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The Digital Morphological Sampling Theorem

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Abstract—There are potential industrial applications for any methodology which inherently reduces processing time and cost and yet produces results sufficiently close to the result of full processing. It is for this reason that a morphological sampling theorem is important.

The morphological sampling theorem described in this paper states: 1) how a digital image must be morphologically filtered before sampling in order to preserve the relevant information after sampling; 2) to what precision an appropriately morphologically filtered image can be reconstructed after sampling; and 3) the relationship between morphologically operating before sampling and the more computationally efficient scheme of morphologically operating on the sampled image with a sampled structuring element.

The digital sampling theorem is developed first for the case of binary morphology, and then it is extended to grayscale morphology through the use of the umbra homomorphism theorems.

I. Introduction

MORPHOLOGICAL operations on images have relevance to conditioning, labeling, grouping, extracting, and matching image processing operations. Thus, from low level to intermediate to high level vision, morphological techniques are important. Indeed, many successful machine vision algorithms employed in industry on the factory floor, processing thousands of images per day in each application, are based on morphological techniques. Among the recent research papers on morphology are [3], [6], [8], [12], and [13]. Reference [21] is comprehensive.

Many well-known relationships worked out in the classical context of the convolution operation have morphological analogs. In this paper, we introduce the digital morphological sampling theorem, which relates to morphology as the standard sampling theorem relates to signal processing and communications. The sampling theorem permits the development of a precise multiresolution approach to morphological processing.

Multiresolution techniques [1], [9], [18], [19] have been useful for at least two fundamental reasons: 1) the representation they provide naturally permits a computational mechanism to focus on objects or features likely to be at least a given specified size [4], [15], [16], [20]; and 2) the computational mechanism can operate on only those resolution levels which just suffice for the detection and localization of objects or features of specified size while significantly reducing the number of operations performed [2], [5], [11].

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The usual resolution hierarchy, called a pyramid, is produced by low-pass filtering and then sampling to generate the next lower resolution level of the hierarchy. The basis for a morphological pyramid requires a morphological sampling theorem which explains how an appropriately morphologically filtered and sampled image relates to the unsampled image. It must explain what kinds of shapes are preserved and what kinds are suppressed or eliminated. It must explain the relationship between performing a less costly morphological filtering operation on the sampled image, and performing the more costly equivalent morphological filtering operations on the original image. It is just these issues which we address in this paper.

We analyze the constraints on sampling and on image objects in order to speed up morphological operations without sacrificing accurate shape analysis. The following results are shown to be true under reasonable morphological sampling conditions. Before sets are sampled, they must be morphologically simplified by an opening or a closing. Such sampled sets can be reconstructed in two ways, by either a closing or dilation. In both reconstructions, the sampled reconstructed sets are equal to the sampled sets. A set contains its reconstruction by closing, and is contained in its reconstruction by dilation; indeed, these are extremal bounding sets. That is, the largest set which downsamples to a given set is its reconstruction by dilation; the smallest is its reconstruction by closing. Furthermore, the distance from the maximal reconstruction to the minimal reconstruction is no more than the diameter of the reconstruction structuring element. Morphological sampling thus provides reconstructions positioned only to within some spatial tolerance which depends on the sampling interval. This spatial limitation contrasts with the sampling reconstruction process in signal processing from which only those frequencies below the Nyquist frequency can be reconstructed.

A number of relationships follow from the morphological sampling theorem. These relationships govern the commutivity between sampling and then performing morphological operations in the sampled domain versus first performing the morphological operations and then sampling. We find that sampling a minimal reconstruction which has been dilated is identical to dilating the sample set with a sampled structuring element. Sampling a maximal reconstruction which has been eroded is identical to eroding the sampled set with a sampled structuring element. These results establish bounds which can be used to determine the difference between morphological oper-

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ations in the sampled domains and operations in the original domain followed by sampling.

All set morphological relationships are immediately generalizable to grayscale morphology via the umbra homomorphism theorems. For grayscale images, the bounds which the reconstruction establishes are bounds in a spatial sense.

In Section II, we review the basic definitions and properties for binary morphology operations. In Section III, we develop the morphological sampling theorem for binary morphology. In Section IV, we derive the relationship between morphologically operating in the original domain and operating in the sampled domain. The homomorphism theorem between binary and grayscale morphology implies that each result in binary morphology has a corresponding result in grayscale morphology. Section V develops these grayscale generalizations. Section VI discusses the computational advantages of operating on morphologically sampled images and shows how successively sampled images can be operated on in a resolution hierarchy called a pyramid. The final section summarizes the key points and contains conclusions.

II. Preliminaries

Let E denote the set of numbers used to index a row or column position on a binary image. We assume that the addition and subtraction operations are defined on E. The binary image itself can then be thought of as a subset of $E \times E$. Pixels are in this subset if and only if they have the binary value one on the image. This correspondence permits us to work with sets rather than with image functions, indeed, with sets in E^N. The first two operations of mathematical morphology are the dual operations of dilation and erosion. The *dilation* of a set $A \subseteq E^N$ with a set $B \subseteq E^N$ is defined by

$$A \oplus B = \{x \mid \text{for some } a \in A \text{ and } b \in B, \; x = a + b\}.$$

The *erosion* of A by B is defined by

$$A \ominus B = \{x \mid \text{for every } b \in B, \quad x + b \in A\}.$$

The careful reader should beware that the symbol $\ominus$ used by Serra [17] does not designate erosion. Rather, it designates Minkowski subtraction.

For any set $A \subseteq E^N$ and $x \in E^N$, let A_x denote the *translation* of A by x

$$A_x = \{y \mid \text{for some } a \in A, \quad y = a + x\}.$$

For any set $A \subseteq E^N$, let $\check{A}$ denote the *reflection* of A about the origin

$$\check{A} = \{x \mid \text{for some } a \in A, \quad x = -a\}.$$

Relationships satisfied by dilation and erosion include the following:

$$A \oplus B = B \oplus A$$

$$(A \oplus B) \oplus C = A \oplus (B \oplus C)$$

$$(A \ominus B) \ominus C = A \ominus (B \oplus C)$$

$$(A \cup B) \oplus C = (A \oplus C) \cup (B \oplus C)$$

$$(A \cap B) \ominus C = (A \ominus C) \cap (B \ominus C)$$

$$A \oplus B = \bigcup_{b \in B} A_b$$

$$A \ominus B = \bigcap_{b \in B} A_{-b}$$

$$A \subseteq B \Rightarrow A \oplus C \subseteq B \oplus C$$

$$A \subseteq B \Rightarrow A \ominus C \subseteq B \ominus C$$

$$(A \cap B) \oplus C \subseteq (A \oplus C) \cap (B \oplus C)$$

$$(A \cup B) \ominus C \supseteq (A \ominus C) \cup (B \ominus C)$$

$$(A \oplus B)^C = A^C \ominus \check{B}$$

$$A \ominus (B \cup C) = (A \ominus B) \cap (A \ominus C).$$

In practice, dilations and erosions are usually employed in pairs; either dilation of an image followed by the erosion of the dilated result, or image erosion followed by dilation. In either case, the result of iteratively applied dilations and erosions is an elimination of specific image detail smaller than the structuring element without the global geometric distortion of unsuppressed features. For example, opening an image with a disk structuring element smooths the contour, breaks narrow isthmuses, and eliminates small islands and sharp peaks or capes. Closing an image with a disk structuring element smooths the contours, fuses narrow breaks and long thin gulfs, eliminates small holes, and fills gaps on the contours.

Of particular significance is the fact that image transformations employing iteratively applied dilations and erosions are idempotent, that is, their reapplication effects no further changes to the previously transformed result. The practical importance of idempotent transformations is that they comprise complete and closed stages of image analysis algorithms because shapes can be naturally described in terms of under what structuring elements they can be opened or can be closed and yet remain the same. Their functionality corresponds closely to the specification of a signal by its bandwidth. Morphologically filtering an image by an opening or closing operation corresponds to the ideal nonrealizable bandpass filters of conventional linear filtering. Once an image is ideal bandpassed filtered, further ideal bandpass filtering does not alter the result.

These properties motivate the importance of opening and closing, concepts first studied by Matheron [14] who was interested in axiomatizing the concept of size. Both Matheron's definitions and Serra's definitions for opening and closing are identical to the ones given here, but their formulas appear different because they use the symbol $\ominus$ to mean Minkowski subtraction rather than erosion.

The morphological filtering operations of opening and closing are made up of dilation and erosion performed in different orders. The *opening* of A by B is defined by

$$A \circ B = (A \ominus B) \oplus B.$$

The *closing* of A by B is defined by

$$A \bullet B = (A \oplus B) \ominus B.$$

Opening and closing satisfy the following basic relationships:

$$(A \circ B) \circ B = A \circ B \qquad (A \bullet B) \bullet B = A \bullet B$$

$$A \circ B \subseteq A \qquad A \subseteq A \bullet B$$

$$A \subseteq B \Rightarrow A \circ C \subseteq B \circ C \qquad A \subseteq B \Rightarrow A \bullet C \subseteq B \bullet C$$

$$(A \circ B)^C = A^C \bullet \check{B} \qquad (A \bullet B)^C = A^C \circ \check{B}.$$

The reason that openings and closings deal directly with shape properties is apparent from the following representation theorem for openings.

$$A \circ B = \{x \mid \text{for some } y, \quad x \in B_y \subseteq A\}.$$

A opened by B contains only those points of A which can be covered by some translation B_y which is, in turn, entirely contained inside A. Thus, x is a member of the opening if it lies in some area inside A which entirely contains a translated copy of the shape B. In this sense, A opened by B is the set of all points of A which can participate in areas of A which match B. If B is a disk of diameter d, for example, then $A \circ B$ would be that part of A which in no place is narrower than d.

The duality relationship $(A \circ B)^C = A^C \bullet \check{B}$ between opening and closing implies a corresponding representation theorem for closing

$$A \bullet B = \{x \mid x \in \check{B}_y \text{ implies } \check{B}_y \cap A \neq \varnothing\}.$$

A closed by B consists of all those points x for which x being covered by some translation $\check{B}_y$ implies that $\check{B}_y$ "hits" or intersects some part of A. A more extensive discussion of these relationships can be found in [8].

III. The Binary Digital Morphological Sampling Theorem

The preliminary part of this section sets the stage, discussing the appropriate morphological simplifying and filtering to be done before sampling. Certain relationships must be satisfied between the sampling set and the structuring element used for reconstruction. The main body of the section discusses two kinds of reconstructions of the sampled images: a maximal reconstruction accomplished by dilation and a minimal reconstruction accomplished by closing. Fundamental set bounding relationships are proved which show that the closing reconstruction of a set must be contained in the set itself which, in turn, must be contained in its dilation reconstruction. The closing reconstruction differs from the dilation reconstruction by just a dilation by the reconstruction structuring element, so the set bound relationships translate to geometric distance relationships. The section concludes by defining a suitable set distance function which measures the distance between the sampled set and the morphologically filtered set. The distance between the minimal reconstruction and the maximal reconstruction, and the distance between the morphologically filtered set and either of its reconstructions, are all less than the sampling distance.

The first conceptual issue which arises in developing a morphological sampling theorem is how to remove small objects, object protrusions, object intrusions, and holes before sampling. It is exactly the presence of this kind of small detail before sampling which causes the sampled result to be unrepresentative of the original. Just as in signal processing, the presence of frequencies higher than the Nyquist frequency causes the sampled signal to be unrepresentative of the original signal. This "aliasing" means that signals must be low-pass filtered before sampling. Likewise in morphology, the sets must be morphologically filtered and simplified before sampling. Small objects and object protrusions can be eliminated by a suitable opening operation. Small object intrusions and holes can be eliminated by a suitable closing. Since opening and closing are duals, we develop our motivation by just considering the opening operation.

