{x}$ for notational convenience unless confusion would arise. The sum of squared residual errors ε^2 can be written as

$$\varepsilon^2 = \beta' \boldsymbol{S} \beta$$

where $\beta = (\,A \quad B \quad C \quad D \quad E \quad F\,)'$ and

$$\boldsymbol{S} = \begin{pmatrix} \sum_{i=1}^{I} x_i^4 & \sum_{i=1}^{I} x_i^3 y_i & \sum_{i=1}^{I} x_i^2 y_i^2 & \sum_{i=1}^{I} x_i^3 & \sum_{i=1}^{I} x_i^2 y_i & \sum_{i=1}^{I} x_i^2 \\ \sum_{i=1}^{I} x_i^3 y_i & \sum_{i=1}^{I} x_i^2 y_i^2 & \sum_{i=1}^{I} x_i y_i^3 & \sum_{i=1}^{I} x_i^2 y_i & \sum_{i=1}^{I} x_i y_i^2 & \sum_{i=1}^{I} x_i y_i \\ \sum_{i=1}^{I} x_i^2 y_i^2 & \sum_{i=1}^{I} x_i y_i^3 & \sum_{i=1}^{I} y_i^4 & \sum_{i=1}^{I} x_i y_i^2 & \sum_{i=1}^{I} y_i^3 & \sum_{i=1}^{I} y_i^2 \\ \sum_{i=1}^{I} x_i^3 & \sum_{i=1}^{I} x_i^2 y_i & \sum_{i=1}^{I} x_i y_i^2 & \sum_{i=1}^{I} x_i^2 & \sum_{i=1}^{I} x_i y_i & \sum_{i=1}^{I} x_i \\ \sum_{i=1}^{I} x_i^2 y_i & \sum_{i=1}^{I} x_i y_i^2 & \sum_{i=1}^{I} y_i^3 & \sum_{i=1}^{I} x_i y_i & \sum_{i=1}^{I} y_i^2 & \sum_{i=1}^{I} y_i \\ \sum_{i=1}^{I} x_i^2 & \sum_{i=1}^{I} x_i y_i & \sum_{i=1}^{I} y_i^2 & \sum_{i=1}^{I} x_i & \sum_{i=1}^{I} y_i & I \end{pmatrix} .$$

$\boldsymbol{S}$ is called the scatter matrix about 0 of the vector $(\,x_i^2 \quad x_iy_i \quad y_i^2 \quad x_i \quad y_i \quad 1\,)'$. Note that $\boldsymbol{S} = \boldsymbol{Q}'\boldsymbol{Q}$ where

$$\boldsymbol{Q} = \begin{pmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ & \vdots & & & \vdots & \\ & & & & & \\ x_I^2 & x_Iy_I & y_I^2 & x_I & y_I & 1 \end{pmatrix} .$$

Let $\boldsymbol{D}$ be the diagonal matrix $\mathbf{diag}\,(\,1 \quad \frac{1}{2} \quad 1 \quad 0 \quad 0 \quad 0\,)$. We want to find β which minimizes $\varepsilon^2 = \beta'\boldsymbol{S}\beta$ subject to $\beta'\boldsymbol{D}\beta = 2$. We partition β into two vectors β_1 and β_2, each of which is of length 3, and $\boldsymbol{S}$ into four 3 by 3 matrices;

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} \\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} \end{pmatrix} .$$

Then

$$\varepsilon^2 = \beta'\boldsymbol{S}\beta = \beta_1'\boldsymbol{S}_{11}\beta_1 + 2\beta_1'\boldsymbol{S}_{12}\beta_2 + \beta_2'\mathbf{S}_{22}\beta_2 .$$

We want to minimize ε^2 subject to $\beta_1'\boldsymbol{D}_1\beta_1 = 2$ where $\boldsymbol{D}_1 = \mathbf{diag}(1, \frac{1}{2}, 1)$. For any fixed β_1, ε^2 is minimal when

$$\frac{\partial \varepsilon^2}{\partial \beta_2} = 2\beta_1'\boldsymbol{S}_{12} + 2\beta_2'\boldsymbol{S}_{22} = 0,$$

which implies

$$\beta_2' = -\beta_1'\boldsymbol{S}_{12}\boldsymbol{S}_{22}^{-1} .$$

In this case, we have

$$\varepsilon^2 = \beta_1'(S_{11} - S_{12}S_{22}^{-1}S_{21})\beta_1 \ .$$

To minimize ε^2 subject to $\beta_1' D_1 \beta_1 = 2$, let λ be a Lagrangian multiplier for the constraint.

$$\varepsilon^2 = \beta_1'(S_{11} - S_{12}S_{22}^{-1}S_{21})\beta_1 - \lambda(\beta_1' D_1 \beta_1 - 2) \ .$$

By taking the derivative and setting it to 0, we have

$$(S_{11} - S_{12}S_{22}^{-1}S_{21})\beta_1 = \lambda D_1 \beta_1 \ .$$

Therefore, the β_1 which minimizes ε^2 subject to $\beta_1' D_1 \beta_1 = 2$ is the eigenvector corresponding to the smallest λ.

5.2 Expected values and variances of estimated parameters

To compute expected values and the variances of the estimated parameters of the ellipse, we will employ the same linearization principle as we did in circle fitting. For ellipses, we have

$$J_{1i} = \begin{pmatrix} 2Ax_i + By_i + D \\ Bx_i + 2Cy_i + E \end{pmatrix} \quad \text{and}$$

$$J_{2i} = \begin{pmatrix} x_i^2 \\ x_i y_i \\ y_i^2 \\ x_i \\ y_i \\ 1 \end{pmatrix} \ .$$

It follows that

$$E[\Delta\beta] = 0 \ , \quad \text{and}$$
$$E[\Delta\beta\Delta\beta'] = \sigma^2 KMK' \ ,$$

where

$$K = (Q'Q)^{-1}Q' \ ,$$

and M is a diagonal matrix whose i-th diagonal element is

$$(2Ax_i + By_i + D)^2 + (Bx_i + 2Cy_i + E)^2 \ .$$

In case the noise comes from a non-identical distribution, we simply define the entries of matrix S as a weighted sum and follow the exact path.

5.3 Expected values and variances of center of ellipse

The variance of the center of the ellipse can be computed using first order Taylor series expansion. If the center of the ellipse is (p, q), then we have

$$p = \frac{2CD - BE}{B^2 - 4AC} \ ,$$
$$q = \frac{2AE - BD}{B^2 - 4AC} \ .$$

Let the estimate of the coordinate of the center of the ellipse be

$$\hat{p} = \frac{2\hat{C}\hat{D} - \hat{B}\hat{E}}{\hat{B}^2 - 4\hat{A}\hat{C}} \ ,$$
$$\hat{q} = \frac{2\hat{A}\hat{E} - \hat{B}\hat{D}}{\hat{B}^2 - 4\hat{A}\hat{C}} \ .$$

Consider $\hat{p}$ and $\hat{q}$ as functions of five variables $\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}$. A first order Taylor expansion around (A, B, C, D, E) produces

$$\begin{aligned}
\hat{p} = p &+ (\hat{A} - A)\frac{4pC}{B^2 - 4AC} - (\hat{B} - B)\frac{E + 2pB}{B^2 - 4AC} \\
&+ (\hat{C} - C)\frac{2D + 4pA}{B^2 - 4AC} + (\hat{D} - D)\frac{2C}{B^2 - 4AC} \\
&- (\hat{E} - E)\frac{B}{B^2 - 4AC} \ , \\
\hat{q} = q &+ (\hat{A} - A)\frac{2E + 4qC}{B^2 - 4AC} - (\hat{B} - B)\frac{E + 2qB}{B^2 - 4AC} \\
&+ (\hat{C} - C)\frac{4qA}{B^2 - 4AC} - (\hat{D} - D)\frac{B}{B^2 - 4AC} \\
&+ (\hat{E} - E)\frac{2A}{B^2 - 4AC} \ .
\end{aligned}$$

