Pattern Recognition Pergamon Press 1977. Vol. 9, pp. 89–94. Printed in Great Britain

PATTERN DISCRIMINATION USING ELLIPSOIDALLY SYMMETRIC MULTIVARIATE DENSITY FUNCTIONS*

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(*Received* 5 *October* 1976)

Abstract—A brief review of ellipsoidally symmetric density functions is done. For the case of monotonic functional forms and distributions with common covariance matrices, a lower bound on the probability of correct classification is calculated in terms of either an incomplete beta or gamma integral, for a class of common functional forms. The lower bound is a monotonically increasing function of the Mahalanobis distance for all monotonic ellipsoidally symmetric forms.

Ellipsoidally symmetric density function Multivariate density function Statistical pattern discrimination Pattern discrimination error bounds

INTRODUCTION

Parametric decision rules based on a multivariate normal density function have been most popular in pattern recognition. It is well known that the normal assumption leads to quadratic discriminant functions. Also known is the fact that quadratic discriminant functions are optimal for the general class of ellipsoidally symmetric density functions.[7] In this note, we review the case of common covariance matrices and linear discriminant functions.[1,7] We briefly discuss the class of ellipsoidally symmetric density functions and provide a lower bound for the correct identification probability. This lower bound is expressible in terms of the incomplete beta or gamma integral for a common class of monotonic ellipsoidally symmetric forms.

Our first task is to define an ellipsoidally symmetric density function. Let f be a non-negative real-valued function defined on R, a subset of $(0, \infty)$. Let N be the dimension of the Euclidean space on which we wish to define an ellipsoidally symmetric density function. We assume that f satisfies

$$\int_{x\in R} x^n f(x)\,\mathrm{d}x < \infty, n \leqslant N + 1.$$

Let A be an NxN symmetric positive definite matrix and x an $Nx1$ vector. An ellipsoidally symmetric function with zero mean is any function of the form

$$f(\sqrt{x'Ax}).$$

Proper normalization of any function of this form determines an ellipsoidally symmetric density function.

NORMALIZATION CONSTANT

The normalization constant c is given by

$$c = \frac{1}{\displaystyle\int_{\sqrt{x'Ax}\in R}\cdots\int f(\sqrt{xAx})\,\mathrm{d}x_1\ldots\mathrm{d}x_N}.$$

To determine the value of the integral we will first make a transformation which rotates and then scales. Let T be an orthonormal matrix satisfying

$$T'AT = D,$$

where D is a diagonal matrix with no non-positive diagonal entries. We make the change of variables

$$x = TD^{-1/2}z.$$

The Jacobian of this transformation is $|A|^{-1/2}$ which is positive since A is positive definite. Hence,

$$\int_{\sqrt{x'Ax}\in R}\cdots\int f(\sqrt{x'Ax})\,\mathrm{d}x_1,\ldots,\mathrm{d}x_N = |A|^{-1/2}$$

$$\int_{\sqrt{z'z}\in R}\cdots\int f(z'z)\,\mathrm{d}z_1\ldots\mathrm{d}z_N.$$

The next step is to change to an N-dimensional spherical coordinate system. Let

$$z_1 = r\cos\theta_1\cos\theta_2\ldots\cos\theta_{N-2}\cos\theta_{N-1}$$

$$z_2 = r\cos\theta_1\cos\theta_2\ldots\cos\theta_{N-2}\sin\theta_{N-1}$$

$$z_3 = r\cos\theta_1\cos\theta_2\ldots\cos\theta_{N-3}\sin\theta_{N-2}$$

$$\vdots$$

$$z_j = r\cos\theta_1\ldots\cos\theta_{N-j}\sin\theta_{N-j+1}$$

$$\vdots$$

$$z_N = r\sin\theta_1.$$

The Jacobian of this transformation is

$$(-1)^N r^{N-1}\cos\theta_1^{N-2}\cos^{N-3}\theta_2\ldots\cos\theta_{N-2}.$$

* This work was supported by NSF grant ENG 74-18981.