Opening a set F by a structuring element K in order to eliminate small details of F raises, in turn, the issue of how K should relate to the sampling set S. If the sample points of S are too finely spaced, little will be accomplished by the reduction in resolution. On the other hand, if S is too coarse relative to K, objects preserved in the opening may be missed by the sampling. S and K can be coordinated by demanding that there be a way to reconstruct the opened image from the sampled opened image. Of course, details smaller than K are removed by the opening and cannot be reconstructed.

One natural way to reconstruct a sampled opening is by dilation. If S and K were coordinated to make the reconstructed image (first opened, then sampled, and then dilated) the same as the opened image, we would have a morphological sampling theorem nearly identical to the standard sampling theorem of signal processing. However, morphology cannot provide a perfect reconstruction, as is illustrated by the following one-dimensional continuous domain example.

Let the image F be the union of three topologically open intervals

$$F = (3.1, 7.4) \cup (11.5, 11.6) \cup (18.9, 19.8),$$

where (x, y) denotes the topologically open interval between x and y. We can remove all details of less than length 2 by opening with the structuring element $K = (-1, 1)$ consisting of the topologically open interval from -1 to 1. Then the opened image $F \circ K = (3.1, 7.4)$. What should the corresponding sample set be? Consider a sampling set $S = \{x \mid x \text{ an integer}\}$, with a sample spacing of unity; other spacings such as 0.2, 0.5, or 0.7 could illustrate the same sampling concept as well. The sampled opened image $(F \circ K) \cap S = \{4, 5, 6, 7\}$. Dilating by K to reconstruct the image produces $[(F \circ K) \cap S] \oplus K = (3, 8)$, an interval which properly contains $F \circ K$. The dilation fills in between the sample points, but cannot "know" to expand on the left end by a length of 0.9 and yet expand by 0.4 on the right end. However, the reconstruction is the largest one for which the sampled reconstruction $\{[(F \circ K) \cap S] \oplus K\} \cap S$ produces the sampled opening $(F \circ K) \cap S = \{4, 5, 6, 7\}$. This is easily seen in the example because substituting the closed interval $[3, 8]$ for the open interval $(3, 8)$ produces the sampled closed interval $[3, 8] \cap S = \{3, 4, 5, 6, 7, 8\}$ which properly contains $(F \circ K) \cap S = \{4, 5, 6, 7\}$.

The difficulty in reconstructing a sampled opened image morphologically can be understood in terms of the standard sampling theorem. Consider the case of a piecewise constant binary valued image. The required morphological simplification means that details smaller than K have been removed from all objects on the opened image, but this removal does not band-limit the image. In fact, the opened image belongs to a special class of infinite bandwidth signals, wherein reconstructing the sampled opened image as specified by the standard sampling theorem cannot produce the kind of aliasing found in moire patterns. The standard sampling theorem reconstruction produces a band-limited signal which passes through the sample points. Thus, the step-like patterns, like the open intervals of F, get reconstructed with ringing throughout and with overshoot and undershoot at step edges. By contrast, the morphological reconstruction cannot produce ringing, but the position of any step edge is uncertain within the sampling interval.

In the remainder of this section, we give a complete derivation of the results illustrated in the example. First, note that to use a structuring element K as a "reconstruction kernel," K must be large enough to ensure that the dilation of the sampling set S by K covers the entire space E^N. For technical reasons apparent in the derivations, we also require that K be symmetric, $K = \check{K}$. In the standard sampling theorem, the period of the highest frequency present must be sampled at least twice in order to properly reconstruct the signal from its sampled form. In mathematical morphology, there is an analogous requirement. The sample spacing must be small enough that the diameter of K is just smaller than these two sample intervals. Hence, the diameter of K is large enough that it can contain two sample points but not three sample points. We express this relationship by requiring that

$$x \in K_y \Rightarrow K_x \cap K_y \cap S \neq \emptyset \quad \text{and} \quad K \cap S = \{0\}.$$

The first condition implies that the dilation of sample points fills the whole space; that is, $S \oplus K = E^N$ when K is not empty. If the points in the sampling set S are spaced no further than d apart, then the corresponding reconstructing kernel K could be the topologically open ball of radius $2d$. In this case, $x \in K_y \Rightarrow K_x \cap K_y \cap S \neq \emptyset$. Notice that two points which are d apart can lie on the diameter of d. But since the ball is topologically open, the diameter cannot contain 3 points spaced d apart. Hence, the radius of K is just smaller than the sampling interval. Also, notice that if a sample point falls in the center of K, K will not contain another sample point.

Why does the morphological sampling theorem we develop here pertain mainly to the digital domain. Consider the two-dimensional continuous case in which there is a regular square grid sampling, with the sample interval in each direction being of length L. To guarantee that $K \cap S = \{0\}$, the biggest possible disk K is the open disk having radius L.

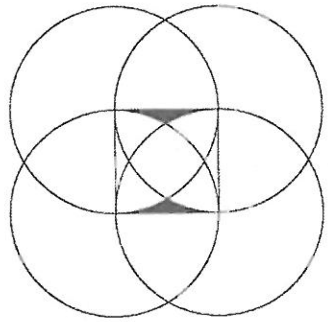


Fig. 1. Two points can be chosen no further apart than the sample distance, yet there is no sample point which is simultaneously less distant than the sample distance to each of them. Take the two points to be opposite each other each interior to one of the shaded regions. Consider the sample points to be the corners of the square.

The difficulty occurs with the condition $x \in K_y \Rightarrow K_x \cap K_y \cap S \neq \emptyset$. Fig. 1 shows a square whose length L side is the sampling interval. It also shows several translates of K, and a disk of the radius L. Select two points which are no further from each other than distance L in the following way. Take one point x to be in the interior of one shaded region of Fig. 1. Take the other point y to be opposite it, interior to the other shaded region. With this selection, the distance between the two points is guaranteed to be less than L. Yet it is apparent from the geometry that since none of the four open disks of diameter L can contain the two points which are distance L apart, the condition $x \in K_y \Rightarrow K_x \cap K_y \cap S \neq \emptyset$ cannot be satisfied. This is because each open disk represents exactly the set of points each having the property that, if an open disk were centered at the point, the open disk would contain a sample point. Hence, with the disk K being defined by the L_2 norm, there can be no morphological sampling theorem in the continuous case. In fact, the only norm by which K can be defined which yields a morphological sampling theorem in the continuous case is the L_∞ norm.

Because the shaded region in Fig. 2 is so narrow, this difficulty does not arise in the digital case. Suppose that the original domain is discrete with nearest points at distance one from each other. Then the condition $x \in K_y \Rightarrow K_x \cap K_y \cap S \neq \emptyset$ is easily satisfied for any $L \in \{2, 3, 4, 5, 6, 7\}$ since in this case $L(1 - \sqrt{3}/2) < 1$. Fig. 2 illustrates the case where the sample interval L is six. Notice that the distance between any pair of digital points, one from a region corresponding to one of the shaded regions of Fig. 1 and the other from the region opposite it, must be greater than L. Hence, for any two such points x and y, it is not the case that $x \in K_y$. So the difficulty with $x \in K_y$ and $K_x \cap K_y \cap S = \emptyset$ cannot arise.

We now prove some propositions which lead to the binary morphological sampling theorem. In what follows, the set $F \subseteq E^N$, the reconstruction structuring element, will be denoted by $K \subseteq E^N$, and the sampling set will be denoted by $S \subseteq E^N$. Although not necessary for every proposition, we assume that S and K obey the following

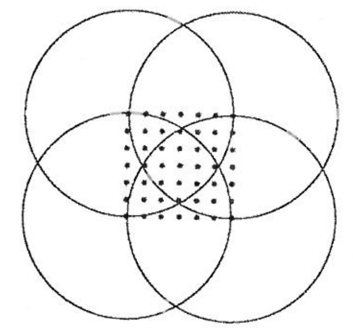


Fig. 2. The condition $x \in K_y \Rightarrow K_x \cap K_y \cap S \neq \varnothing$ can be satisfied in the digital case where K is a circular disk. Notice that for any pair of digital points in the regions corresponding to the shaded region of Fig. 1, the distance between them is not less than the radius of the open disk K. Hence, the difficulty illustrated in Fig. 1 cannot arise.

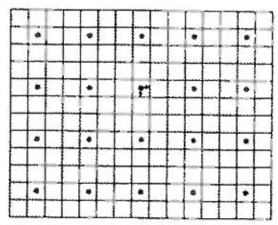


Fig. 3. Sampling every third pixel by row and by column. The sampling set S is represented by all points which are shown as •.

five conditions:

1) $S = S \oplus S$,
2) $S = \check{S}$,
3) $K \cap S = \{0\}$,
4) $K = \check{K}$,
5) $a \in K_b \Rightarrow K_a \cap K_b \cap S \neq \varnothing$.

Fig. 3 illustrates the S associated with a 3 to 1 down-sampling. Fig. 4 illustrates a structuring element K satisfying (3), (4), and (5). Since the dilation operation is commutative and associative, conditions 1)–3) imply that the sampling set S with the dilation operation comprises an abelian group with the origin being its unit element. Thus, if $x \in S$, then $S_x = S$, and also since $K \cap S = \{0\}$, $x \in S$ implies $K_x \cap S = \{x\}$. Both these facts are utilized in a number of the proofs to follow.

A. The Set Bounding Relationships

It is obvious that since $0 \in K$, the reconstruction of a sampled set $F \cap S$ by dilation with K produces a superset of the sampled set $F \cap S$. That is, $F \cap S \subseteq (F \cap S) \oplus K$. The reconstruction by dilation is open so that $[(F \cap S) \oplus K] \circ K = (F \cap S) \oplus K$. Moreover, as proved in the next proposition, the erosion and dilation of the original image F by K bounds the reconstructed sampled image.

Proposition 1: Let $F, K, S \subseteq E^N$. Suppose $S \oplus K = E^N$ and $K = \check{K}$. *Then* $F \ominus K \subseteq (F \cap S) \oplus K \subseteq F \oplus K$.

Proof: Since $F \cap S \subseteq F$, $(F \cap S) \oplus K \subseteq F \oplus K$. To show $F \ominus K \subseteq (F \cap S) \oplus K$, we show $[(F \cap S) \oplus K]^C \subseteq (F \ominus K)^C = F^C \oplus \check{K} = F^C \oplus K$. Let $x \in [(F \cap S) \oplus K]^C = (F \cap S)^C \ominus \check{K} = (F \cap S)^C \ominus K = (F^C \cup S^C) \ominus K$. Hence, $K_x \subseteq F^C \cup S^C$. It is not the case that $K_x \subseteq S^C$ because $S^C \ominus K = S^C \ominus \check{K} = (S \oplus K)^C = \varnothing$. Thus, $K_x \cap F^C \neq \varnothing$ so there exists a $k \in K$ such that $x + k \in F^C$, and $x = (x + k) + (-k) \in F^C \oplus \check{K} = (F \ominus K)^C$. ■

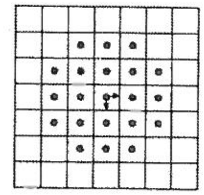


Fig. 4. A symmetric structuring element K which is a digital disk of radius $\sqrt{5}$. For the sampling set S of Fig. 3, $K \cap S = \{0\}$ and $x \in K_y$ implies $K_x \cap K_y \cap S \neq \varnothing$.

Proposition 1 shows that the reconstruction by dilation cannot be too far away from F since the reconstruction is constrained to lie between F eroded by K and F dilated by K. Proposition 2 strengthens the closeness between F and the dilation reconstruction $(F \cap S) \oplus K$. Sampling F and sampling the dilation reconstruction of F produce identical results.

Proposition 2: $F \cap S = [(F \cap S) \oplus K] \cap S$.