Since $E[\Delta\beta] = 0$, we have $E[\hat{p}] = p$ and $E[\hat{q}] = q$. For the variances of $\hat{p}$ and $\hat{q}$, we have

$$\begin{aligned}
V(\hat{p}) &= E[(\hat{p} - p)^2] \\
&= \frac{1}{h^2}\left[a_p^2 V(\hat{A}) + b_p^2 V(\hat{B}) + c_p^2 V(\hat{C}) + d_p^2 V(\hat{D}) + e_p^2 V(\hat{E})\right. \\
&- 2a_p b_p \text{cov.}(\hat{A}, \hat{B}) + 2a_p c_p \text{cov.}(\hat{A}, \hat{C}) \\
&+ 2a_p d_p \text{cov.}(\hat{A}, \hat{D}) - 2a_p e_p \text{cov.}(\hat{A}, \hat{E}) - 2b_p c_p \text{cov.}(\hat{B}, \hat{C}) \\
&- 2b_p d_p \text{cov.}(\hat{B}, \hat{D}) + 2b_p e_p \text{cov.}(\hat{B}, \hat{E}) \\
&+ 2c_p d_p \text{cov.}(\hat{C}, \hat{D}) - 2c_p e_p \text{cov.}(\hat{C}, \hat{E}) \\
&\left. - 2d_p e_p \text{cov.}(\hat{D}, \hat{E})\right] \ , \\
V(\hat{q}) &= E[(\hat{q} - q)^2] \\
&= \frac{1}{h^2}\left[a_q^2 V(\hat{A}) + b_q^2 V(\hat{B}) + c_q^2 V(\hat{C}) + d_q^2 V(\hat{D}) + e_q^2 V(\hat{E})\right. \\
&- 2a_q b_q \text{cov.}(\hat{A}, \hat{B}) + 2a_q c_q \text{cov.}(\hat{A}, \hat{C}) \\
&- 2a_q d_q \text{cov.}(\hat{A}, \hat{D}) + 2a_q e_q \text{cov.}(\hat{A}, \hat{E}) \\
&- 2b_q c_q \text{cov.}(\hat{B}, \hat{C}) + 2b_q d_q \text{cov.}(\hat{B}, \hat{D}) \\
&- 2b_q e_q \text{cov.}(\hat{B}, \hat{E}) - 2c_q d_q \text{cov.}(\hat{C}, \hat{D}) \\
&\left. + 2c_q e_q \text{cov.}(\hat{C}, \hat{E}) - 2d_q e_q \text{cov.}(\hat{D}, \hat{E})\right] , \\
\text{cov.}(\hat{p}, \hat{q}) &= E[(\hat{p} - p)(\hat{q} - q)] \\
&= \frac{1}{h^2}\left[a_p a_q V(\hat{A}) + b_p b_q V(\hat{B}) + c_p c_q V(\hat{C}) - d_p d_q V(\hat{D})\right. \\
&- e_p e_q V(\hat{E}) - (a_p b_q + a_q b_p)\text{cov.}(\hat{A}, \hat{B}) \\
&+ (a_p c_q + a_q c_p)\text{cov.}(\hat{A}, \hat{C}) \\
&+ (a_q d_p - a_p d_q)\text{cov.}(\hat{A}, \hat{D}) + (a_p e_q - a_q e_p)\text{cov.}(\hat{A}, \hat{E}) \\
&- (b_p c_q + b_q c_p)\text{cov.}(\hat{B}, \hat{C}) + (b_p d_q - b_q d_p)\text{cov.}(\hat{B}, \hat{D}) \\
&+ (b_q e_p - b_p e_q)\text{cov.}(\hat{B}, \hat{E}) + (c_q d_p - c_p d_q)\text{cov.}(\hat{C}, \hat{D}) \\
&\left. + (c_p e_q - c_q e_p)\text{cov.}(\hat{C}, \hat{E}) + (d_p e_q + d_q e_p)\text{cov.}(\hat{D}, \hat{E})\right] ,
\end{aligned}$$

where

$$\begin{aligned} a_p &= 4pC\,, & a_q &= 2E+4qC\,,\\ b_p &= E+2pB\,, & b_q &= E+2qB\,,\\ c_p &= 2D+4pA\,, & c_q &= 4qA\,,\\ d_p &= 2C\,, & d_q &= B\,,\\ e_p &= B\,, & e_q &= 2A\,,\\ h &= B^2-4AC\,. \end{aligned}$$

6 Variances of measurements of vision tasks

We have discussed how the edge position uncertainties propagate to the parameters of the lines, circles, or ellipses. We derived formulas for the variances and covariances of the estimates for each coordinate of the corner point, which is an intersection of two straight lines. We also derived formulas for the variances and covariances of the estimates for each coordinate of the centers of circles and ellipses. Now we want to determine the variances of the estimate for the final measurements. We will discuss two cases: (1) distance and (2) angle.

6.1 Distance between two points

Let (r_1, c_1) and (r_2, c_2) be the coordinates of the two points between which we measure the distance. These two points may be either a corner point or the center of a circle or an ellipse. The real distance between these two points is

$$d = \sqrt{(r_1-r_2)^2+(c_1-c_2)^2}\,.$$

Let $(\hat{r}_1, \hat{c}_1)$ and $(\hat{r}_2, \hat{c}_2)$ be the estimates of (r_1, c_1) and (r_2, c_2). Then the estimate of the distance d is

$$\hat{d} = \sqrt{(\hat{r}_1-\hat{r}_2)^2+(\hat{c}_1-\hat{c}_2)^2}\,.$$

Again we proceed with a first-order Taylor expansion around (r_1, c_1, r_2, c_2).

$$\begin{aligned} \hat{d} &= d + \frac{(\hat{r}_1-r_1)(r_1-r_2)}{d} + \frac{(\hat{c}_1-c_1)(c_1-c_2)}{d}\\ &\quad - \frac{(\hat{r}_2-r_2)(r_1-r_2)}{d} - \frac{(\hat{c}_2-c_2)(c_1-c_2)}{d}\\ &= d + \frac{1}{d}\big[(r_1-r_2)[(\hat{r}_1-r_1)-(\hat{r}_2-r_2)]\\ &\quad + (c_1-c_2)[(\hat{c}_1-c_1)-(\hat{c}_2-c_2)]\big]\\ &= \frac{1}{d}[(r_1-r_2)(\hat{r}_1-\hat{r}_2)+(c_1-c_2)(\hat{c}_1-\hat{c}_2)]\,. \end{aligned}$$

Hence the variance of the estimate of the distance d is

$$\begin{aligned} V(\hat{d}) &= E[(\hat{d}-d)^2]\\ &= \frac{1}{d^2}E\big[\big((r_1-r_2)(\hat{r}_1-r_2)+(c_1-c_2)(\hat{c}_1-\hat{c}_2)\big)^2\big]\\ &\quad - 2E\big[(r_1-r_2)(\hat{r}_1-r_2)+(c_1-c_2)(\hat{c}_1-\hat{c}_2)\big] + d^2\\ &= \frac{1}{d^2}\big[(r_1-r_2)^2E[(\hat{r}_1-\hat{r}_2)^2]+(c_1-c_2)^2E[(\hat{c}_1-\hat{c}_2)^2]\\ &\quad + 2(r_1-r_2)(c_1-c_2)E[(\hat{r}_1-\hat{r}_2)(\hat{c}_1-\hat{c}_2)]\big] - d^2\,. \end{aligned}$$

Using the fact that the two points are independent, *i.e.* (cov.$(\hat{r}_1,\hat{r}_2)$ = cov.$(\hat{c}_1,\hat{c}_2) = 0$), we have

$$\begin{aligned} &E[(\hat{r}_1-\hat{r}_2)^2] = V(\hat{r}_1)+V(\hat{r}_2)+(r_1-r_2)^2,\\ &E[(\hat{c}_1-\hat{c}_2)^2] = V(\hat{c}_1)+V(\hat{c}_2)+(c_1-c_2)^2,\\ &\text{and cov.}(\hat{r}_i,\hat{c}_i) = E[\hat{r}_i\hat{c}_i]-r_ic_i,\quad i=1,2\,. \end{aligned}$$

Therefore, after some algebra, we have

$$\begin{aligned} V(\hat{d}) &= \frac{1}{d^2}\big[(r_1-r_2)^2[V(\hat{r}_1)+V(\hat{r}_2)]+(c_1-c_2)^2\\ &\quad \times [V(\hat{c}_1)+V(\hat{c}_2)] + 2(r_1-r_2)(c_1-c_2)\\ &\quad \times [\text{cov.}(\hat{r}_1,\hat{c}_1)+\text{cov.}(\hat{r}_2,\hat{c}_2)]\big]\,. \end{aligned} \tag{18}$$

6.2 Angle

In this subsection, we will analyze the variance of the angle measurements. In a 2D image, two straight lines make an angle. Also three different points that do not lie on the same straight line make an angle. The angle that two straight lines make can be measured by finding the angle that each line makes with the x-axis and subtracting the smaller angle from the bigger one. We will discuss the variances of the angle that a line makes with the x-axis first and then the variance of the angular difference.