Hence,

$$\int \cdots \int_{\sqrt{x'Ax} \in R} f(\sqrt{x'Ax})\, dx \ldots dx_N$$

$$= |A|^{-1/2} \int_{r \in R} \int_{\theta_1 = (-\pi)/2}^{\pi/2} \cdots \int_{\theta_{N-2} = (-\pi)/2}^{\pi/2} \int_{\theta_{N-1} = 0}^{2\pi}$$

$$\times\, f(r) r^{N-1} \cos^{N-2} \theta_1 \cos^{N-3} \theta_2 \ldots \cos \theta_{N-2}$$

$$\times\, dr\, d\theta_1 \ldots d\theta_{N-1}.$$

Since

$$\int_{\theta = (-\pi)/2}^{\pi/2} \cos^n \theta\, d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma(\frac{1}{2})}{\Gamma\left(\frac{n+2}{2}\right)},$$

the θ integrals are readily evaluated and there results

$$\int \cdots \int_{\sqrt{x'Ax} \in R} f(\sqrt{x'Ax})\, dx_1 \ldots dx_N = \frac{2(\pi)^{N/2}}{|A|^{1/2}\,\Gamma\left(\frac{N}{2}\right)}$$

$$\times \int_{r \in R} r^{N-1} f(r)\, dr.$$

For example, for f functions of the form $e^{-u^2/2}$ and $(1+u^2)^{-m}$ defined on the non-negative real line, we obtain the well known forms for the multivariate normal and the multivariate Pearson Type VII:

$$f(u) = e^{-u^2/2} \text{ implies } f(\sqrt{x'Ax}) = \frac{1}{(2\pi)^{N/2}|A'|^{-1/2}} e^{-x'Ax/2} \quad (1)$$

$$f(u) = (1+u^2)^{-m} \text{ implies } f(\sqrt{x'Ax}) = \frac{\Gamma(m)|A|^{1/2}}{\pi^{N/2}\,\Gamma\left(m - \frac{N}{2}\right)} (1 + x'Ax)^{-m}, m > \frac{N}{2}. \quad (2)$$

COVARIANCE MATRIX

The covariance matrix is also easily calculated for the ellipsoidally symmetric density function and we obtain the not so surprising result that the covariance matrix Σ is proportional to A^{-1}. Assuming the mean to be zero, we have

$$\Sigma = E[x'x] = c \int \cdots \int_{x'Ax \in R} x'x f(\sqrt{x'Ax})\, dx_1, \ldots, dx_N,$$

where c is the normalizing constant. Letting T be an orthonormal matrix satisfying

$$T'AT = D,$$

where D is a diagonal matrix, we may use the transformation

$$x = TD^{-1/2} z$$

to simplify the integral.

$$\Sigma = c|A|^{-1/2} TD^{-1/2} \int \cdots \int_{\sqrt{z'z} \in R}$$

$$\times\, zz' f(\sqrt{z'z})\, dz_1, \ldots, dz_N D^{-1/2} T'.$$

Notice that the $(i,j)^{\text{th}}$ term of the matrix defined by the integral is 0.

$$\int \cdots \int_{\sqrt{z'z} \in R} z_i z_j f(\sqrt{z'z})\, dz_1 \ldots dz_N = 0.$$

This happens because the integration is carried out for an odd function over even limits. The diagonal terms of the matrix defined by the integral are all equal from symmetry consideration. We can evaluate

$$\int \cdots \int_{\sqrt{z'z} \in R} z_i^2 f(\sqrt{z'z})\, dz_1 \ldots dz_N,$$

by changing to spherical coordinates. After evaluating the integrals we find

$$\int \cdots \int_{\sqrt{z'z} \in R} z_i^2 f(\sqrt{z'z})\, dz_1 \ldots dz_N = \frac{2\pi^{N/2}}{N\Gamma\left(\frac{N}{2}\right)} \int_{r \in R} r^{N+1} f(r)\, dr.$$