Proof:

$$\begin{aligned} [(F \cap S) \oplus K] \cap S &= \Big[\bigcup_{x \in F \cap S} K_x\Big] \cap S \\ &= \bigcup_{x \in F \cap S} K_x \cap S \\ &= \bigcup_{x \in F \cap S} K_x \cap S_x \\ &= \bigcup_{x \in F \cap S} (K \cap S)_x \\ &= \bigcup_{x \in F \cap S} \{0\}_x \\ &= F \cap S. \end{aligned}$$ ■

From this result, it rapidly follows that sampling followed by a dilation reconstruction is an idempotent operation. That is, $([(F \cap S) \oplus K] \cap S) \oplus K = (F \cap S) \oplus K$.

Considering sampling followed by reconstruction as an operation, we discover that it is an increasing operation, distributes over union but not over intersection. That is,

1) $F_1 \subseteq F_2$ implies $(F_1 \cap S) \oplus K \subseteq (F_2 \cap S) \oplus K$

2) $((F_1 \cup F_2) \cap S) \oplus K = [(F_1 \cap S) \oplus K] \cup [(F_2 \cap S) \oplus K]$

3) $((F_1 \cap F_2) \cap S) \oplus K \subseteq [(F_1 \cap S) \oplus K] \cap [(F_2 \cap S) \oplus K]$.

Proposition 3 states that the dilation reconstruction of a sampled F is always a superset of F opened by the reconstruction structuring element K. Hence, if F is open under K, then F is contained in its dilation reconstruction.

Proposition 3: $F \circ K \subseteq (F \cap S) \oplus K$.

Proof: Let $x \in F \circ K$. Then for some y, $x \in K_y \subseteq F$. Since $x \in K_y$, there exists $z \in K_x \cap K_y \cap S$. Now $z \in K_y \subseteq F$ and $z \in S$ implies $z \in F \cap S$. Also, $z \in K_x$ implies that there exists $k \in K$ such that $z = x + k$. Then $x = z - k$. Since $K = \check{K}$, $k \in K$ implies $-k \in K$. Now since $z \in F \cap S$ and since $-k \in K$, $x = z - k \in (F \cap S) \oplus K$. ■

Corollary: $F \circ K \subseteq [(F \circ K) \cap S] \oplus K$.

Thus, the reconstruction of the opened sampled image F is bounded by $F \circ K$ on the low side and $F \circ K$ dilated by K on the high side.

$$F \circ K \subseteq \left[(F \circ K) \cap S\right] \oplus K \subseteq (F \circ K) \oplus K.$$

If F is morphologically simplified and filtered so that $F = F \circ K$, then the previous bounds reduce to

$$F \subseteq (F \cap S) \oplus K \subseteq F \oplus K.$$

By reconsidering our example $F = (3.1, 7.4) \cup (11.5, 11.6) \cup (18.9, 19.8)$ which is not open under $K = (-1, 1)$, we can see that such an F is not necessarily a lower bound for the reconstruction. In this case, $F \cap S = \{4, 5, 6, 7, 19\}$ and the reconstruction $(F \cap S) \oplus K = (3, 8) \cup (18, 20)$, which does not contain F. This suggests that the condition that F be open under K is essential in order to have $F \subseteq (F \cap S) \oplus K$.

We now show one last relation between the reconstruction $(F \cap S) \oplus K$ and F. The reconstruction $(F \cap S) \oplus K$ is the largest open set which when sampled produces $F \cap S$.

Proposition 4: Let $A \subseteq E^N$ satisfy $A \cap S = F \cap S$ and $A = A \circ K$. Then $A \supseteq (F \cap S) \oplus K$ implies $A = (F \cap S) \oplus K$.

Proof: Suppose $A \supseteq (F \cap S) \oplus K$ and $A \cap S = F \cap S$ and $A = A \circ K$. Since $A \cap S = F \cap S$, $(A \cap S) \oplus K = (F \cap S) \oplus K$. But $A = A \circ K$ implies $A \subseteq (A \cap S) \oplus K = (F \cap S) \oplus K$. Now $A \subseteq (F \cap S) \oplus K$ together with the supposition $A \supseteq (F \cap S) \oplus K$ implies $A = (F \cap S) \oplus K$. ■

Thus, we have established the maximality of the reconstruction $(F \cap S) \oplus K$ with respect to the two properties of being open and downsampling to $F \cap S$. What about a minimal reconstruction? Certainly we would expect a minimal reconstruction to be contained in the maximal reconstruction and contain the sampled image. Since closing is extensive, we immediately have $F \cap S \subseteq (F \cap S) \bullet K$. Since $0 \in K$, erosion is an antiextensive operation. Hence, $(F \cap S) \bullet K = [(F \cap S) \oplus K] \ominus K \subseteq (F \cap S) \oplus K$. These relations suggest the possibility of a reconstruction by closing. The next proposition shows that a closing reconstruction has set bounds similar to the dilation reconstruction.

Proposition 5: Let $F, K, S \subseteq E^N$. If $K = \check{K}$ *and* $x \in K_y$ implies $K_x \cap K_y \cap S \neq \varnothing$ and $0 \in K$ then $F \ominus K \subseteq (F \cap S) \bullet K \subseteq (F \cap S) \oplus K \subseteq F \oplus K$.

Proof: Let $x \in F \ominus K$. Then $K_x \subseteq F$. To show $x \in (F \cap S) \bullet K$, we will show that $x + k \in (F \cap S) \oplus K$ for every $k \in K$. So let $k \in K$. Then $x + k \in K_x$. But $x + k \in K_x$ implies $K_x \cap K_{x+k} \cap S \neq \varnothing$. Hence, there exists $s \in K_x \cap K_{x+k} \cap S$. Now $s \in K_x$ and $K_x \subseteq F$ implies $s \in F$. Hence, $s \in F \cap S$. Furthermore, $s \in K_{x+k}$ implies there exists $k' \in K$ such that $s = x + k + k'$. Rearranging $s - k' = x + k$. But $K = \check{K}$ so $k' \in K$ implies $-k' \in K$. Now $s \in F \cap S$ and $-k' \in K$ implies $s - k' \in (F \cap S) \oplus K$. Therefore, $F \ominus K \subseteq (F \cap S) \bullet K$. Since $0 \in K$, $(F \cap S) \bullet K \subseteq (F \cap S) \oplus K$. Since $F \cap S \subseteq F$, $(F \cap S) \oplus K \subseteq F \oplus K$. ■

For true reconstruction, the sampled reconstruction should be identical to the sampled image. Indeed, this is the case.

Proposition 6: $[(F \cap S) \bullet K] \cap S = F \cap S$.

Proof: $F \cap S \subseteq (F \cap S) \bullet K \subseteq (F \cap S) \oplus K$. Now taking an intersection with S, we have $F \cap S \subseteq [(F \cap S) \bullet K] \cap S = [(F \cap S) \oplus K] \cap S$. But $[(F \cap S) \oplus K] \cap S = F \cap S$. ■

Consider our example $F = (3.1, 7.4) \cup (11.5, 11.6) \cup (18.9, 19.8)$, which is closed under $K = (-1, 1)$. If the sampling set S is the integers, then $F \cap S = \{4, 5, 6, 7, 19\}$. Closing $F \cap S$ with K can be visualized via the opening/closing duality $(F \cap S) \bullet K = ((F \cap S)^C \circ \check{K})^C$. Opening the set $(F \cap S)^C$ with $\check{K} = K$ produces $(F \cap S)^C \circ K = \{x \neq 19 \mid x < 4 \text{ or } > 7\}$. Hence, $(F \cap S) \bullet K = ((F \cap S)^C \circ K)^C = \{x \mid x = 19 \text{ or } 4 \le x \le 7\}$, and sampling produces $[(F \cap S) \bullet K] \cap S = \{4, 5, 6, 7, 19\} = F \cap S$.

From the previous proposition, it rapidly follows that sampling followed by a reconstruction by closing is an idempotent operation. That is, $[(F \cap S) \bullet K] \cap S \bullet K = (F \cap S) \bullet K$.

A reconstruction by closing is obviously closed under K. Moreover, it can be quickly determined that

$$F_1 \subseteq F_2 \text{ implies } (F_1 \cap S) \bullet K \subseteq (F_2 \cap S) \bullet K$$

$$\left[(F_1 \cup F_2) \cap S\right] \bullet K \supseteq \left[(F_1 \cap S) \bullet K\right] \cup \left[(F_2 \cap S) \bullet K\right]$$

$$\left[(F_1 \cap F_2) \cap S\right] \bullet K \subseteq \left[(F_1 \cap S) \bullet K\right] \cap \left[(F_2 \cap S) \bullet K\right].$$

Furthermore, the closing reconstruction of a sampled F is always a subset of F closed by the reconstruction structuring element K. That is, $(F \cap S) \bullet K \subseteq F \bullet K$, so that $((F \bullet K) \cap S) \bullet K \subseteq F \bullet K$. Hence, a closing reconstruction of an image which is closed before sampling will be a subset of the closed image.

By considering a simple example $F = \{0, 1\}$, which is not closed under $K = (-1, 1)$, we can see that F is not necessarily an upper bound for the reconstruction. In this case, $F \cap S = \{0, 1\} = F$ and the reconstruction $(F \cap S) \bullet K = F \bullet K = [0, 1]$ which properly contains F. This suggests that the condition that F be closed under K is essential in order to have $(F \cap S) \bullet K \subseteq F$.

We now show one last relation between the reconstruction $(F \cap S) \bullet K$ and F. The reconstruction $(F \cap S) \bullet K$ is the smallest closed set which when sampled produces $F \cap S$.

Proposition 7: Let $A \subseteq E^N$ satisfy $A \cap S = F \cap S$ and $A = A \bullet K$. Then $A \subseteq (F \cap S) \bullet K$ implies $A = (F \cap S) \bullet K$.

Proof: Suppose $A \subseteq (F \cap S) \bullet K$. Now $A \cap S = F \cap S$ implies $(A \cap S) \bullet K = (F \cap S) \bullet K$. Since $(A \cap S) \bullet K \subseteq A \bullet K$ and $A \bullet K = A$, we obtain $(F \cap S) \bullet K \subseteq A$. But $A \subseteq (F \cap S) \bullet K$ and $A \supseteq (F \cap S) \bullet K$ imply $A = (F \cap S) \bullet K$. ■

B. Examples

To better illustrate the bounding relationships developed in the previous section between a set and its sample reconstructions, we show three simple examples. The domain of these examples is defined as $E \times E$ where E is the set of integers. The sample set S is chosen as the set of even numbers in both row and column directions. Thus,

$$S = \{(r, c) \mid r \in E \text{ and is even}; \quad c \in E \text{ and is even}\}.$$

K is chosen as a box of size 3×3 whose center is defined as the origin. The sets S, K, and the three example sets F_1, F_2, and F_3 are shown in Fig. 5. The sets F_1, F_2, and F_3 are 3×3 boxes having different origins, and the condition $F = F \circ K$ holds for all these example sets.

The results of $F \ominus K$, $F \cap S$, $(F \cap S) \bullet K$, $(F \cap S) \oplus K$, and $F \oplus K$ for sets F_1, F_2, and F_3 are shown in Figs. 6, 7, and 8, respectively.