6.2.1 Direction cosine

Suppose a line whose equation is

$$\alpha x + \beta y + \gamma = 0,\quad \alpha^2+\beta^2 = 1\,,$$

is given. This line equation coincides with the line fitting model defined in Sect. 3, and α and β are direction cosines to the x- and y-axes, respectively. Let θ be the angle that the given line makes with the x-axis. Then it is not hard to see that the expression for $E\left[(\hat{\alpha}-\alpha)^2\right]$ and $E\left[(\hat{\beta}-\beta)^2\right]$ can be used to generate an expression for the variance for the angle $\hat{\theta}$ defined by $\cos\hat{\theta} = \hat{\alpha}, \sin\hat{\theta} = \hat{\beta}$.

A first order expansion of $\cos\hat{\theta}$ around θ, where $\cos\theta = \alpha$, and a first order expansion of $\sin\hat{\theta}$ around θ, where $\sin\theta = \beta$ gives

$$\begin{aligned} \cos\hat{\theta} &= \cos\theta + (\hat{\theta}-\theta)(-\sin\theta)\,,\\ \sin\hat{\theta} &= \sin\theta + (\hat{\theta}-\theta)\cos\theta\,. \end{aligned}$$

Using this approximation, we have

$$\begin{aligned} &E\left[(\cos\hat{\theta}-\cos\theta)^2\right] + E\left[(\sin\hat{\theta}-\sin\theta)^2\right]\\ &\quad = E\left[(\hat{\theta}-\theta)^2\right](\sin^2\theta+\cos^2\theta) = E\left[(\hat{\theta}-\theta)^2\right]\,. \end{aligned}$$

Since we have

$$\begin{aligned} &E\left[(\cos\hat{\theta}-\cos\theta)^2\right] + E\left[(\sin\hat{\theta}-\sin\theta)^2\right]\\ &\quad = E\left[(\hat{\alpha}-\alpha)^2\right] + E\left[(\hat{\beta}-\beta)^2\right]\,, \end{aligned}$$

upon substituting, we get

$$E\left[(\hat{\theta}-\theta)^2\right] = E\left[(\hat{\alpha}-\alpha)^2 + (\hat{\beta}-\beta)^2\right]$$
$$= V(\hat{\alpha}) + V(\hat{\beta}) \,.$$

Therefore, the variance of the angle that the given line makes with the x-axis is the sum of the variances of the direction cosines of the line.

6.2.2 Angle between two straight lines

Suppose that two straight lines L_1 and L_2 are given,

$$\alpha_1 x + \beta_1 y + \gamma_1 = 0 \quad (L_1)$$
$$\alpha_2 x + \beta_2 y + \gamma_2 = 0 \quad (L_2)$$

and we are to measure the angle between them. Let the angle that L_1 makes with the x-axis be θ_1 and the angle that L_2 makes with the x-axis be θ_2. Then the angle we want to measure is $\theta = \theta_1 - \theta_2$. Let the estimates of θ_1 and θ_2 be $\hat{\theta}_1$ and $\hat{\theta}_2$, respectively, and let $\hat{\theta}$ be the estimate of θ. Since $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent, the variance of $\hat{\theta}$ is

$$\begin{aligned} V(\hat{\theta}) &= E[(\hat{\theta}-\theta)^2] \\ &= E\left[[(\hat{\theta}_1-\hat{\theta}_2)-(\theta_1-\theta_2)]^2\right] \\ &= E\left[(\hat{\theta}_1-\theta_1)^2\right] + E\left[(\hat{\theta}_2-\theta_2)^2\right] \\ &= V(\hat{\theta}_1)+V(\hat{\theta}_2) \\ &= V(\hat{\alpha}_1)+V(\hat{\beta}_1)+V(\hat{\alpha}_2)+V(\hat{\beta}_2) \,. \end{aligned} \tag{19}$$

If we have already estimated parameters of the two given lines, there is no problem since we know that

$$V(\hat{\alpha}) + V(\hat{\beta}) = \frac{\sigma_x^2 + \sigma_y^2 + \frac{I}{W}}{\tilde{W}(\sigma_x^2+\sigma_y^2)^2} \,.$$

When $\frac{I}{W} \ll \sigma_x^2 + \sigma_y^2$, we have

$$E\left[(\hat{\theta}-\theta)^2\right] \le \frac{1}{\tilde{W}(\sigma_x^2+\sigma_y^2)} \,.$$

However, if we have only the coordinates of three points, then we must estimate the parameters of the line that connects two points.

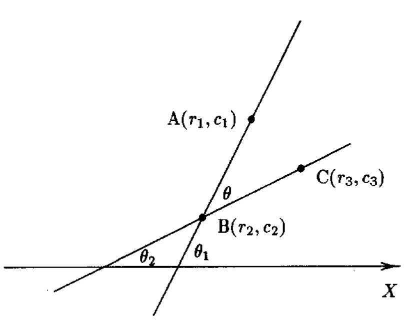

Fig. 2. An angle made by three points

6.2.3 Angle made by three points

Suppose we are given three points A, B, and C, and want to measure the angle $\angle ABC$(see Fig. 2). To determine the variance of the angle $\angle ABC$, we need to know the variances of parameters of two lines passing through both A and B, and both B and C, respectively. Since the line equations are not available, we need to estimate the line equations from the given points.

Suppose we have two points, (r_1, c_1) and (r_2, c_2). Then the line that passes through these two points is

$$(c_2 - c_1)x - (r_2 - r_1)y + (r_2 c_1 - r_1 c_2) = 0 \,.$$

Since $\alpha^2 + \beta^2 = 1$, we have

$$\alpha = \frac{c_2 - c_1}{d} \,,$$
$$\beta = -\frac{r_2 - r_1}{d} \,,$$
$$\gamma = \frac{r_2 c_1 - r_1 c_2}{d} \,,$$

where

$$d = \sqrt{(r_1 - r_2)^2 + (c_1 - c_2)^2} \,.$$

Let the estimates of α and β be

$$\hat{\alpha} = \frac{\hat{c}_2 - \hat{c}_1}{\hat{d}} \,,$$
$$\hat{\beta} = -\frac{\hat{r}_2 - \hat{r}_1}{\hat{d}} \,,$$

where

$$\hat{d} = \sqrt{(\hat{r}_1 - \hat{r}_2)^2 + (\hat{c}_1 - \hat{c}_2)^2} \,.$$

Using the first order approximation, we have

$$\begin{aligned} \hat{\alpha} &= \alpha + \frac{(r_2-r_1)(c_2-c_1)}{d^2}[(\hat{r}_1 - r_1) - (\hat{r}_2 - r_2)] \\ &\quad - \frac{(r_2-r_1)^2}{d^2}[(\hat{c}_1 - c_1) - (\hat{c}_2 - c_2)] \,. \\ \hat{\beta} &= \beta + \frac{(c_2-c_1)^2}{d^2}[(\hat{r}_1 - r_1) - (\hat{r}_2 - r_2)] \\ &\quad - \frac{(r_2-r_1)(c_2-c_1)}{d^2}[(\hat{c}_1 - c_1) - (\hat{c}_2 - c_2)] \,. \end{aligned}$$

Using the fact that the two points are independent, the variance of $\hat{\alpha}$ is

$$V(\hat{\alpha}) = E[(\hat{\alpha} - \alpha)^2] = \frac{(r_2 - r_1)^2(c_2 - c_1)^2}{d^4}$$
$$\times E\left[(\hat{r}_1 - r_1)^2 - 2(\hat{r}_1 - r_1)(\hat{r}_2 - r_2) + (\hat{r}_2 - r_2)^2\right]$$
$$+ \frac{(r_2 - r_1)^4}{d^4} E\left[(\hat{c}_1 - c_1)^2 - 2(\hat{c}_1 - c_1)(\hat{c}_2 - c_2)\right.$$
$$\left.+(\hat{c}_2 - c_2)^2\right] - \frac{2(r_2 - r_1)^3(c_2 - c_1)}{d^4}$$
$$\times E\left[(\hat{r}_1 - r_1)(\hat{c}_1 - c_1) + (\hat{r}_2 - r_2)(\hat{c}_2 - c_2)\right.$$
$$\left.- (\hat{r}_1 - r_1)(\hat{c}_2 - c_2) - (\hat{r}_2 - r_2)(\hat{c}_1 - c_1)\right]$$
$$= \frac{(r_2 - r_1)^2(c_2 - c_1)^2}{d^4}[V(\hat{r}_1) + V(\hat{r}_2)]$$
$$+ \frac{(r_2 - r_1)^4}{d^4}[V(\hat{c}_1) + V(\hat{c}_2)]$$
$$- \frac{2(r_2 - r_1)^3(c_2 - c_1)}{d^4}[\text{cov.}(\hat{r}_1, \hat{c}_1) + \text{cov.}(\hat{r}_2, \hat{c}_2)] .$$