Substituting this back in and using the correct value for the normalizing constant c, there results

$$\Sigma = \frac{A^{-1} \int_{r \in R} r^{N+1} f(r)\, dr}{N \int_{r \in R} r^{N-1} f(r)\, dr}.$$

For example for f functions of the form $e^{-u^2/2}$ and $(1+u^2)^{-m}$, defined on the non-negative real line, we obtain the well known relation between Σ and A for the multivariate normal and multivariate Pearson Type VII density functions:

$$(1)\ f(u) = e^{-u^2/2} \text{ implies } \Sigma = A^{-1}$$

$$(2)\ f(u) = (1+u^2)^{-m} \text{ implies}$$

$$\times\, \Sigma = \frac{1}{\Gamma\left(m - \frac{N}{2} - 1\right)} A^{-1}, m > \frac{N}{2} + 1.$$

Table 1 lists the common forms for f functions, their normalizing constants and the relationship between A and Σ.

CORRECT CLASSIFICATION BOUND

The general ellipsoidally symmetric density function with mean u can be written as

$$\frac{|A|^{-1/2}\,\Gamma\left(\frac{N}{2}\right)}{2(\pi)^{N/2} \int_{r \in R} r^{N-1} f(r)\, dr} f(\sqrt{(x-u)'A(x-u)}).$$

Table 1. Lists normalizing constants and covariance matrices for common ellipsoidally symmetric forms

Functional Form f	Ellipsoidally symmetric functional forms $cf(x'Ax)$ Normalizing constant c	Relationship between Σ and A
$f(u), u \in R$	$c = \dfrac{\Gamma\left(\frac{N}{2}\right)}{2(\pi)^{N/2} \int_{r \in R} r^{N-1} f(r)\,dr} \lvert A \rvert^{1/2}$	$A = \dfrac{\int_{r \in R} r^{N+1} f(r)\,dr}{N \int_{r \in R} r^{N-1} f(r)\,dr} \Sigma^{-1}$
$e^{-u^2/2}, u \geq 0$	$c = \dfrac{1}{(2\pi)^{N/2}} \lvert A \rvert^{1/2}$	$A = \Sigma^{-1}$
$(1 + u^2)^{-m}, u \geq 0, m > N/2 + 1$	$c = \dfrac{\Gamma(m)}{\pi^{N/2} \Gamma\left(m - \frac{N}{2}\right)} \lvert A \rvert^{1/2}$	$A = \dfrac{1}{2m - N - 2} \Sigma^{-1}$
$u^{k-1} e^{-u}, u \geq 0$	$c = \dfrac{\Gamma\left(\frac{N}{2}\right)}{2(\pi)^{N/2} \Gamma(N + m - 1)} \lvert A \rvert^{1/2}$	$A = \dfrac{(N + m - 1)(N + m - 2)}{N} \Sigma^{-1}$
$(1 - u^2)^m, u^2 \leq 1$	$c = \dfrac{\Gamma\left(m + \frac{N}{2} + 1\right)}{\pi^{N/2} \Gamma(m + 1)} \lvert A \rvert^{1/2}$	$A = \dfrac{1}{2m + N + 1} \Sigma^{-1}$

Let u_1 and u_2 be the mean vectors for categories 1 and 2 and let A_1 and A_2 be positive definite matrices proportional to the inverse covariance matrix for categories 1 and 2. If f is a monotonically decreasing function and the covariance matrices for categories 1 and 2 have the same determinant, then a maximum likelihood rule will determine a quadratic discriminant function and assign the vector x to category 1 when

$$(x - u_1)' A_1 (x - u_1) < (x - u_2)' A_2 (x - u_2).^{(5)}$$

Anderson and Bahadur[2] have discussed error probabilities in this case for the normal distribution.