1) Example 1: All the pixels contained in the vertical boundaries of F_1 have even column coordinates, and those in the horizontal boundaries of F_1 have even row coordinates. Since the sample set S consists of pairs of even numbers, and F_1 is a 3×3 box, the set $F_1 \cap S$ consists of the four corner points of F_1 and is contained in the boundary set of F_1. Hence, the closing reconstruction of $F_1 \cap S$ recovers F_1, and the dilation reconstruction of $F_1 \cap S$ is equivalent to $F \oplus K$. In fact, the following two equalities hold only when 1) the sampling is every other row and column, 2) a set's vertical boundaries have even column coordinates, and 3) its horizontal boundaries have even row coordinates

$$(F \cap S) \bullet K = F \quad \text{and} \quad (F \cap S) \oplus K = F \oplus K.$$

The bounding relationships for F_1, illustrated in Fig. 6, are

$$F_1 \ominus K \subseteq (F_1 \cap S) \bullet K$$
$$= F_1 \subseteq (F_1 \cap S) \oplus K = F_1 \oplus K.$$

2) Example 2: Since all pixels contained in the vertical boundaries of F_2 have odd column coordinates, and those in the horizontal boundaries of F_2 have odd row coordinates, and F_2 is a small 3×3 box, the set $F_2 \cap S$ does not contain any part of the boundary of F_2. Thus, the closing reconstruction of $F_2 \cap S$ equals $F_2 \ominus K$, and the dilation reconstruction of $F_2 \cap S$ is equivalent to F_2. Similar to the Example 1, the following equalities hold only when the sampling is every other row and column, and has its odd column coordinates in its vertical boundaries and its odd row coordinates in its horizontal boundaries.

$$F \ominus K = (F \cap S) \bullet K \quad \text{and} \quad F = (F \cap S) \oplus K.$$

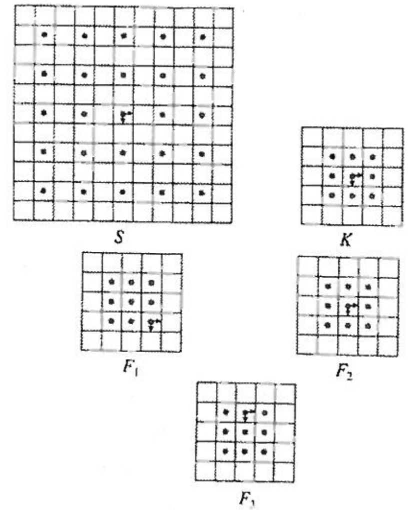


Fig. 5. A sampling set S, a reconstruction structuring element K, and three sets, F_1, F_2, and F_3, each of which is open under K.

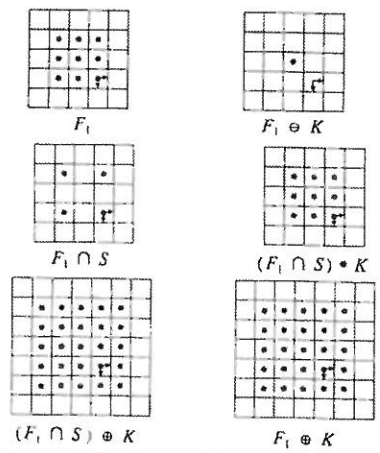


Fig. 6. The erosion and dilation of F_1 bound the minimal reconstruction $(F_1 \cap S) \bullet K$ and the maximal reconstruction $(F_1 \cap S) \oplus K$, respectively, which in turn bound F_1 because F_1 is both open and closed under K.

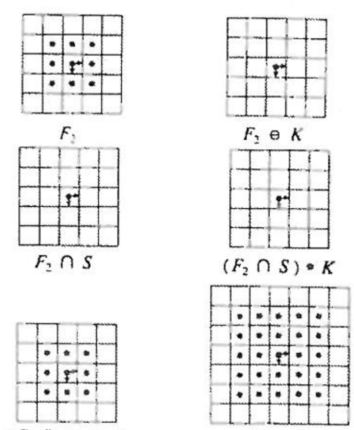


Fig. 7. A second example of how the erosion and dilation of F_2 bound the minimal reconstruction $(F_2 \cap S) \bullet K$ and the maximal reconstruction $(F_2 \cap S) \oplus K$, respectively, which in turn bound F_2.

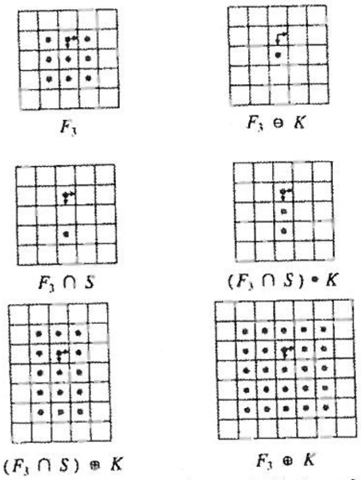


Fig. 8. A third example of how erosion and dilation of F_3 bound (in this case properly) the minimal reconstruction $(F_3 \cap S) \bullet K$ and the maximal reconstruction $(F_3 \cap S) \oplus K$, respectively, which in turn bound (in this case properly) F_3.

The bounding relationships for F_2, illustrated in Fig. 7, are

$$F_2 \ominus K = (F_2 \cap S) \bullet K \subseteq F_2$$
$$= (F_2 \cap S) \oplus K \subseteq F_2 \oplus K.$$

3) Example 3: The pixels contained in the vertical boundaries of F_3 have odd column coordinates, and the pixels in the horizontal boundaries of F_3 have even row coordinates. Hence, no equalities should exist in the bounding relationship. This is illustrated in Fig. 8. The bounding relationships for F_3 are

$$F_3 \ominus K \subseteq (F_3 \cap S) \bullet K \subseteq F_3 \subseteq (F_3 \cap S) \oplus K \subseteq F_3 \oplus K.$$

To show why the opening condition $F = F \circ K$ is needed for the bounding relationships involving F, we show an example set F_4 which deviates from the set F_3 by adding six extra points to it (see Fig. 9). The sample and reconstruction results of F_4, $F_4 \cap S$, $(F_4 \cap S) \bullet K$, and $(F_4 \cap S) \oplus K$ are exactly the same as the results for F_3. However, no bounding relationships between F_4 and its sample reconstructions are applicable. If we open F_4 by K, the bounding relationships exist because $F_4 \circ K = F_3$.

C. The Distance Relationships

We have established the maximality of the reconstruction $(F \cap S) \oplus K$ with respect to the property of being open and downsampling to $F \cap S$, and the minimality of the reconstruction $(F \cap S) \bullet K$ with respect to the property of being closed and downsampling to $F \cap S$. We now give a more precise characterization of how far $F \ominus K$ is from $F \bullet K$, how far $F \circ K$ is from $F \oplus K$, and how far $(F \cap S) \bullet K$ is from $(F \cap S) \oplus K$. This is important to know since $F \ominus K \subseteq (F \cap S) \bullet K \subseteq F$ when $F = F \bullet K$, and $F \subseteq (F \cap S) \oplus K \subseteq F \oplus K$ when $F = F \circ K$, and $(F \cap S) \bullet K \subseteq F \subseteq (F \cap S) \oplus K$ when $F = F \circ K = F \bullet K$. Notice that in all three cases, the difference between the lower and the upper set bound is just a dilation by K. This motivates us to define a distance function to measure the distance between two sets and to work out the relation between the distance between a set and its dilation by K with the size of the set K. In this section, we show that with a suitable definition of distance, all these distances are less than the radius of K. Since K is related to the sampling distance, all the above-mentioned distances are less than the sampling interval.

For the size of a set B, denoted by $r(B)$, we use the radius of its circumscribing disk. Thus, $r(B) = \min_{x \in B} \max_{y \in B} \|x - y\|$. The more mathematically correct forms of inf for min and sup for max may be substituted when the space E is the real line. In this case, the proofs in this section require similar modifications. For a set A which contains a set B, a natural pseudodistance from A to B is defined by $\rho(A, B) = \max_{x \in A} \min_{y \in B} \|x - y\|$. Proposition 8 proves that 1) $\rho(A, B) \geq 0$, 2) $\rho(A, B) = 0$ implies $A \subseteq B$, and 3) $\rho(A, C) \leq \rho(A, B) + \rho(B, C) + r(B)$. The asymmetric relation 2) is weaker than the corresponding metric requirement that $\rho(A, B) = 0$ if and only if $A = B$, and relation 3) is weaker than the metric triangle inequality.

Proposition 8:

1) $\rho(A, B) \geq 0$
2) $\rho(A, B) = 0$ if and only if $A \subseteq B$
3) $\rho(A, C) \leq \rho(A, B) + \rho(B, C) + r(B)$.

Proof:

1) $\rho(A, B) \geq 0$ since $\rho(A, B) = \max_{x \in A} \min_{y \in B} \|x - y\|$ and $\|x - y\| \geq 0$.

2) Suppose $\rho(A, B) = 0$. Then $\max_{a \in A} \min_{b \in B} \|a - b\| = 0$. Since $\|a - b\| \geq 0$, $\max_{a \in A} \min_{b \in B} \|a - b\| = 0$ implies for every $a \in A$, $\min_{b \in B} \|a - b\| = 0$. But $\|a - b\| = 0$ if and only if $a = b$. Hence, for every $a \in A$, there exists a $b \in B$ satisfying $a = b$, i.e., $A \subseteq B$. Suppose $A \subseteq B$. Then for each $a \in A$, $\min_{b \in B} \|a - b\| = 0$. Hence, $\max_{a \in A} \min_{b \in B} \|a - b\| = 0$.

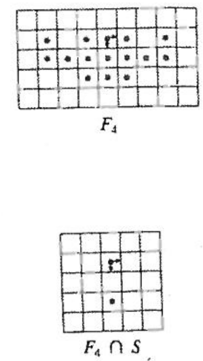


Fig. 9. A set F_4 which is not open under K. Its sampling $F_4 \cap S$ is identical to the sampling of F_3 yet the maximal reconstruction $(F_4 \cap S) \oplus K$ does not constitute an upper bound for F_4 as in the previous examples.

3)

$$\rho(A, C) = \max_{a\in A} \min_{c\in C} \|a - c\|$$
$$\leq \max_{a\in A} \min_{c\in C} \|a - b\| + \|b - c\| \quad \text{for every } b \in B$$
$$\leq \max_{a\in A} \left\{ \|a - b\| + \min_{c\in C} \|b - c\| \right\} \quad \text{for every } b \in B$$
$$\leq \max_{a\in A} \left\{ \|a - b\| + \max_{b'\in B} \min_{c\in C} \|b' - c\| \right\} \quad \text{for every } b \in B$$
$$\leq \rho(B, C) + \max_{a\in A} \|a - b\| \text{ for every } b \in B.$$

But $\max_{a\in A} \|a - b\|$

$$= \max_{a\in A} \|a - b' + b' - b\| \text{ for every } b, b' \in B$$
$$\leq \max_{a\in A} \|a - b'\| + \|b' - b\| \text{ for every } b, b' \in B.$$

Finally $\max_{a\in A} \|a - b\|$

$$\leq \|b' - b\| + \max_{a\in A} \|a - b'\| \text{ for every } b, b' \in B.$$

Thus, $\max_{a\in A} \|a - b\|$

$$\leq \max_{b'\in B} \|b' - b\| + \min_{b'\in B} \max_{a\in A} \|a - b'\| \quad \text{for every } b \in B$$
$$\leq \max_{b'\in B} \|b' - b\| + \rho(A, B).$$

Finally $\rho(A, C)$

$$\leq \rho(B, C) + \max_{b'\in B} \|b' - b\| \text{ for every } b \in B$$
$$\leq \rho(A, B) + \rho(B, C) + \min_{b\in B} \max_{b'\in B} \|b' - b\|$$
$$\leq \rho(A, B) + \rho(B, C) + r(B).$$

The pseudodistance ρ has a very direct interpretation. $\rho(A, B)$ is the radius of the smallest disk which when used as a structuring element to dilate B produces a result which contains A.

Proposition 9: Let $disk\,(r) = \{x \mid \|x\| \leq r\}$ and $A, B \subseteq F^N$. Then $\max_{a\in A} \min_{b\in B} \|a - b\| = inf\{r \mid A \subseteq B \oplus disk(r)\}$.