Similarly, the variance of $\hat{\beta}$ is

$$V(\hat{\beta}) = E[(\hat{\beta} - \beta)^2]$$
$$= \frac{(c_2 - c_1)^4}{d^4}[V(\hat{r}_1) + V(\hat{r}_2)]$$
$$+ \frac{(r_2 - r_1)^2(c_2 - c_1)^2}{d^4}[V(\hat{c}_1) + V(\hat{c}_2)]$$
$$- \frac{2(r_2 - r_1)(c_2 - c_1)^3}{d^4}[\text{cov.}(\hat{r}_1, \hat{c}_1) + \text{cov.}(\hat{r}_2, \hat{c}_2)] .$$

Therefore, the variance of the angle that the line makes with the x-axis is

$$V(\hat{\theta}) = V(\hat{\alpha}) + V(\hat{\beta})$$
$$= \frac{(c_2 - c_1)^2}{d^2}[V(\hat{r}_1) + V(\hat{r}_2)] + \frac{(r_2 - r_1)^2}{d^2}$$
$$\times [V(\hat{c}_1) + V(\hat{c}_2)] - \frac{2(r_2 - r_1)(c_2 - c_1)}{d^2}$$
$$\times [\text{cov.}(\hat{r}_1, \hat{c}_1) + \text{cov.}(\hat{r}_2, \hat{c}_2)] . \quad (20)$$

Now we turn to the question of determining the variance of the angle $\angle ABC$. Let the coordinates of A, B, C be $(r_1, c_1), (r_2, c_2)$ and (r_3, c_3), respectively and the angle $\angle ABC = \theta$. Let the angle each line makes with the x-axis be θ_1 and θ_2, respectively(see Fig. 2). Then, using Eq. (20), simple algebra produces

$$V(\hat{\theta}) = V(\hat{\theta}_1) + V(\hat{\theta}_2)$$
$$= \frac{(c_2 - c_1)^2}{d_1^2}[V(\hat{r}_1) + V(\hat{r}_2)] + \frac{(r_2 - r_1)(c_2 - c_1)}{d_1^2}$$
$$\times [V(\hat{c}_1) + V(\hat{c}_2)] - \frac{2(r_2 - r_1)(c_2 - c_1)}{d_1^2}$$
$$\times [\text{cov.}(\hat{r}_1, \hat{c}_1) + \text{cov.}(\hat{r}_2, \hat{c}_2)] + \frac{(c_2 - c_3)^2}{d_2^2}$$
$$\times [V(\hat{r}_3) + V(\hat{r}_2)] + \frac{(r_2 - r_3)(c_2 - c_3)}{d_2^2}$$
$$\times [V(\hat{c}_3) + V(\hat{c}_2)] - \frac{2(r_2 - r_3)(c_2 - c_3)}{d_2^2}$$
$$\times [\text{cov.}(\hat{r}_3, \hat{c}_3) + \text{cov.}(\hat{r}_2, \hat{c}_2)]$$

where

$$d_1 = \sqrt{(r_1 - r_2)^2 + (c_1 - c_2)^2} ,$$
$$d_2 = \sqrt{(r_3 - r_2)^2 + (c_3 - c_2)^2} .$$

7 Summary of the derivations

We summarize all the results in the preceding sections here.

7.1 Variances of least squares estimates of line parameters

$$V(\hat{\alpha}) = \beta^2 T ,$$

$$V(\hat{\beta}) = \alpha^2 T ,$$

$$V(\hat{\gamma}) = \left(\mu_x^2 + \mu_y^2 - \gamma^2 + \frac{I}{W}\right) T + \frac{I}{W} ,$$

$$\text{cov.}(\hat{\alpha}, \hat{\beta}) = -\alpha\beta T ,$$

$$\text{cov.}(\hat{\alpha}, \hat{\gamma}) = \beta(-\mu_x\beta + \mu_y\alpha)T ,$$

$$\text{cov.}(\hat{\beta}, \hat{\gamma}) = \alpha(\mu_x\beta - \mu_y\alpha)T ,$$

$$T = \frac{\sigma_x^2 + \sigma_y^2 + \frac{I}{W}}{\tilde{W}(\sigma_x^2 + \sigma_y^2)^2} .$$

7.2 Variance of corner point position

$$V(\hat{r}) = \frac{1}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2}$$
$$\times \left[\beta_2^2 E[(\hat{\alpha}_1 r + \hat{\beta}_1 c + \hat{\gamma}_1)^2] + \beta_1^2 E[(\hat{\alpha}_2 r + \hat{\beta}_2 c + \hat{\gamma}_2)^2]\right] ,$$

$$V(\hat{c}) = \frac{1}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2}$$
$$\times \left[\alpha_2^2 E[(\hat{\alpha}_1 r + \hat{\beta}_1 c + \hat{\gamma}_1)^2] + \alpha_1^2 E[(\hat{\alpha}_2 r + \hat{\beta}_2 c + \hat{\gamma}_2)^2]\right] ,$$

$$\text{cov.}(\hat{r}, \hat{c}) = \frac{1}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2}\left[-\alpha_2\beta_2 E[(\hat{\alpha}_1 r + \hat{\beta}_1 c + \hat{\gamma}_1)^2]\right.$$
$$\left.- \alpha_1\beta_1 E[(\hat{\alpha}_2 r + \hat{\beta}_2 c + \hat{\gamma}_2)^2]\right] ,$$

where

$$E[(\hat{\alpha}r + \hat{\beta}c + \hat{\gamma})^2] = r^2 V(\hat{\alpha}) + c^2 V(\hat{\beta}) + V(\hat{\gamma})$$
$$+ 2rc\text{cov.}(\hat{\alpha}, \hat{\beta}) + 2r\text{cov.}(\hat{\alpha}, \hat{\gamma}) + 2c\text{cov.}(\hat{\beta}, \hat{\gamma}) .$$

7.3 Variance of circle center position

$$E[\Delta\beta\Delta\beta'] = 4(r + p^2 + q^2)\begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{pmatrix},$$

where

$$K_{11} = \sum_{i=1}^{I} w_i(2Ax_i + 2By_i + C)^2 ,$$

$$K_{12} = \sum_{i=1}^{I} w_i(2Ax_i + 2By_i + C)(2Bx_i + 2Dy_i + E) ,$$

$$K_{13} = \sum_{i=1}^{I} w_i(2Ax_i + 2By_i + C)(2Cx_i + 2Ey_i + F) ,$$

$$K_{22} = \sum_{i=1}^{I} w_i(2Bx_i + 2Dy_i + E)^2 ,$$

$$K_{23} = \sum_{i=1}^{I} w_i(2Bx_i + 2Dy_i + E)(2Cx_i + 2Ey_i + F) ,$$

$$K_{33} = \sum_{i=1}^{I} w_i(2Cx_i + 2Ey_i + F)^2 .$$

7.4 Variance of distance between two points

$$\begin{aligned} V(\hat{d}) = \frac{1}{d^2}\big[&(r_1 - r_2)^2[V(\hat{r}_1) + V(\hat{r}_2)] + (c_1 - c_2)^2 \\ &\times [V(\hat{c}_1) + V(\hat{c}_2)] + 2(r_1 - r_2)(c_1 - c_2) \\ &\times [\text{cov.}(\hat{r}_1, \hat{c}_1) + \text{cov.}(\hat{r}_2, \hat{c}_2)]\big] . \end{aligned}$$

7.5 Variance of angle between two straight lines

$$V(\hat{\theta}) = V(\hat{\alpha}_1) + V(\hat{\beta}_1) + V(\hat{\alpha}_2) + V(\hat{\beta}_2) .$$