This inequality can be further simplified when the categories share a common covariance matrix. The decision region R_1 containing all vectors assigned to category 1 is then defined by

$$R_1 = \left\{ x \mid (u_2 - u_1)' A \left(x - \frac{u_1 + u_2}{2} \right) \leq 0 \right\}.$$

The discriminant function has changed from quadratic to linear. Anderson[1] discusses this case for a normal distribution assumption and determined the correct classification probability for category 1 to be

$$\int_{-\infty}^{1/2 r_0} \frac{1}{\sqrt{2\pi}} e^{-1/2 x^2}\,dx,$$

where

$$r_0^2 = (u_2 - u_1)' A (u_2 - u_1).$$

This result follows from the fact that if x has a $N(u_1, A^{-1})$ distribution, then $(u_2 - u_1)Ax$ has a $N((u_2 - u_1)' A u_1, (u_2 - u_1)' A (u_2 - u_1))$ distribution. Integration of this density function over the region R_1 yields the correct classification probability.

For the general ellipsoidally symmetric density function it is easy to calculate the mean and variance of $(u_2 - u_1)Ax$. But the distribution for $(u_2 - u_1)Ax$ is not normal and, in general, may be difficult to determine. It is for this general case that we compute a lower bound on the probability of correct identification.

Let T be an orthonormal matrix satisfying $T'AT = D$, where D is a diagonal matrix. The region R_1 can be rewritten as

$$R_1 = \left\{ x \mid (u_2 - u_1)' A x \leq (u_2 - u_1)' A \left(\frac{u_1 + u_2}{2} \right) \right\}.$$

The fraction p_1 of category 1 correctly identified by the maximum likelihood rule can be computed as

$$p_1 = c \int \cdots \int_{x \in R_1} f\left(\sqrt{(x - u_1)' A (x - u_1)}\right) dx_1 \ldots dx_N,$$

where c is the normalizing constant. Making the transformation $y = D^{1/2} T' (x - u_1)$, there results

$$p_1 = \frac{\Gamma\left(\frac{N}{2}\right)}{2(\pi)^{N/2} \int_{r \in R} r^{N-1} f(r)\,dr} \int \cdots \int_{u_1 + TD^{-1/2} y \in R_1} \times f(\sqrt{y'y})\,dy_1 \ldots dy_N$$

$$= \frac{\Gamma\left(\frac{N}{2}\right)}{2(\pi)^{N/2} \int_{r \in R} r^{N-1} f(r)\,dr} \int \cdots \int_{w'y \leq w'w} \times f(\sqrt{y'y})\,dy_1 \ldots dy_N,$$

where

$$w = D^{1/2}T'\left(\frac{u_2 - u_1}{2}\right).$$

Now by the Schwarz inequality,

$$|w'y|^2 \leqslant w'w\, y'y.$$

Hence $\{y \mid w'y \leqslant 0\} \cup \{y \mid y'y \leqslant w'w\} \subseteq \{y \mid w'y \leqslant w'w\}$. By integrating the non-negative function over two smaller non-overlapping areas consisting of a half space and half of an ellipsoid, we can obtain a lower bound for p_1.

$$p_1 \geqslant \frac{1}{2} + \frac{1}{2}\frac{\Gamma\left(\frac{N}{2}\right)}{2(\pi)^{N/2}\int_{r\in R} r^{N-1} f(r)\,\mathrm{d}r}\int\cdots\int_{y'y \leqslant w'w} \times f(\sqrt{y'y})\,\mathrm{d}y_1\ldots\mathrm{d}y_N.$$

This integral is easier to evaluate because instead of having to integrate an ellipsoidally symmetric function over all points to one side of a hyperplane, we just have to integrate over an ellipsoid. The integration can be done by a transformation to a spherical co-ordinate system.