Proof: Let $\rho_0 = \max_{a\in A} \min_{b\in B} \|a - b\|$ and $r_0 = inf\{r \mid A \subseteq B \oplus disk(r)\}$. Let $a \in A$ be given. Let $b_0 \in B$ satisfy $\|a - b_0\| = \min_{b\in B} \|a - b\|$. Now, $\rho_0 = \max_{x\in A} \min_{y\in B} \|x - y\| \geq \min_{b\in B} \|a - b\|$. Hence, $\rho_0 \geq \|a - b_0\|$ so that $a - b_0 \in disk(\rho_0)$. Now, $b_0 \in B$ and $a - b_0 \in disk(\rho_0)$ implies $a = b_0 + (a - b_0) \in B \oplus disk(\rho_0)$. Hence, $A \subseteq B \oplus disk(\rho_0)$. Since $r_0 = inf\{r \mid A \subseteq B \oplus disk(r)\}$, $r_0 \leq \rho_0$. Suppose $A \subseteq B \oplus disk(r_0)$. Then $\max_{a\in A} \min_{b\in B \oplus disk(r_0)} \|a - b\| = 0$. Hence, $\max_{a\in A} \min_{b\in B} \min_{y\in disk(r_0)} \|a - b - y\| = 0$. But $\|(a - b) - y\| \geq \|a - b\| - \|y\|$. Therefore,

$$0 \geq \max_{a\in A} \min_{b\in B} \min_{y\in disk(r_0)} \|a - b\| - \|y\|$$
$$\geq \max_{a\in A} \min_{b\in B} \|a - b\| + \min_{y\in disk(r_0)} - \|y\|$$
$$\geq \max_{a\in A} \min_{b\in B} \|a - b\| - \max_{y\in disk(r_0)} \|y\|.$$

Now $\rho_0 = \max_{a\in A} \min_{b\in B} \|a - b\|$ and $r_0 = \max_{y\in disk(r_0)} \|y\|$ implies $0 \geq \rho_0 - r_0$ so that $r_0 \geq \rho_0$. Finally, $r_0 \leq \rho_0$ and $r_0 \geq \rho_0$ implies $r_0 = \rho_0$. ■

The pseudodistance ρ can be used as the basis for a true set metric by making it symmetric. We define the set metric $\rho_M(A, B) = \max\{\rho(A, B), \rho(B, A)\}$, also called the Hausdorf metric. The proof that ρ_M is indeed a metric follows rapidly after noting that $\rho_M(A, B) = inf\{r \mid A \subseteq B \oplus disk(r) \text{ and } B \subseteq A \oplus disk(r)\}$. This happens since

$$\rho_m(A, B) = \max\left\{\rho(A, B), \rho(B, A)\right\}$$
$$= \max\left\{inf\left\{r \mid A \subseteq B \oplus disk(r)\right\}, inf\left\{r \mid B \subseteq A \oplus disk(r)\right\}\right\}$$
$$= inf\left\{r \mid A \subseteq B \oplus disk(r) \text{ and } B \subseteq A \oplus disk(r)\right\}.$$

A strong relationship between the set distance and the dilation of sets must be developed to translate set bounding relationships to distance bounding relationships. We show that $\rho(A \oplus B, C \oplus D) \leq \rho(A, C) + \rho(B, D)$ and then quickly extend the result to $\rho_M(A \oplus B, C \oplus D) \leq \rho_M(A, C) + \rho_M(B, D)$.

Proposition 10:

1) $\rho(A \oplus B, C \oplus D) \leq \rho(A, C) + \rho(B, D)$

2) $\rho_M(A \oplus B, C \oplus D) \leq \rho_M(A, C) + \rho_M(B, D)$.

Proof:

1) $\rho(A \oplus B, C \oplus D)$

$$= \max_{x\in A\oplus B} \min_{y\in C\oplus D} \|x - y\|$$
$$= \max_{a\in A} \max_{b\in B} \min_{c\in C} \min_{d\in D} \|a + b - c - d\|$$
$$\leq \max_{a\in A} \max_{b\in B} \min_{d\in D} \min_{c\in C} [\|a - c\| + \|b - d\|]$$
$$\leq \max_{a\in A} \max_{b\in B} \min_{d\in D} [(\min_{c\in C} \|a - c\|) + \|b - d\|]$$
$$\leq \max_{a\in A} \min_{c\in C} \|a - c\| + \max_{b\in B} \min_{d\in D} \|b - d\|$$
$$\leq \rho(A, C) + \rho(B, D)$$

2) $\rho_M(A \oplus B, C \oplus D)$

$$= \max\{\rho(A \oplus B, C \oplus D), \rho(C \oplus D, A \oplus B)\}$$
$$\leq \max\{\rho(A, C) + \rho(B, D), \rho(C, A) + \rho(D, B)\}$$
$$\leq \max\{\rho(A, C), \rho(C, A)\} + \max\{\rho(B, D), \rho(D, B)\}$$
$$\leq \rho_M(A, C) + \rho_m(B, D).$$ ■

From this last result, it is apparent that dilating two sets with the same structuring element cannot increase the distance between the sets. Dilation tends to suppress differences between sets, making them more similar. More precisely, if $B = D = K$, then $\rho_M(A \oplus K, C \oplus K) \leq \rho_M(A, C)$. It is also apparent that $\rho_M(A, A \oplus K) = \rho_M(A \oplus \{0\}, A \oplus K) \leq \rho_M(A, A) + \rho_M(\{0\}, K) = \rho_M(\{0\}, K) \leq \max_{k \in K} \|k\|$. Indeed, since the reconstruction structuring element $K = \check{K}$ *and* $\emptyset \in K$, the radius of the circumscribing disk is precisely $\max_{k \in K} \|k\|$. Hence, the distance between A and $A \oplus K$ is more than the radius of the circumscribing disk of K.

Proposition 11: If $K = \check{K}$ and $0 \in K$, then $r(K) = \max_{k \in K} \|k\|$.

Proof:

$$r(K) = \min_{x \in K} \max_{y \in K} \|x - y\| \leq \max_{y \in K} \|0 - y\|$$
$$\leq \max_{y \in K} \|y\| \quad \text{and} \quad \max_{y \in K} \|y\|$$
$$= \tfrac{1}{2} \max_{y \in K} \|y - x + x + y\| \text{ for } x \in K$$
$$\leq \tfrac{1}{2}\left\{\max_{y \in K} \|y - x\| + \max_{y \in K} \|x + y\|\right\} \text{ for } x \in K$$
$$\leq \tfrac{1}{2}\left\{\max_{y \in K} \|x - y\| + \max_{y \in K} \|x - y\|\right\} \text{ for } x \in K$$
$$\leq \max_{y \in K} \|x - y\| \text{ for } x \in K$$
$$\leq \min_{x \in K} \max_{y \in K} \|x - y\| = r(K).$$ ■

Since $\rho_M(A, A \oplus K) \leq \max_{k \in K} \|k\|$ and $\max_{k \in K} \|k\| = r(K)$, we have $\rho_M(A, A \oplus K) \leq r(K)$. Also, since $A \bullet K \supseteq A$, $\rho_M(A \bullet K, A) = \rho(A \bullet K, A)$. Since $0 \in K$, $A \bullet K \subseteq A \oplus K$. Hence, $\rho_M(A \bullet K, A) = \rho(A \bullet K, A) \leq \rho((A \bullet K) \oplus K, A) = \rho(A \oplus K, A) \leq r(K)$.

It immediately follows that the distance between the minimal and maximal reconstructions, which differ only by a dilation by K, is no greater than the size of the reconstruction structuring element; that is, $\rho_M((F \cap S) \bullet K, (F \cap S) \oplus K) \leq r(K)$. When $F = F \circ K = F \bullet K$, $(F \cap S) \bullet K \subseteq F \subseteq (F \cap S) \oplus K$. Since the distance between the minimal and maximal reconstruction is no greater than $r(K)$, it is unsurprising that the distance between F and either of the reconstructions is no greater than $r(K)$.

Proposition 12: If $A \subseteq B \subseteq C$, then 1) $\rho_M(A, B) \leq \rho_M(A, C)$, and 2) $\rho_M(B, C) \leq \rho_M(A, C)$.

Proof:

1) Since $A \subseteq B$, $\rho_M(A, B) = \rho(B, A)$, then

$$\rho(B, A) = \max_{b \in B} \min_{a \in A} \|b - a\|$$
$$\leq \max_{c \in C} \min_{a \in A} \|c - a\| \text{ since } B \subseteq C$$
$$\leq \rho(C, A) = \rho_M(A, C) \text{ since } A \subseteq C.$$

2) The proof of (2) is similar to (1) with B taking the role of A and C taking the role of B. ■

Now it immediately follows that if $F = F \circ K = F \bullet K$, $\rho_M(F, (F \cap S) \oplus K) \leq r(K)$ and $\rho_M(F, (F \cap S) \bullet K) \leq r(K)$.

These distance bounds can actually be shown under slightly less restrictive conditions. Suppose that $F = F \circ K$. Then it follows that $F \subseteq (F \cap S) \oplus K$. Since $F \cap S \subseteq F$, $(F \cap S) \oplus K \subseteq F \oplus K$. Hence, $F \subseteq (F \cap S) \oplus K \subseteq F \oplus K$. But $\rho_M(F, F \oplus K) \leq r(K)$. Hence, $\rho_M(F, (F \cap S) \oplus K) \leq r(K)$ and $\rho_M((F \cap S) \oplus K, F \oplus K) \leq r(K)$. It goes similarly with the closing reconstruction.

When the image F is open under K, the distance between F and its sampling $F \cap S$ can be no greater than $r(K)$. Why? It is certainly the case that $F \cap S \subseteq F \subseteq (F \cap S) \oplus K$. Hence, $\rho_M(F, F \cap S) \leq \rho_M(F \cap S, (F \cap S) \oplus K) \leq r(K)$.

If two sets are both open under the reconstruction structuring element K, then the distance between the sets must be no greater than the distance between their samplings plus the size of K.

Proposition 13: If $A = A \circ K$ and $B = B \circ K$, then $\rho_M(A, B) \leq \rho_M(A \cap S, B \cap S) + r(K)$.

Proof: Consider $\rho(A, B)$. $\rho(A, B) \leq \rho(A, B \cap S)$. Since $A = A \circ K$, $A \subseteq (A \cap S) \oplus K$. Hence, $\rho(A, B) \leq \rho(A, B \cap S) \leq \rho((A \cap S) \oplus K, B \cap S) \leq \rho(A \cap S, B \cap S) + r(K)$. Similarly, since $B = B \circ K$, $\rho(B, A) \leq \rho(B \cap S, A \cap S) + r(K)$.

Now

$$\rho_M(A, B) = \max\{\rho(A, B), \rho(B, A)\}$$
$$\leq \max\{\rho(A \cap S, B \cap S) + r(k), \rho(B \cap S, A \cap S) + r(K)\}$$
$$= r(K) + \max\{\rho(A \cap S, B \cap S), \rho(B \cap S, A \cap S)\}$$
$$= r(K) + \rho_M(A \cap S, B \cap S).$$ ■

From this last result, it is easy to see that if F is closed under K, then the distance between F and its minimal reconstruction $(F \cap S) \bullet K$ is no greater than $r(K)$. Consider

$$\begin{aligned}\rho_M(F, (F \cap S) \bullet K) &\leq \rho_M\big(F \cap S, \big((F \cap S) \bullet K\big) \cap S\big) + r(K) \\ &= \rho_M(F \cap S, F \cap S) + r(K) = r(K).\end{aligned}$$

These distance relationships mean that just as the standard sampling theorem cannot produce a reconstruction with frequencies higher than the Nyquist frequency, the morphological sampling theorem cannot produce a reconstruction whose positional accuracy is better than the radius of the circumscribing disk of the reconstruction structuring element K. Since the diameter of this disk is just short of being large enough to contain two sample intervals, the morphological sampling theorem cannot produce a reconstruction whose positional accuracy is better than the sampling interval.