7.6 Variance of angle made by three points

$$\begin{aligned} V(\hat{\theta}) &= V(\hat{\theta}_1) + V(\hat{\theta}_2) \\ &= \frac{(c_2 - c_1)^2}{d_1^2}[V(\hat{r}_1) + V(\hat{r}_2)] + \frac{(r_2 - r_1)(c_2 - c_1)}{d_1^2} \\ &\quad \times [V(\hat{c}_1) + V(\hat{c}_2)] - \frac{2(r_2 - r_1)(c_2 - c_1)}{d_1^2} \\ &\quad \times [\text{cov.}(\hat{r}_1, \hat{c}_1) + \text{cov.}(\hat{r}_2, \hat{c}_2)] + \frac{(c_2 - c_3)^2}{d_2^2} \\ &\quad \times [V(\hat{r}_3) + V(\hat{r}_2)] + \frac{(r_2 - r_3)(c_2 - c_3)}{d_2^2} \\ &\quad \times [V(\hat{c}_3) + V(\hat{c}_2)] - \frac{2(r_2 - r_3)(c_2 - c_3)}{d_2^2} \\ &\quad \times [\text{cov.}(\hat{r}_3, \hat{c}_3) + \text{cov.}(\hat{r}_2, \hat{c}_2)] , \end{aligned}$$

where

$$d_1 = \sqrt{(r_1 - r_2)^2 + (c_1 - c_2)^2} ,$$
$$d_2 = \sqrt{(r_3 - r_2)^2 + (c_3 - c_2)^2} .$$

8 Experiments and results

In our derivation of expected variance of measurement, we used an approximation employing a first-order Taylor expansion to linearize the equations. In this section, we will demonstrate that the formula for the variance of the measurement derived using the first-order Taylor approximation is good enough for practical use, at least in a statistical sense. We will briefly discuss the statistical test method first, and then we will describe our experimental protocol followed by a discussion of the results.

8.1 Statistical test

The test we are conducting concerns whether the variances of the estimates of line parameters and/or the center of a circle or an ellipse have the form of the formulas presented in this paper. We describe the test involving line parameter $\hat{\alpha}$; other tests are similar.

Suppose we want to test if $\sigma_{\hat{\alpha}}^2$, which is the variance of the random variable $\hat{\alpha}$, is equal to $V(\hat{\alpha})$. Then the null hypothesis and alternate hypothesis would be

$$\text{H}_0 : \sigma_{\hat{\alpha}}^2 = V(\hat{\alpha}) ,$$

$$\text{H}_1 : \sigma_{\hat{\alpha}}^2 \neq V(\hat{\alpha}) ,$$

and the test statistic is

$$\text{TS} = \frac{\sigma_{\hat{\alpha}}^2 - V(\hat{\alpha})}{\sigma_{\sigma_{\hat{\alpha}}^2}} .$$

Since we do not know the distribution of $\sigma_{\hat{\alpha}}^2$, we need to estimate the mean and variance of $\sigma_{\hat{\alpha}}^2$. We use the sample mean and variance of the statistic $\hat{V}(\hat{\alpha})$ for the estimates of the mean and variance of $\sigma_{\hat{\alpha}}^2$. Let the mean and variance of samples of $\hat{V}(\hat{\alpha})$ be $\bar{\hat{V}}$ and s^2 respectively. Then the test statistic would be

$$\text{TS} = \frac{\bar{\hat{V}} - V(\hat{\alpha})}{s/\sqrt{n}} \tag{21}$$

where n is the sample size. Although TS possesses a t distribution only if the sample is selected from a normal distribution, the t distribution provides a reasonable approximation to the distribution of TS [9]. As n increases, TS asymptotically approaches the standard normal distribution, $N(0, 1)$ by the central limit theorem [8]. Therefore we cannot reject the null hypothesis if $\text{TS}_c \leq Z_{\alpha/2}$ with level of significance α, where TS_c is the computed test statistic, and $Z_{\alpha/2}$ is the standard normal distribution table entry for $\alpha/2$.

8.2 Experimental protocol

This subsection describes how the samples for the estimate of variance of the random variables are generated and how the test statistic is computed. We present only the procedure for line parameters, since the others are similar. For simplicity, we assume that the random noise comes from an independent and identical distribution with mean 0 and variance σ^2. In this case, we have the following identity:

$$\frac{I}{W} = \sigma^2 \ .$$

Suppose a line $\alpha x + \beta y + \gamma = 0$ is given. Since $V(\hat{\alpha})$ is a function of five independent variables, $\alpha, \beta, \gamma, \sigma_x^2 + \sigma_y^2$, and σ^2, we will vary the values of one variable with the others fixed. Since $\alpha^2 + \beta^2 = 1$, α and β are on the unit circle, and can be controlled by an angle. If the angle made by the vector (α, β) with the x-axis is θ, then we have $\alpha = \cos\theta$ and $\beta = \sin\theta$. We define

$$\Sigma^2 = \{0.5, 1.0, 2.0, 3.0\} \ ,$$
$$\Sigma_{xy}^2 = \{50, 100, 500, 1000\} \ ,$$
$$\Gamma = \{-125, -25, -5, -1, 0, 1, 5, 25, 125\} \ ,$$
$$\Theta = \{0°, 30°, 60°, 90°, 120°, 150°, 180°, 210°, 240°, 270°, 300°, 330°\} \ .$$

The top-level procedure is shown in Fig. 3, and the procedure that computes the test statistic is shown in Fig. 4.
In summary, we use $I = M = N = 100$. Therefore, for each combination of $(\sigma^2, \sigma_x^2 + \sigma_y^2, \gamma, \theta)$,

1. A set of 100 true points on the given line is generated.
2. For this set of points, 100 sets of 100 noisy points are constructed.
3. For each set of noisy points, the least squares estimates of line parameters are determined.
4. For these 100 sets of noisy points, the sample means and variances are computed.
5. By repeating step 2 to step 3 100 times to reduce the estimation error, 100 samples for the variance are obtained.
6. From the 100 samples of the variance, the sample mean and variance of the estimate of the line parameter variance are computed.

Therefore, $4\times4\times9\times12\times100\times100 = 17{,}280{,}000$ experiments are performed.

for each $\sigma^2 \in \Sigma^2$ **do**
  for each $\sigma_x^2 + \sigma_y^2 \in \Sigma_{xy}^2$ **do**
    for each $\gamma \in \Gamma$ **do**
      for each $\theta \in \Theta$ **do**
        compute TS
      end
    end
  end
end

Fig. 3. The main control procedure for the statistical test in which line parameter α is involved

step 1: (Generate exact line)
  generate I (x_i, y_i)'s such that $\sigma_x^2 + \sigma_y^2$ is a given constant.
step 2: (Generate estimated lines and compute sample mean and variance of the variance of the least squares estimate of the line parameters)
  for $j = 1, M$ **do**
    for $k = 1, N$ **do**
      add random gaussian noise σ^2 to the (x_i, y_i)'s generated in **step 1**.
      compute the least square estimate of the line parameters, $\hat{\alpha}_k, \hat{\beta}_k, \hat{\gamma}_k$.
    end
    compute the mean of the estimate of the variance of the least squares estimate of the line parameters, for instance,
    $$\bar{\hat{\alpha}}_j = \frac{1}{N}\sum_{k=1}^{N}\alpha_j \text{ and } \hat{V}_j(\hat{\alpha}) = \frac{1}{N-1}\sum_{k=1}^{N}(\hat{\alpha}_k - \bar{\hat{\alpha}}_j)^2.$$
  end
  compute mean and variance of the estimate of the variance of the least squares estimate of the line parameters, for instance,
  $$\bar{\hat{V}}(\hat{\alpha}) = \frac{1}{M}\sum_{j=1}^{M}\hat{V}_j(\hat{\alpha}), \text{ and } s_{\bar{\hat{V}}(\hat{\alpha})}^2 = \frac{1}{M-1}\sum_{j=1}^{M}(\hat{V}_j(\hat{\alpha}) - \bar{\hat{V}}(\hat{\alpha}))^2.$$

Fig. 4. The subprocedure that computes the test statistic for the test in which line parameter α is involved

8.3 Implementation

We implemented the experimental protocol described in the previous subsection using the C programming language on a SUN4 computer. Details of the implementation are discussed in this subsection.

8.3.1 Generation of normal random variates

There are several algorithms known for generating pseudo random normal variates. The method we use here is a modified Box-Mueller algorithm [6, 12]. The algorithm consists of the following steps:

1. generate two uniform random numbers, U_1 and U_2,
2. let $V_1 = 2 * U_1 - 1$, and $V_2 = 2 * U_2 - 1$,
3. let $S = V_1^2 + V_2^2$,
4. if $S >= 1$, go back to step 1,
5. otherwise, let $T = \sqrt{-2 * log(S)/S}$,
6. let $X_1 = V_1 * T * \sigma + \mu$, and $X_2 = V_2 * T * \sigma + \mu$.