$$p_1 \geqslant \frac{1}{2} + \frac{1}{2}\frac{\Gamma\left(\frac{N}{2}\right)}{2(\pi)^{N/2}\int_{r\in R} r^{N-1} f(r)\,\mathrm{d}r}\int_{r=0}^{(r_0)/2} \times r^{N-1} f(r)\,\mathrm{d}r \int_{\theta_1=(-\pi)/2}^{\pi/2}\int_{\theta_{N-2}=(-\pi)/2}^{\pi/2} \times \int_{\theta_{N-1}=0}^{2\pi} \cos^{N-2}\theta_1 \cos^{N-3}\theta_2 \ldots \cos\theta_{N-2} \times \mathrm{d}\theta_1\ldots\mathrm{d}\theta_{N-1},$$

where

$$r_0^2 = (u_2 - u_1)'A(u_2 - u_1).$$

The cosine integrals are readily evaluated since

$$\int_{\theta=(-\pi)/2}^{\pi/2} \cos^n\theta\,\mathrm{d}\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}.$$

Hence, there results

$$p_1 \geqslant \frac{1}{2} + \frac{1}{2}\frac{1}{\int_{r\in R} r^{N-1} f(r)\,\mathrm{d}r}\int_{r=0}^{(r_0)/2} r^{N-1} f(r)\,\mathrm{d}r.$$

Since the lower bound is a monotonically increasing function of r_0, the Mahalanobis distance, we have shown that r_0 can be the basis of a good feature selection procedure in the general ellipsoidally symmetric case.

For the special case of $f(u) = \mathrm{e}^{-u^2/2}$ and where the domain of f is taken to be the non-negative numbers,

$$p_1 \geqslant \frac{1}{2} + \frac{1}{2}\frac{\int_{r=0}^{(r_0)/2} r^{N-1}\,\mathrm{e}^{-1/2r^2}\,\mathrm{d}r}{\int_0^\infty r^{N-1}\,\mathrm{e}^{-1/2r^2}\,\mathrm{d}r} = \frac{1}{2} + \frac{1}{2}\frac{1}{\Gamma\left(\frac{N}{2}\right)}\int_{u=0}^{(r_0^2/8} u^{N/2-1}\,\mathrm{e}^{-u}\,\mathrm{d}u.$$

Recognizing the integral as the incomplete gamma integral,

$$p_1 \geqslant \frac{1}{2} + \frac{1}{2}P\left(\chi_N^2 \leqslant \frac{r_0^2}{4}\right)$$

where χ_N^2 is a chi-squared random variable with N degrees of freedom. Pearson and Hartley[8] is one place where tables may be found for this probability distribution. Wilson and Hilferty[9] provide the following approximation in terms of the normal integral.

$$P(\chi_v^2 \leqslant x) \approx \Phi\left(\left[\left(\frac{x}{v}\right)^{1/3} - 1 + \frac{2}{9v}\right]\sqrt{\frac{9v}{2}}\right),$$

where

$$\Phi(a) = \int_{-\infty}^{a} \frac{\mathrm{e}^{-u^2/2}}{\sqrt{2\pi}}\,\mathrm{d}y.$$

For the special case $f(u) = (1 + u^2)^{-m}$ and where the domain of f is taken to be the non-negative numbers,

$$p_1 \geqslant \frac{1}{2} + \frac{1}{2}\frac{\int_0^{r_0^2} r^{N-1}(1 + r^2)^{-m}\,\mathrm{d}r}{\int_0^\infty r^{N-1}(1 + r^2)^{-m}\,\mathrm{d}r} = \frac{1}{2} + \frac{1}{2}\frac{\int_0^{r_0^2/4+r_0^2} u^{N/2-1}(1-u)^{m-N/2-1}\,\mathrm{d}u}{\frac{\Gamma\left(\frac{N}{2}\right)\Gamma(m)}{\Gamma\left(\frac{N}{2}+m\right)}}.$$

The integral is the incomplete beta integral. By successive integration by parts we can establish the correspondence between it and the binomial distribution. Hence,

$$p_1 \geqslant \tfrac{1}{2} + \tfrac{1}{2}P\left(x \geqslant \frac{N}{2}\right),$$

where x has the binomial distribution with parameters

$$\left(m - 1, \frac{r_0^2}{4 + r_0^4}\right).$$

Tables for the binomial distribution are numerous.