D. Examples

We use the example sets F_1, F_2, F_3, and F_4 in computing the distance between the original images and the sample reconstruction images. The values $\max_{y \in K} \|x - y\|$ for each $x \in K$ are shown in Fig. 10. The minimum value among them, $\sqrt{2}$, is the radius $r(K)$ since $r(K) = \min_{x \in K} \max_{y \in K} \|x - y\|$.

We now measure the distance between two sample reconstructions for all the example sets. To compute $\rho_M((F_1 \cap S) \bullet K, (F_1 \cap S) \oplus K)$, we first compute $\rho((F_1 \cap S) \oplus K, (F_1 \cap S) \bullet K)$ and $\rho((F_1 \cap S) \bullet K, (F_1 \cap S) \oplus K)$. The values $\min_{y \in (F_1 \cap S) \bullet K} \|x - y\|$ for all $x \in (F_1 \cap S) \oplus K$ are shown in Fig. 11. The maximum value among them, $\sqrt{2}$, is the distance $\rho((F_1 \cap S) \oplus K, (F_1 \cap S) \bullet K)$. Similarly, we can compute $\rho((F_1 \cap S) \bullet K, (F_1 \cap S) \oplus K)$ which equals 0. Thus, $\rho_M((F_1 \cap S) \bullet K, (F_1 \cap S) \oplus K)$ equals $\sqrt{2}$ which is exactly the radius $r(K)$. Similarly, the distance between two reconstructions for sets F_2, F_3, and F_4 can be measured and they are all equal to $r(K)$.

What is the distance $\rho_M(F, (F \cap S) \oplus K)$ for the example sets? Since $F_1 = (F_1 \cap S) \bullet K$, $\rho_M(F_1, (F_1 \cap S) \oplus K) = \rho_M((F_1 \cap S) \bullet K, (F_1 \cap S) \oplus K) = r(K)$. It is easy to see $\rho_M((F_2, (F_2 \cap S) \oplus K) = 0$ because $F_2 = (F_2 \cap S) \oplus K$. Fig. 12 shows the values $\min_{y \in F_3} \|x - y\|$ for all $x \in (F_3 \cap S) \oplus K$, their maximum value being $\rho((F_3 \cap S) \oplus K, F_3) = 1$. Since $F_3 \subseteq (F_3 \cap S) \oplus K$, $\rho(F_3, (F_3 \cap S) \oplus K)$ equals 0. Hence, $\rho_M((F_3 \cap S) \oplus K, F_3) = 1 < r(K)$.

The distance $\rho(F_4, (F_4 \cap S) \oplus K)$ is interesting since $F_4 \neq F_4 \circ K$. The $\min_{y \in (F_4 \cap S) \oplus K} \|x - y\|$ values for all $x \in F_4$ are shown in Fig. 13(a), their maximum value being $\rho(F_4, (F_4 \cap S) \oplus K) = 2$. The $\min_{y \in F_4} \|x - y\|$ values for all $x \in (F_4 \cap S) \oplus K$ are shown in Fig. 11(b), the maximum value is $\rho((F_4 \cap S) \oplus K, F_4) = 1$. Thus, the distance $\rho_M(F_4, (F_4 \cap S) \oplus K)$ is equal to 2 which is greater than $r(K)$. This shows why the condition $F = F \circ K$ is required to bound the difference between F and its maximum reconstruction $(F \cap S) \oplus K$. Similarly, we

	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{2}$	
	$\sqrt{5}$	$\sqrt{2}$	$\sqrt{5}$	
	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{2}$	

Fig. 10. The $\max_{y \in K} \|x - y\|$ values for all $x \in K$, where K is the digital disk having radius $\sqrt{2}$.

	$\sqrt{2}$	1	1	1	$\sqrt{2}$	
	1	0	0	0	1	
	1	0	0	0	1	
	1	0	0	0	1	
	$\sqrt{2}$	1	1	1	$\sqrt{2}$	

Fig. 11. The $\min_{y \in (F_1 \cap S) \bullet K} \|x - y\|$ for all $x \in (F_1 \cap S) \oplus K$.

	1	1	1	
	0	0	0	
	0	0	0	
	0	0	0	
	1	1	1	

Fig. 12. $\min_{y \in F_3} \|x - y\|$ for each $x \in (F_3 \cap S) \oplus K$.

	2		0	0	0		2	
	2	1	0	0	0	1	2	
			0	0	0			

(a)

	1	1	1	
	0	0	0	
	0	0	0	
	0	0	0	
	1	1	1	

(b)

Fig. 13. (a) Values for $\min_{y \in (F_4 \cap S) \oplus K} \|x - y\|$ for all $x \in F_4$. (b) Values for $\min_{y \in F_4} \|x - y\|$ for all $x \in (F_4 \cap S) \oplus K$. The maximum among all these values is 2. Hence, $\rho_M((F_4 \cap S) \oplus k) = 2 > r(K)$.

find

$$\rho_M\big((F_1 \cap S) \oplus K, F_1 \oplus K\big) = 0 < r(K)$$

$$\rho_M\big((F_2 \cap S) \oplus K, F_2 \oplus K\big) = \sqrt{2} = r(K)$$

$$\rho_M\big((F_3 \cap S) \oplus K, F_3 \oplus K\big) = 1 < r(K)$$

$$\rho_M\big((F_4 \cap S) \oplus K, F_4 \oplus K\big) = 2 > r(K).$$

Note that since $F_4 \neq F_4 \circ K$, $\rho_M((F_4 \cap S) \oplus K, F_4 \oplus K) \not\leq r(K)$. Using the minimum reconstruction, the positional accuracy for the example sets are

$$\rho_M\big(F_1, (F_1 \cap S) \bullet K\big) = 0 < r(K)$$

$$\rho_M\big(F_2, (F_2 \cap S) \bullet K\big) = \sqrt{2} = r(K)$$

$$\rho_M\big(F_3, (F_3 \cap S) \bullet K\big) = 1 < r(K)$$

$$\rho_M\big(F_4, (F_4 \cap S) \bullet K\big) = 3 > r(K).$$

Also, since $F_4 \neq F_4 \bullet K$, $\rho_m(F_4, (F_4 \cap S) \bullet K) \not\leq r(K)$.

E. Binary Digital Morphological Sampling Theorem

This subsection summarizes the results developed in the previous subsections. These results constitute the binary digital morphological sampling theorem.

Theorem 1—Binary Digital Morphological Sampling Theorem: Let $F, K, S \subseteq E^N$. Suppose K and S satisfy the sampling conditions

1) $S \oplus S = S$
2) $S = \check{S}$
3) $K \cap S = \{0\}$
4) $K = \check{K}$
5) $x \in K_y$ implies $K_x \cap K_y \cap S \neq \varnothing$.

Then

1) $F \cap S = [(F \cap S) \bullet K] \cap S$.
2) $F \cap S = [(F \cap S) \oplus K] \cap S$.
3) $(F \cap S) \bullet K \subseteq F \bullet K$.
4) $F \circ K \subseteq (F \cap S) \oplus K$.
5) If $F = F \circ K = F \bullet K$, then $(F \cap S) \bullet K \subseteq F \subseteq (F \cap S) \oplus K$.
6) If $A = A \bullet K$ and $A \cap S = F \cap S$, then $A \subseteq (F \cap S) \bullet K$ implies $A = (F \cap S) \bullet K$.
7) If $A = A \circ K$ and $A \cap S = F \cap S$, then $A \supseteq (F \cap S) \oplus K$ implies $A = (F \cap S) \oplus K$.
8) If $F = F \bullet K$, then $\rho_M(F, (F \cap S) \bullet K) \leq r(K)$.
9) If $F = F \circ K$, then $\rho_M((F \cap S) \oplus K, F) \leq r(K)$.

IV. Morphologically Operating in the Sampled Domain

Section III established the relationship between the information contained in the sampled set and the information contained in the unsampled set. It shows that a minimal and maximal reconstruction can be computed from the sampled set. When the set is smooth enough with respect to the sampling S (that is, when the set is both open and closed under the reconstruction structuring element), then the minimal and maximal reconstructions bound the unsampled set, differing from it by no more than the sampling interval length.

Not addressed in Section III is the relationship between the computationally more efficient procedure of morphologically operating in the sampled domain versus the less computationally efficient procedure of morphologically operating in the unsampled domain. In this section we quantify just exactly how close a morphological operation in the sampled domain can come to the corresponding morphological operation in the original domain. Thus, we answer the question of how to compute the largest length of sampling interval which can produce an answer close enough to the desired answer when morphologically operating in the sampled domain.

The first proposition shows that a sampled dilation contains the dilation of the sampled sets and a sampled erosion is contained in the erosion of the sampled sets.

Proposition 14: Let $B \subseteq E^N$ be the structuring element employed in the dilation or erosion. Then

1) $(F \cap S) \oplus (B \cap S) \subseteq (F \oplus B) \cap S$.
2) $(F \cap S) \ominus (B \cap S) \supseteq (F \ominus B) \cap S$.

Proof:

1) $F \cap S \subseteq F$ and $B \cap S \subseteq B$. Hence, $(F \cap S) \oplus (B \cap S) \subseteq F \oplus B$. Also, $F \cap S \subseteq S$ and $B \cap S \subseteq S$. Hence, $(F \cap S) \oplus (B \cap S) \subseteq S \oplus S$. But $S \oplus S = S$. Then, $(F \cap S) \oplus (B \cap S) \subseteq (F \oplus B) \cap S$.

2) By (1) $[(F \ominus B) \cap S] \oplus (B \cap S) \subseteq [(F \ominus B) \oplus B] \cap S = (F \circ B) \cap S \subseteq F \cap S$. But $[(F \ominus B) \cap S] \oplus (B \cap S) \subseteq (F \cap S)$ if and only if $(F \cap S) \ominus (B \cap S) \supseteq (F \ominus B) \cap S$. ■

Unfortunately, the containment relations cannot, in general, be strengthened to equalities. But we can determine the conditions under which the equality occurs and the distance between sets such as $(F \cap S) \oplus (B \cap S)$ and $(F \oplus B) \cap S$. In the sampled domain, we compare the scheme of sampling and then performing the dilation in the sampled domain to dilating first and then sampling. We also inquire about how different things could be in the unsampled domain by comparing performing the dilation in the sampled space and then reconstructing versus performing the dilation in the unsampled domain. The next proposition shows that this difference in the sampled domain cannot be more than $2r(K)$.

Proposition 15: If $F = F \circ K$ and $B = B \circ K$, then $\rho_M((F \oplus B) \cap S, (F \cap S) \oplus (B \cap S)) \leq 2r(K)$.

Proof: First consider $\rho((F \oplus B) \cap S, (F \cap S) \oplus (B \cap S)) \leq \rho(F \oplus B, (F \cap S) \oplus (B \cap S))$. Since $F = F \circ K$ and $B = B \circ K$, $F \subseteq (F \cap S) \oplus K$ and $B \subseteq (B \cap S) \oplus K$. Hence,

$$\begin{aligned}\rho(F \oplus B, (F \cap S) \oplus (B \cap S)) \\ \leq \rho((F \cap S) \oplus K \oplus (B \cap S) \oplus K, (F \cap S) \\ \oplus (B \cap S))\end{aligned}$$

$$\leq \rho\big([(F \cap S) \oplus (B \cap S)] \oplus K \oplus K, (F \cap S) \oplus (B \cap S)\big)$$

$$\leq r(K \oplus K) \leq 2r(K).$$

Next note that $\rho((F \cap S) \oplus (B \cap S), (F \oplus B) \cap S) = 0$. Since $(F \cap S) \oplus (B \cap S) \subseteq (F \oplus B) \cap S$. Now $\rho_M((F \oplus B) \cap S, (F \cap S) \oplus (B \cap S)) = \max\{\rho((F \oplus B) \cap S, (F \cap S) \oplus (B \cap S)), \rho((F \cap S) \oplus (B \cap S), (F \oplus B) \cap S)\} \leq \max\{2r(K), 0\} = 2r(K)$. ■

Whereas dilation tends to suppress differences, erosion tends to accentuate differences. Consider the following example. Let F be a disk of radius 12, and let B be a disk of radius 10. Then $F \ominus B$ is a disk of radius 2. Now define F' to be a disk of radius 12 with its center point deleted. Notice that the pseudoset distance between F and F' is zero. But although F' close to F, $F' \ominus B = \varnothing$. The difference of one point makes all the difference.