X_1 and X_2 are two random variates from the normal distribution $N(\mu, \sigma^2)$. We used the pseudo random number generator in the standard C library to generate uniform random numbers.

8.3.2 Generation of points lying exactly on the given line

Suppose a straight line $\alpha x + \beta y + \gamma = 0$ is given, and the condition $\alpha^2 + \beta^2 = 1$ holds. The geometric meaning of this straight line is illustrated in Fig. 5. It is perpendicular to the unit vector $\boldsymbol{u} = (\alpha, \beta)$ and $-\gamma$ away from the origin. We want to generate I points such that the sum of variances of the x and y coordinates of those points are equal to some given constant, say T. Let Z be a random variable from a uniform distribution with mean 0 and variance T^2. Then we have

$$E[Z] = 0 \ , \quad E[Z^2] = T^2 \ .$$

Let z_i be an instance of Z. Let (x_0, y_0) be the coordinate of the closest point on the given line from the origin in the direction of

Table 1. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 0.5$ and $\sigma_x^2 + \sigma_y^2 = 50.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	2.56	0.04	0.00	0.09	0.14	0.04	2.36	0.19	0.00	0.03	0.14	0.01
25	2.64	0.02	0.04	0.07	0.03	0.20	2.70	0.19	0.05	0.06	0.06	0.11
5	2.89	0.04	0.03	0.05	0.05	0.01	2.88	0.06	0.11	0.01	0.02	0.02
1	2.21	0.04	0.03	0.02	0.10	0.02	3.11	0.09	0.00	0.05	0.11	0.08
0	2.56	0.02	0.05	0.26	0.15	0.07	3.04	0.05	0.11	0.02	0.05	0.04
−1	2.61	0.11	0.09	0.04	0.12	0.22	2.53	0.03	0.06	0.04	0.26	0.24
−5	2.47	0.10	0.06	0.06	0.25	0.02	2.81	0.04	0.09	0.10	0.17	0.04
−25	2.44	0.13	0.03	0.03	0.11	0.03	2.67	0.16	0.16	0.13	0.09	0.07
−125	2.53	0.01	0.05	0.10	0.03	0.07	2.67	0.11	0.05	0.00	0.07	0.01

Table 2. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 0.5$ and $\sigma_x^2 + \sigma_y^2 = 100.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	2.78	0.07	0.02	0.01	0.10	0.08	2.91	0.04	0.10	0.06	0.05	0.12
25	3.04	0.04	0.07	0.08	0.12	0.06	3.09	0.03	0.03	0.03	0.06	0.12
5	2.81	0.15	0.06	0.00	0.07	0.06	2.80	0.09	0.16	0.05	0.10	0.07
1	2.93	0.09	0.21	0.06	0.02	0.11	2.93	0.05	0.03	0.13	0.24	0.05
0	2.42	0.13	0.01	0.05	0.12	0.04	2.50	0.02	0.11	0.03	0.09	0.01
−1	2.71	0.17	0.10	0.12	0.12	0.04	2.71	0.02	0.10	0.01	0.06	0.02
−5	2.74	0.04	0.04	0.11	0.03	0.08	2.52	0.03	0.22	0.01	0.10	0.08
−25	2.45	0.01	0.05	0.17	0.12	0.01	2.75	0.07	0.10	0.14	0.09	0.06
−125	2.74	0.13	0.03	0.04	0.16	0.11	2.77	0.01	0.09	0.05	0.05	0.03

Table 3. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 0.5$ and $\sigma_x^2 + \sigma_y^2 = 500.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	2.70	0.04	0.23	0.15	0.27	0.03	2.70	0.21	0.07	0.06	0.06	0.01
25	3.23	0.11	0.08	0.08	0.14	0.02	2.57	0.16	0.00	0.06	0.08	0.12
5	2.74	0.13	0.03	0.06	0.11	0.04	3.05	0.04	0.13	0.02	0.10	0.00
1	2.76	0.13	0.17	0.13	0.00	0.12	2.92	0.01	0.13	0.01	0.11	0.02
0	2.66	0.17	0.13	0.10	0.03	0.08	2.56	0.09	0.12	0.00	0.13	0.04
−1	2.73	0.10	0.04	0.18	0.01	0.04	3.20	0.07	0.06	0.02	0.01	0.12
−5	2.44	0.03	0.00	0.02	0.12	0.10	2.49	0.09	0.12	0.02	0.13	0.13
−25	2.82	0.09	0.12	0.12	0.14	0.04	2.83	0.03	0.01	0.03	0.09	0.03
−125	3.33	0.12	0.14	0.10	0.19	0.16	2.77	0.07	0.06	0.14	0.13	0.09

Table 4. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 0.5$ and $\sigma_x^2 + \sigma_y^2 = 1000.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	2.09	0.01	0.02	0.06	0.08	0.05	2.71	0.15	0.02	0.04	0.01	0.09
25	2.37	0.08	0.14	0.10	0.05	0.11	2.21	0.10	0.00	0.02	0.05	0.10
5	3.00	0.04	0.10	0.00	0.05	0.03	2.87	0.02	0.12	0.06	0.06	0.16
1	2.81	0.23	0.03	0.10	0.04	0.08	3.04	0.10	0.02	0.11	0.11	0.02
0	2.84	0.04	0.03	0.02	0.28	0.15	2.68	0.23	0.10	0.04	0.06	0.23
−1	2.79	0.01	0.02	0.11	0.09	0.05	3.48	0.06	0.04	0.16	0.10	0.03
−5	2.47	0.12	0.09	0.03	0.08	0.04	2.64	0.01	0.01	0.04	0.12	0.07
−25	3.05	0.03	0.08	0.07	0.13	0.10	2.35	0.02	0.04	0.05	0.03	0.09
−125	2.82	0.13	0.07	0.02	0.08	0.01	2.89	0.13	0.03	0.11	0.06	0.04

Table 5. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 1.0$ and $\sigma_x^2 + \sigma_y^2 = 50.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	2.23	0.22	0.01	0.01	0.16	0.23	2.47	0.07	0.05	0.24	0.01	0.03
25	2.83	0.10	0.01	0.07	0.00	0.06	2.66	0.09	0.18	0.02	0.11	0.01
5	2.94	0.01	0.09	0.02	0.02	0.14	2.50	0.03	0.09	0.06	0.02	0.06
1	2.82	0.12	0.16	0.06	0.11	0.06	2.82	0.10	0.10	0.0860.05	0.01	
0	3.01	0.05	0.04	0.06	0.13	0.11	2.41	0.12	0.15	0.06	0.09	0.05
−1	2.66	0.31	0.14	0.05	0.02	0.22	2.84	0.06	0.03	0.08	0.10	0.12
−5	2.38	0.16	0.19	0.09	0.09	0.07	2.87	0.17	0.04	0.09	0.03	0.01
−25	2.65	0.03	0.05	0.17	0.19	0.01	2.61	0.04	0.09	0.04	0.06	0.19
−125	2.86	0.09	0.03	0.09	0.06	0.14	2.68	0.03	0.05	0.12	0.03	0.171

Table 6. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 1.0$ and $\sigma_x^2 + \sigma_y^2 = 100.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	2.98	0.05	0.14	0.24	0.02	0.05	2.60	0.01	0.14	0.03	0.03	0.15
25	2.20	0.00	0.02	0.05	0.00	0.03	2.51	0.20	0.04	0.02	0.02	0.02
5	2.60	0.10	0.07	0.07	0.12	0.07	2.69	0.20	0.03	0.02	0.04	0.11
1	2.50	0.12	0.10	0.06	0.02	0.03	2.69	0.23	0.00	0.00	0.07	0.08
0	2.62	0.01	0.01	0.06	0.05	0.09	3.00	0.14	0.03	0.05	0.20	0.03
−1	2.46	0.08	0.04	0.09	0.09	0.10	2.89	0.05	0.09	0.22	0.10	0.09
−5	2.59	0.05	0.04	0.05	0.05	0.10	2.59	0.12	0.03	0.27	0.12	0.01
−25	2.34	0.14	0.15	0.09	0.14	0.16	2.86	0.25	0.04	0.09	0.01	0.09
−125	2.89	0.04	0.09	0.11	0.03	0.20	2.72	0.01	0.18	0.04	0.00	0.02