Table 2. Lists the relationship between some common functional forms for ellipsoidally symmetric functions and a lower bound for the probability of correct classification. $r_0^2 = (u_2 - u_1)'A(u_2 - u_1)$

Function form of f	Lower bound of probability of correct classification
$f(u), u \in R$	$p_1 \geqslant \frac{1}{2} + \frac{1}{2} \frac{\int_0^{r_0/2} r^{N-1} f(r)\,\mathrm{d}r}{\int_{r \in R} r^{N-1} f(r)\,\mathrm{d}r}$
$e^{-1/2 u^2}, u \geqslant 0$	$p_1 \geqslant \frac{1}{2} + \frac{1}{2} \frac{\int_0^{r_0^2/8} u^{N/2-1} e^{-u}\,\mathrm{d}u}{\Gamma\left(\frac{N}{2}\right)}$
$(1 + u^2)^m, u \geqslant 0$	$p_1 \geqslant \frac{1}{2} + \frac{1}{2} \frac{\int_0^{r_0^2/4+r_0^2} u^{N/2-1}(1-u)^{m-N/2-1}\,\mathrm{d}u}{\frac{\Gamma\left(\frac{N}{2}\right)\Gamma(m)}{\Gamma\left(\frac{N}{2}+m\right)}}$
$u^{k-1} e^{-u}, u \geqslant 0$	$p_1 \geqslant \frac{1}{2} + \frac{1}{2} \frac{\int_0^{r_0/2} u^{(N+k-1)-1} e^{-u}\,\mathrm{d}u}{\Gamma(N+k-1)}$
$(1 - u^2)^{+m}, u^2 \leqslant 1$	$p_1 \geqslant \frac{1}{2} + \frac{1}{2} \frac{\int_0^{r_0^2/4} u^{N/2-1}(1-u)^{(m+1)-1}\,\mathrm{d}u}{\frac{\Gamma\left(\frac{N}{2}\right)\Gamma(m+1)}{\Gamma\left(\frac{N}{2}+m+1\right)}}$

The Harvard University Press in 1955 printed a volume called Tables of the Cumulative Binomial Probability Distribution. Bahadur[3] obtained the following bounds for the cumulative distribution of a random variable x having the binomial distribution with parameter (N, p).

$$\left[1 + \frac{Np(1-p)}{(k-Np)^2}\right]^{-1} \frac{(1-p)(k+1)}{k+1-(N+1)p} \leqslant \frac{P(x \geqslant k)}{\binom{N}{k} p^k (1-p)^{N-k}} \leqslant \frac{(1-p)(k+1)}{(k+1)-(N+1)P}.$$

Tables for the incomplete beta function itself can be found in Pearson and Hartley.[8]

Table 2 summarizes the relationship between common functional forms for ellipsoidally symmetric functions and lower bounds on the probability of correct classification. Notice that for these common forms, the lower bounds can be expressed either in terms of the incomplete gamma integral or the incomplete beta integral.

CONCLUSION

We have reviewed the ellipsoidally symmetric density function. We have indicated that when the functional forms on which they are based are monoionic and when the distributions have covariance matrix with same determinant then the quadratic form is the optimal discriminant function. In case the covariance matrices are the same, the optimal discriminant function becomes a linear one. For this case we computed a lower bound on the probability of correct identification. For all ellipsoidally symmetric forms this bound is a monotically increasing function of the Mahalanobis distance between the distributions. For a common class of functional forms the bound is expressible as an incomplete beta or gamma integral.

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