More formally, consider the difference between the erosion of F and the erosion of $F \oplus K$.

$$\rho_M\big((F \oplus K) \ominus B, F \ominus B\big) = \rho\big((F \oplus K) \ominus B, F \ominus B\big) \geq \rho\big((F \ominus B) \oplus K, F \ominus B\big)$$

since $(F \oplus K) \ominus B \subseteq (F \ominus B) \oplus K$ where $\rho((F \ominus B) \oplus K, F \ominus B)$ is no greater than, and could be as close as possible to, $r(K)$.

Thus, we cannot expect that the difference between performing an erosion in the sampled domain versus performing a sampling of the erosion in the unsampled domain is no greater than the size of K. However, we do obtain set bounding relationships for dilation and erosion using the following relationships.

Dilating (eroding) a sampled set by a sampled structuring element is equivalent to sampling the dilation (erosion) of the unsampled set by the sampled structuring element.

Lemma:

1) $(F \cap S) \oplus (B \cap S) = [F \oplus (B \cap S)] \cap S$
2) $(F \cap S) \ominus (B \cap S) = [F \ominus (B \cap S)] \cap S.$

Proof:

1) $$[F \oplus (B \cap S)] \cap S = \Big(\bigcup_{x \in B \cap S} F_x\Big) \cap S = \bigcup_{x \in B \cap S} (F_x \cap S).$$

But $x \in S$ implies $S = S_x$. Hence,

$$[F \oplus (B \cap S)] \cap S = \bigcup_{x \in B \cap S} F_x \cap S_x = \bigcup_{x \in B \cap S} (F \cap S)_x = (F \cap S) \oplus (B \cap S).$$

2) $$[F \ominus (B \cap S)] \cap S = \Big(\bigcap_{x \in B \cap S} F_{-x}\Big) \cap S = \Big(\bigcap_{x \in \check{B} \cap \check{S}} F_x\Big) \cap S = \bigcap_{x \in \check{B} \cap \check{S}} (F_x \cap S)$$

$$= \bigcap_{x \in \check{B} \cap \check{S}} (F_x \cap S_x) \text{ since } x \in S \text{ implies } S_x = S$$

$$= \bigcap_{x \in \check{B} \cap \check{S}} (F \cap S)_x = (F \cap S) \ominus (B \cap S).$$ ■

Moreover, the dilation of the minimal reconstruction by a structuring element B open under K is contained in the dilation of the maximal reconstruction by the sampled structuring element $B \cap S$.

Lemma: Let $B = B \circ K$. Then $[(F \cap S) \bullet K] \oplus B \subseteq [(F \cap S) \oplus K] \oplus (B \cap S)$.

Proof: Let $x \in [(F \cap S) \bullet K] \oplus B$. Then there exists an $f \in (F \cap S) \bullet K$ and $b \in B$ such that $x = f + b$. Since $B = B \circ K$, $b \in B$ implies there exists a y such that $b \in K_y \subseteq B$. But because of the sampling constraint between K and S, $b \in K_y$ implies $K_b \cap K_y \cap S \neq \varnothing$. Therefore, there exists a $z \in K_b \cap S$. Now $z \in K_b$ implies that $z = k + b$ for some $k \in K$. Since it is also the case that $z \in K_y$, it must be that $z \in B$ because $K_y \subseteq B$. Recall that $x = f + b = f + z - k = (f - k) + z$. Since $f \in (F \cap S) \bullet K = [(F \cap S) \oplus K] \ominus K$ and since $-k \in \check{K} = K$, $f - k \in ([(F \cap S) \oplus K] \ominus K) \oplus K = (F \cap S) \oplus K$. Since $z \in B$ and $z \in S$, $z \in B \cap S$. Finally, $f - k \in (F \cap S) \oplus K$ and $z \in B \cap S$ imply $x = (f - k) + z \in [(F \cap S) \oplus K] \oplus (B \cap S)$. ■

Now we see that dilation in the sampled domain and dilation in the unsampled domain are equivalent exactly when the structuring element B of the dilation is open under K, and when the image F is its minimal reconstruction.

Theorem 2: Let $B = B \circ K$. Then $(F \cap S) \oplus (B \cap S) = \{[(F \cap S) \bullet K] \oplus B\} \cap S$.

Proof: $(F \cap S) \oplus (B \cap S) = ((F \cap S) \oplus B) \cap S$ is always true. Since $F \cap S \subseteq (F \cap S) \bullet K$, $((F \cap S) \oplus B) \cap S \subseteq \{[(F \cap S) \bullet K] \oplus B\} \cap S$. But $[(F \cap S) \bullet K] \oplus B \subseteq [(F \cap S) \oplus K] \oplus (B \cap S)$ when $B = B \circ K$. Hence, $(F \cap S) \oplus (B \cap S) \subseteq \{[(F \cap S) \bullet K] \oplus B\} \cap S \subseteq \{[(F \cap S) \oplus K] \oplus (B \cap S)\} \cap S$. Now $\{[(F \cap S) \oplus K] \oplus (B \cap S)\} \cap S = \{[(F \cap S) \oplus K] \cap S\} \oplus (B \cap S)$. Since $[(F \cap S) \oplus K] \cap S = F \cap S$ always holds under the sampling conditions, there results $(F \cap S) \oplus (B \cap S) \subseteq [(F \cap S) \bullet K] \oplus (B \cap S) \subseteq (F \cap S) \oplus (B \cap S)$ so that $(F \cap S) \oplus (B \cap S) = [(F \cap S) \bullet K] \oplus (B \cap S)$. ■

The equality relationship established in the theorem immediately leads to a set bounding relationship for dilation.

$$[(F \ominus K) \oplus B] \cap S \subseteq \{[(F \cap S) \bullet K] \oplus B\} \cap S$$
$$= (F \cap S) \oplus (B \cap S)$$
$$\subseteq (F \oplus B) \cap S.$$

Also from the theorem, it becomes apparent that the difference between the maximally reconstructed dilation and the dilation of the minimal reconstruction can be no more than the size of K when B is open under K. This happens because

$$\rho_M([(F \cap S) \oplus (B \cap S)] \oplus K, [(F \cap S) \bullet K] \oplus B)$$
$$\leq \rho_M(\{[(F \cap S) \oplus (B \cap S)] \oplus K\} \cap S, \{[(F \cap S) \bullet K] \oplus B\} \cap S) + r(K)$$
$$\leq \rho_M((F \cap S) \oplus (B \cap S), (F \cap S) \oplus (B \cap S)) + r(K) = r(K).$$

Similarly, eroding a sampled image by a sampled structuring element is equivalent to eroding the maximal reconstruction by the structuring element and then sampling when the structuring element is open under K.

Theorem 3: Let $B = B \circ K$. Then $(F \cap S) \ominus (B \cap S) = \{[(F \cap S) \oplus K] \ominus B\} \cap S$.

Proof: The sampling conditions imply $[(F \cap S) \oplus K] \cap S = F \cap S$. Hence,

$$(F \cap S) \ominus (B \cap S)$$
$$= \{[(F \cap S) \oplus K] \cap S\} \ominus (B \cap S)$$
$$= \{[(F \cap S) \oplus K] \ominus (B \cap S)\} \cap S$$
$$\supseteq \{[(F \cap S) \oplus K] \ominus B\} \cap S. \quad \blacksquare$$

Under the sampling conditions, $(F \cap S) \ominus (B \cap S) \subseteq S$. So to complete the equality, we need to show that $(F \cap S) \ominus (B \cap S) \subseteq [(F \cap S) \oplus K] \ominus B$. Let $x \in (F \cap S) \ominus (B \cap S)$. Then $(B \cap S)_x \subseteq F \cap S$. Since $B = B \circ K$, $B \subseteq (B \cap S) \oplus K$. Hence, $B_x \subseteq (B \cap S)_x \oplus K$. But $(B \cap S)_x \subseteq F \cap S$ so that $B_x \subseteq (F \cap S) \oplus K$. Now by definition of erosion, if $B_x \subseteq (F \cap S) \oplus K$, then $x \in [(F \cap S) \oplus K] \ominus B$.

Theorem 3 immediately leads to some set bounding relationships for erosion

$$(F \ominus B) \cap S \subseteq (F \cap S) \ominus (B \cap S)$$
$$= \{[(F \cap S) \oplus K] \ominus B\} \cap S$$
$$\subseteq [(F \oplus K) \ominus B] \cap S.$$

Theorem 3 also makes it apparent that the difference between the maximally reconstructed erosion and the erosion of the maximal reconstruction can be no more than the size of K when both B and the erosion of the maximal reconstruction are open under K. This happens because

$$\rho_M([(F \cap S) \ominus (B \cap S)] \oplus K, [(F \cap S) \oplus K] \ominus B)$$
$$\leq \rho_M(\{[(F \cap S) \ominus (B \cap S)] \oplus K\} \cap S, \{[(F \cap S) \oplus K] \ominus B\} \cap S) + r(K)$$
$$\leq \rho_M((F \cap S) \ominus (B \cap S), (F \cap S) \ominus (B \cap S)) + r(K) = r(K).$$

Just as it was the case that dilating (eroding) a sampled set by a sampled structuring element is equivalent to sampling the dilation (erosion) of the unsampled set by the sampled structuring element, so it is also the case that opening (closing) a sampled set by a sampled structuring element is equivalent to sampling the opening (closing) of the unsampled set by the sampled structuring element. These relationships are useful in establishing when the opening and closing operation are equivalent in the sampled and unsampled domain.

Proposition 16: $[F \circ (B \cap S)] \cap S = (F \cap S) \circ (B \cap S)$.

Proof: Let $x \in [F \circ (B \cap S)] \cap S$. Then $x \in F \circ (B \cap S)$ and $x \in S$. But $x \in F \circ (B \cap S)$ if and only if for some $y \in F \ominus (B \cap S)$, $x \in (B \cap S)_y \subseteq F$. Now $x \in (B \cap S)_y$ implies $x = b + y$ where $b \in B \cap S$. Then $y = x - b$. Since $b \in S$ and since $S - \check{S}$, $-b \in S$. Since $x \in S$ and $-b \in S$, $x - b \in S \oplus S$. But $S \oplus S = S$. Then $y \in S$. Now we show that $y \in S$ and $(B \cap S)_y \subseteq F$ imply $(B \cap S)_y \subseteq F \cap S$. Let $z \in (B \cap S)_y$ since $(B \cap S)_y \subseteq F$, $z \in F$. Now $z \in (B \cap S)_y = B_y \cap S_y = B_y \cap S$ since $y \in S$. Hence, $z \in S$. Hence, $z \in F \cap S$. Finally, $x \in (B \cap S)_y \subseteq F \cap S$ implies $x \in (F \cap S) \circ (B \cap S)$. Thus, $[F \circ (B \cap S)] \cap S \subseteq (F \cap S) \circ (B \cap S)$.