Table 7. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 1.0$ and $\sigma_x^2 + \sigma_y^2 = 500.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	2.87	0.10	0.05	0.02	0.12	0.10	2.89	0.02	0.05	0.06	0.04	0.11
25	2.35	0.09	0.09	0.06	0.09	0.11	3.04	0.06	0.07	0.14	0.11	0.07
5	2.75	0.06	0.09	0.07	0.08	0.06	3.58	0.09	0.15	0.05	0.02	0.08
1	2.62	0.02	0.13	0.02	0.06	0.08	3.08	0.13	0.01	0.05	0.11	0.28
0	2.56	0.01	0.01	0.03	0.15	0.12	2.82	0.09	0.02	0.04	0.02	0.17
−1	2.98	0.05	0.12	0.11	0.01	0.07	2.55	0.23	0.12	0.07	0.09	0.05
−5	3.21	0.01	0.06	0.06	0.01	0.04	2.50	0.15	0.05	0.10	0.03	0.04
−25	3.32	0.17	0.01	0.14	0.16	0.01	2.60	0.12	0.14	0.10	0.21	0.13
−125	2.18	0.05	0.01	0.10	0.20	0.05	3.31	0.28	0.12	0.01	0.05	0.09

Table 8. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 1.0$ and $\sigma_x^2 + \sigma_y^2 = 1000.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	3.00	0.0t	0.01	0.10	0.08	0.18	2.90	0.24	0.03	0.07	0.11	0.19
25	3.39	0.22	0.07	0.04	0.01	0.02	2.24	0.15	0.04	0.06	0.00	0.01
5	2.66	0.04	0.11	0.04	0.06	0.02	2.68	0.18	0.01	0.14	0.07	0.01
1	2.38	0.13	0.02	0.13	0.18	0.11	2.95	0.05	0.02	0.05	0.11	0.00
0	2.51	0.12	0.07	0.14	0.08	0.14	2.79	0.09	0.00	0.12	0.00	0.00
−1	2.36	0.12	0.01	0.04	0.14	0.01	2.57	0.20	0.04	0.08	0.08	0.03
−5	2.79	0.01	0.08	0.07	0.15	0.10	2.53	0.11	0.15	0.02	0.06	0.01
−25	2.70	0.04	0.01	0.04	0.12	0.04	2.61	0.14	0.16	0.08	0.03	0.04
−125	2.70	0.02	0.08	0.05	0.05	0.26	2.64	0.03	0.14	0.04	0.00	0.05

Table 9. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 2.0$ and $\sigma_x^2 + \sigma_y^2 = 50.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	2.67	0.00	0.28	0.11	0.22	0.10	2.72	0.11	0.11	0.03	0.03	0.00
25	2.97	0.06	0.05	0.24	0.04	0.06	2.47	0.29	0.08	0.06	0.12	0.04
5	3.00	0.06	0.04	0.13	0.01	0.25	2.69	0.18	0.11	0.02	0.12	0.01
1	2.62	0.08	0.11	0.08	0.03	0.06	3.47	0.18	0.14	0.12	0.05	0.03
0	2.72	0.18	0.19	0.04	0.09	0.08	2.47	0.04	0.00	0.01	0.06	0.12
−1	2.82	0.06	0.04	0.14	0.09	0.07	2.81	0.17	0.06	0.06	0.00	0.11
−5	2.76	0.08	0.04	0.01	0.20	0.10	3.10	0.06	0.04	0.07	0.08	0.21
−25	2.72	0.16	0.01	0.06	0.16	0.02	2.57	0.01	0.26	0.03	0.02	0.04
−125	2.73	0.16	0.14	0.06	0.25	0.18	2.58	0.06	0.04	0.07	0.03	0.07

Table 10. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 2.0$ and $\sigma_x^2 + \sigma_y^2 = 100.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	3.19	0.13	0.17	0.04	0.17	0.05	2.47	0.12	0.19	0.05	0.03	0.03
25	2.39	0.01	0.15	0.10	0.08	0.29	2.60	0.01	0.07	0.08	0.03	0.08
5	2.82	0.20	0.04	0.07	0.08	0.06	2.73	0.06	0.04	0.0960.04	0.06	
1	2.59	0.1.3	0.01	0.04	0.06	0.05	2.33	0.08	0.02	0.00	0.01	0.16
0	2.58	0.04	0.14	0.01	0.13.	0.10	2.80	0.02	0.06	0.04	0.06	0.22
−1	2.73	0.24	0.03	0.02	0.07	0.15	2.63	0.13	0.11	0.11	0.19	0.02
−5	2.79	0.27	0.02	0.17	0.18	0.05	3.19	0.14	0.00	0.01	0.01	0.05
−25	2.79	0.11	0.06	0.05	0.07	0.21	3.13	0.11	0.04	0.12	0.01	0.04
−125	2.30	0.02	0.05	0.12	0.13	0.11	2.76	0.17	0.06	0.08	0.09	0.11

Table 11. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 2.0$ and $\sigma_x^2 + \sigma_y^2 = 500.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	2.76	0.15	0.12	0.05	0.18	0.05	2.65	0.04	0.03	0.03	0.07	0.09
25	2.83	0.15	0.02	0.10	0.15	0.09	2.68	0.03	0.07	0.03	0.04	0.05
5	3.12	0.04	0.26	0.02	0.17	0.10	2.67	0.01	0.10	0.04	0.03	0.15
1	2.43	0.02	0.07	0.11	0.06	0.07	2.36	0.12	0.06	0.06	0.06	0.09
0	2.37	0.17	0.06	0.11	0.15	0.02	2.57	0.00	0.09	0.03	0.07	0.09
−1	2.89	0.20	0.07	0.04	0.01	0.01	2.52	0.01	0.12	0.17	0.08	0.08
−5	2.46	0.03	0.11	0.13	0.13	0.28	2.61	0.17	0.15	0.13	0.02	0.05
−25	2.12	0.15	0.00	0.04	0.03	0.08	2.81	0.15	0.00	0.08	0.06	0.05
−125	2.70	0.10	0.00	0.06	0.06	0.04	2.42	0.08	0.03	0.06	0.23	0.25

Table 12. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 2.0$ and $\sigma_x^2 + \sigma_y^2 = 1000.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	2.50	0.09	0.02	0.05	0.01	0.02	2.84	0.08	0.02	0.11	0.03	0.20
25	3.35	0.06	0.05	0.01	0.07	0.08	2.87	0.22	0.04	0.07	0.07	0.07
5	2.95	0.16	0.10	0.12	0.17	0.08	2.29	0.08	0.07	0.04	0.03	0.00
1	2.53	0.10	0.03	0.10	0.05	0.08	2.57	0.10	0.18	0.02	0.01	0.00
0	2.73	0.02	0.13	0.02	0.12	0.05	2.85	0.06	0.25	0.27	0.06	0.08
−1	2.32	0.00	0.10	0.08	0.04	0.14	2.34	0.12	0.03	0.10	0.03	0.16
−5	2.99	0.01	0.14	0.05	0.08	0.22	2.50	0.05	0.08	0.00	0.01	0.09
−25	2.67	0.12	0.26	0.15	0.10	0.08	2.62	0.04	0.11	0.02	0.23	0.13
−125	2.98	0.04	0.00	0.02	0.23	0.06	2.98	0.05	0.19	0.01	0.12	0.05

Table 13. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 3.0$ and $\sigma_x^2 + \sigma_y^2 = 50.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	2.92	0.07	0.01	0.05	0.09	0.11	2.52	0.17	0.03	0.10	0.01	0.01
25	2.55	0.06	0.12	0.06	0.10	0.14	2.47	0.11	0.02	0.06	0.14	0.03
5	2.29	0.23	0.07	0.13	0.26	0.18	2.38	0.17	−0.07	0.00	0.01	0.04
1	2.62	0.05	0.02	0.12	0.00	0.04	2.85	0.05	0.01	0.10	0.04	0.08
0	2.59	0.06	0.07	0.02	0.07	0.16	2.46	0.02	0.06	0.04	0.10	0.04
−1	2.39	0.15	0.11	0.03	0.11	0.18	2.70	0.14	0.08	0.02	0.05	0.09
−5	2.62	0.16	0.09	0.04	0.21	0.07	2.64	0.06	0.20	0.00	0.08	0.09
−25	2.40	0.06	0.19	0.06	0.10	0.07	2.23	0.05	0.17	0.04	0.08	0.06
−125	2.51	0.11	0.04	0.09	0.03	0.08	2.55	0.00	0.11	0.09	0.03	0.09

Table 14. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 3.0$ and $\sigma_x^2 + \sigma_y^2 = 100.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	2.91	0.01	0.02	0.08	0.12	0.03	2.95	0.10	0.20	0.08	0.06	0.05
25	2.73	0.14	0.13	0.07	0.14	0.02	2.72	0.11	0.10	0.05	0.17	0.08
5	2.61	0.10	0.10	0.09	0.11	0.08	3.37	0.06	0.03	0.22	0.15	0.05
1	2.97	0.09	0.02	0.13	0.06	0.14	2.63	0.18	0.04	0.04	0.04	0.06
0	2.74	0.04	0.01	0.18	0.01	0.08	2.81	0.10	0.03	0.05	0.11	0.10
−1	2.79	0.10	0.13	0.13	0.09	0.14	2.71	0.04	0.10	0.09	0.14	0.04
−5	2.78	0.12	0.03	0.09	0.02	0.01	2.28	0.14	0.12	0.04	0.10	0.06
−25	2.81	0.01	0.01	0.01	0.12	0.12	2.73	0.06	0.00	0.02	0.06	0.02
−125	2.43	0.21	0.01	0.18	0.07	0.19	2.55	0.08	0.15	0.08	0.26	0.01

Table 15. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 3.0$ and $\sigma_x^2 + \sigma_y^2 = 500.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	2.85	0.17	0.09	0.14	0.16	0.17	2.55	0.06	0.19	0.10	0.10	0.04
25	2.56	0.13	0.18	0.04	0.10	0.00	2.94	0.18	0.00	0.03	0.03	0.09
5	2.86	0.09	0.06	0.04	0.13	0.07	2.91	0.13	0.04	0.01	0.07	0.09
1	2.83	0.03	0.02	0.10	0.08	0.01	2.72	0.00	0.06	0.01	0.02	0.03
0	2.89	0.17	0.03	0.11	0.03	0.13	2.96	0.17	0.03	0.07	0.10	0.05
−1	2.28	0.07	0.08	0.05	0.07	0.02	2.50	0.20	0.01	0.01	0.25	0.02
−5	2.52	0.04	0.11	0.07	0.18	0.10	2.65	0.03	0.19	0.01	0.15	0.04
−25	2.88	0.07	0.07	0.05	0.15	0.07	2.88	0.02	0.12	0.16	0.08	0.14
−125	2.90	0.11	0.03	0.21	0.04	0.01	2.66	0.11	0.19	0.09	0.04	0.16

Table 16. Test statistic values of the test for $V(\hat{\alpha})$ when $\sigma^2 = 3.0$ and $\sigma_x^2 + \sigma_y^2 = 1000.0$

γ	θ where $\alpha = \cos\theta$ and $\beta = \sin\theta$											
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
125	2.79	0.06	0.07	0.11	0.13	0.07	2.92	0.04	0.01	0.01	0.00	0.06
25	2.61	0.00	0.06	0.05	0.05	0.03	2.18	0.17	0.16	0.03	0.07	0.14
5	2.49	0.14	0.05	0.20	0.22	0.08	3.16	0.03	0.25	0.05	0.03	0.23
1	2.64	0.12	0.16	0.17	0.06	0.15	2.29	0.14	0.04	0.07	0.06	0.11
0	2.21	0.00	0.21	0.13	0.02	0.05	2.78	0.02	0.04	0.21	0.04	0.15
−1	2.69	0.18	0.06	0.03	0.09	0.02	2.64	0.02	0.02	0.14	0.13	0.11
−5	2.69	0.26	0.03	0.02	0.05	0.01	2.73	0.08	0.05	0.20	0.06	0.35
−25	2.89	0.04	0.17	0.25	0.21	0.12	2.61	0.15	0.05	0.03	0.10	0.05
−125	2.94	0.17	0.06	0.00	01.02	0.13	2.82	0.02	0.17	0.15	0.01	0.251

$\boldsymbol{u}$ and (x_i, y_i) be the coordinate of the point on the line that is z_i away from (x_0, y_0). Then we have the following relationships:

$$(x_0, y_0) = -\gamma \boldsymbol{u} = (-\gamma\alpha, -\gamma\beta) ,$$
$$x_i = x_0 + \sin\theta z_i = x_0 + \beta z_i ,$$
$$y_i = y_0 - \cos\theta z_i = y_0 - \alpha z_i .$$

Let $\bar{x}$ and $\bar{y}$ be the mean of the x_i's and the y_i's, respectively. Since we have

$$E\left[(x_i - \bar{x})^2\right] + E\left[(y_i - \bar{y})^2\right] = T^2,$$

the (x_i, y_i)'s are lying exactly on the line $\alpha x + \beta y + \gamma = 0$ for which $\sigma_x^2 + \sigma_y^2 = T^2$.

The uniform distribution on the interval $[-\sqrt{3}T, \sqrt{3}T]$ has variance T^2. To generate (x_i, y_i)'s lying exactly on the given line, when $\sigma_x^2 + \sigma_y^2$ is equal to T^2, we do the following:

1. A uniform random number z from the interval $[-\sqrt{3}T, \sqrt{3}T]$ is generated.
2. The coordinate of the desired point is $(-\gamma\alpha + \beta z, -\gamma\beta - \alpha z)$.

8.4 Results and discussion

The computed test statistic values for each combination of control variables for the test where $\hat{\alpha}$ is involved are illustrated in Tables 1–16. Similar related results are given in [13]. From the standard normal distribution table, we have the following:

when $\alpha = 0.001$, $Z_{\alpha/2} = 3.0$,
when $\alpha = 0.01$, $Z_{\alpha/2} = 2.56$,
when $\alpha = 0.05$, $Z_{\alpha/2} = 1.96$.

Since all the values in the tables are far less than 1.0 except when $\theta = 0°$ or $180°$, the test of the null hypothesis H_0: $\sigma_{\hat{\alpha}}^2 = V(\hat{\alpha})$, when $\theta \neq 0°, 180°$, is not significant with the level of significance 0.4.

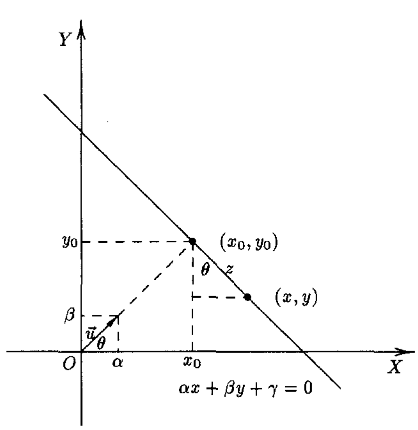

Fig. 5. When $\alpha^2 + \beta^2 = 1$, the geometric meaning of a straight line $\alpha x + \beta y + \gamma = 0$ is illustrated. It is perpendicular to the unit vector $\boldsymbol{u} = (\alpha, \beta)$ and $-\gamma$ away from the origin

There are two interesting facts that deserve mention here. One is the lack of symmetry in the Tables 1–16, and the other is the higher test statistic values when $\alpha = 1$ and $\beta = 0$.

1. Since $\alpha = \cos\theta$ and $\cos(180° + \theta) = \cos\theta$, values for θ are supposed to be the same as those for $180° + \theta$. However, this symmetry is not found in the tables. This is because the sample size is finite. In one sample, the constraint $\sigma_x^2 + \sigma_y^2 = T^2$ holds statistically, but there are variations due to the finite sample size.

2. When $\theta = 0°$ or $180°$, we have $\alpha = 1$ and $\beta = 0$. In this case, most of the test statistic values are larger than 2.56, which is the critical value for the level of significance 0.01. The test seems to be significant. However, this is due to the numerical error which may be avoided only by explicit control. Since $\sin 0°$ that the C library routine of SUN4 computes is not a true zero, the constraint, $\sigma_x^2 = T^2$ and $\sigma_y^2 = 0$, could not be achieved in our experiments. Considering the effect of the numerical error, we can tell that the test is not significant.

Based on the results of statistical tests, we conclude that the derivations using the first order Taylor series expansion in the preceding sections are valid.

9 Conclusion

We have extensively analyzed the propagation of the uncertainty in edge point position to the 2D measurements in machine vision. The errors in 2D measurements, in which 2D points are involved, are dependent upon the errors in the positions of these 2D points. The position of a 2D point has error related to the errors in the parameters of 2D curves which determine the point. The parameters of 2D curves are affected by the errors in the participating edge point positions. We have extended the work of Haralick on the relationship between random perturbation of edge point position and variance of the least squares estimates of line parameters from independent and identically distributed noise to independent, but not necessarily identically distributed, noise and from lines only to lines, circles, and ellipses. We also analyzed the relationship between the variance of estimated curve parameters and the variances of 2D points. Finally, the variance of the 2D measurement was given in terms of the variances of participating 2D points.

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