Now suppose $x \in (F \cap S) \circ (B \cap S)$. Then there exists a $y \in (F \cap S) \ominus (B \cap S)$ such that $x \in (B \cap S)_y \subseteq F \cap S$. But $F \cap S \subseteq F$. Then $x \in (B \cap S)_y \subseteq F$ and this implies that $x \in F \circ (B \cap S)$. Also, $x \in F \cap S$ implies $x \in S$. Then $x \in [F \circ (B \cap S)] \cap S$. This establishes that $(F \cap S) \circ (B \cap S) \subseteq [F \circ (B \cap S)] \cap S$. $\blacksquare$

Proposition 17: $[F \bullet (B \cap S)] \cap S = (F \cap S) \bullet (B \cap S)$.

Proof: Let $x \in [F \bullet (B \cap S)] \cap S$. Then $x \in F \bullet (B \cap S)$ and $x \in S$. But $x \in F \bullet (B \cap S)$ if and only if $x \in (B \check{\cap} S)_y$ implies $x \in (B \check{\cap} S)_y \cap F \neq \varnothing$. Let y satisfy $x \in (B \check{\cap} S)_y$. Then $x = b + y$ where $b \in \check{B} \cap \check{S}$. Then $y = x - b$. Since $x \in S$ and $-b \in S$, $y \in S \oplus S = S$. Now if $y \in S$, then $(B \cap S)_y \cap F = \check{B}_y \cap \check{S}_y \cap F = \check{B}_y \cap \check{S}_y \cap S \cap F = (B \check{\cap} S)_y \cap (F \cap S)$. Now $x \in (B \check{\cap} S)_y$ implies $(B \check{\cap} S)_y \cap F \neq \varnothing$. Since $(B \check{\cap} S)_y \cap F = (B \check{\cap} S)_y \cap (F \cap S)$, $(B \check{\cap} S)_y \cap (F \cap S) \neq \varnothing$. This implies that $x \in (F \cap S) \bullet (B \cap S)$.

Let $x \in (F \cap S) \bullet (B \cap S)$. Then $x \in (B \check{\cap} S)_y$ implies $(B \check{\cap} S)_y \cap (F \cap S) \neq \varnothing$. But $(B \check{\cap} S)_y \cap (F \cap S) \subseteq (B \check{\cap} S)_y \cap F$. Hence, $(B \check{\cap} S)_y \cap F \neq$

$\varnothing$ and this implies that $x \in F \bullet (B \cap S)$. Also,

$$\begin{aligned}(F \cap S) \bullet (B \cap S) &= [(F \cap S) \ominus (B \cap S)] \oplus (B \cap S) \\ &= \{[F \ominus (B \cap S)] \cap S\} \oplus (B \cap S) \\ &\subseteq S \oplus (B \cap S) \subseteq S \oplus S \subseteq S.\end{aligned}$$

Hence, $x \in S$. Finally, $x \in F \bullet (B \cap S)$ and $x \in S$ imply $x \in [F \bullet (B \cap S)] \cap S$. ■

The bounding relationships between the sampled and unsampled domains for the opening and closing operation now follow immediately.

Theorem 4: Suppose $B = B \circ K$, then

1) $\{F \circ [(B \cap S) \oplus K]\} \cap S \subseteq (F \cap S) \circ (B \cap S) \subseteq \{[(F \cap S) \oplus K] \circ B\} \cap S$

2) $\{[(F \cap S) \bullet K] \bullet B\} \cap S \subseteq (F \cap S) \bullet (B \cap S) \subseteq \{F \bullet [(B \cap S) \oplus K]\} \cap S$.

Proof:

1) Notice that $[(B \cap S) \oplus K] \circ (B \cap S) = (B \cap S) \oplus K$. Under this condition, $\{F \circ [(B \cap S) \oplus K]\} \cap S \subseteq [F \circ (B \cap S)] \cap S$. But by a previous proposition $[F \circ (B \cap S)] \cap S = (F \cap S) \circ (B \cap S)$. Now suppose $x \in (F \cap S) \circ (B \cap S)$. Then there exists a y such that $x \in (B \cap S)_y \subseteq F \cap S$. But $(B \cap S)_y \subseteq (F \cap S)$ implies $(B \cap S)_y \oplus K \subseteq (F \cap S) \oplus K$ since dilation is an increasing operation. Hence, $[(B \cap S) \oplus K]_y \subseteq (F \cap S) \oplus K$. Since $B = B \circ K$, $B \subseteq (B \cap S) \oplus K$. Then, $B_y \subseteq (F \cap S) \oplus K$. Also, $x \in (B \cap S)_y$ implies $x \in B_y$. Finally, $x \in B_y \subseteq (F \cap S) \oplus K$ implies $x \in [(F \cap S) \oplus K] \circ B$.

2) By a previous proposition $(F \cap S) \bullet (B \cap S) = [F \bullet (B \cap S)] \cap S$. Since $[(B \cap S) \oplus K] \circ (B \cap S) = (B \cap S) \oplus K$, $[F \bullet (B \cap S)] \cap S \subseteq \{F \bullet [(B \cap S) \oplus K]\} \cap S$. Let $R = (F \cap S) \bullet K$. Since $B = B \circ K$, $R \oplus B$ is open under K. Hence, $R \oplus B \subseteq [(R \oplus B) \cap S] \oplus K$. Now

$$\begin{aligned}(R \bullet B) \cap S &= [(R \oplus B) \ominus B] \cap S \\ &\subseteq (\{[(R \oplus B) \cap S] \oplus K\} \ominus B) \cap S.\end{aligned}$$

But the sampled erosion of a maximal reconstruction is the erosion of the sampled set by the sampled structuring element. Hence,

$$\begin{aligned}&(\{[(R \oplus B) \cap S] \oplus K\} \ominus B) \cap S \\ &\quad = ([(R \oplus B) \cap S]) \ominus (B \cap S).\end{aligned}$$

And the sampled dilation of a minimal reconstruction is the dilation of the sampled set by the sampled structuring element. Hence,

$$\begin{aligned}&[(R \oplus B) \cap S] \ominus (B \cap S) \\ &\quad = [(R \cap S) \oplus (B \cap S)] \ominus (B \cap S).\end{aligned}$$

Finally, $R \cap S = [(F \cap S) \bullet K] \cap S = F \cap S$ so that $([(F \cap S) \bullet K] \bullet B) \cap S \subseteq (F \cap S) \bullet (B \cap S)$. ■

The bounding relationships immediately imply the following equivalence for the opening and closing operations between the sampled and unsampled domains.

Theorem 5: Suppose $B = B \circ K$.

1) If $F = (F \cap S) \oplus K$ and $B = (B \cap S) \oplus K$, then $(F \cap S) \circ (B \cap S) = (F \circ B) \cap S$.

2) If $F = (F \cap S) \bullet K$ and $B = (B \cap S) \oplus K$, then $(F \cap S) \bullet (B \cap S) = (F \bullet B) \cap S$.

Proof:

1) If $F = (F \cap S) \oplus K$ and $B = (B \cap S) \oplus K$, the bounding relationship for opening becomes

$$(F \circ B) \cap S \subseteq (F \cap S) \circ (B \cap S) \subseteq (F \circ B) \cap S$$

from which we immediately obtain $(F \circ B) \cap S = (F \cap S) \circ (B \cap S)$.

2) If $F = (F \cap S) \bullet K$ and $B = (B \cap S) \oplus K$, the bounding relationship for closing becomes

$$(F \bullet B) \cap S \subseteq (F \cap S) \bullet (B \cap S) \subseteq (F \bullet B) \cap S$$

from which we immediately obtain $(F \bullet B) \cap S = (F \cap S) \bullet (B \cap S)$. ■

A. Examples

A simple example illustrates the bounding relationships of morphological operations operating in the pre- and postsampled domain. The sample set S and the set K we used are those defined in the previous examples (see Fig. 5). The sets F, B, and K are defined in Fig. 14. It is clear that $B = B \circ K$. In Fig. 15, we show the results of down-sampling every other row and every other column, $F \cap S$, $B \cap S$, and the sampled domain morphological operations, $(F \cap S) \oplus (B \cap S)$, $(F \cap S) \ominus (B \cap S)$. The results $[(F \cap S) \bullet K] \oplus B$, $[(F \cap S) \oplus K] \ominus B$, $\{[(F \cap S) \bullet K] \oplus B\} \cap S$, and $\{[(F \cap S) \oplus K] \ominus B\} \cap S$ are shown in Fig. 16. Note that the following equalities hold:

$$(F \cap S) \oplus (B \cap S) = \{[(F \cap S) \bullet K] \oplus B\} \cap S$$

and

$$(F \cap S) \ominus (B \cap S) = \{[(F \cap S) \oplus K] \ominus B\} \cap S.$$

Fig. 17 shows $(F \oplus B) \cap S$, $(F \oplus B) \cap S$, $(F \oplus (B \cap S))$, and $(F \ominus (B \cap S))$. Note that the following are true:

$$(F \cap S) \oplus (B \cap S) \subseteq (F \oplus B) \cap S$$

and

$$(F \cap S) \ominus (B \cap S) \supseteq (F \ominus B) \cap S.$$

It can be easily verified that

$$(F \cap S) \oplus (B \cap S) = [F \oplus (B \cap S)] \cap S$$

and

$$(F \cap S) \ominus (B \cap S) = [F \ominus (B \cap S)] \cap S.$$

In practical multiresolution image processing applications, we would like to perform morphological operations

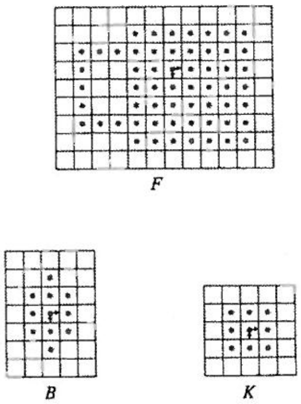


Fig. 14. The sets F, B, and K.

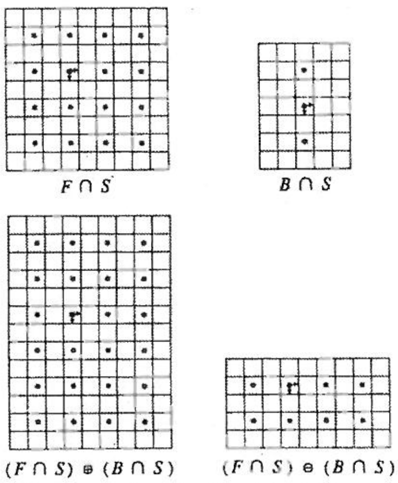


Fig. 15. The results of sampling the F and B of Fig. 14 and performing the dilation and erosion of $F \cap S$ by $B \cap S$ in the sampled domain.

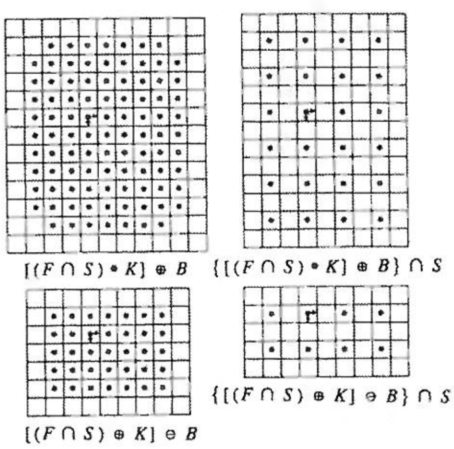


Fig. 16. The dilation and erosion of the minimal and maximal reconstruction of F by the structuring element B, and also the sampling of this dilation and erosion.

in the sampled domain to reduce the computational expense. How well can a morphological operation be performed in the sampled domain rather than the original domain can be answered by the relationships and distances between $(F \cap S) \oplus (B \cap S)$ and $(F \oplus B) \cap S$ as well as $(F \cap S) \ominus (B \cap S)$ and $(F \ominus B) \cap S$. Unfortunately, the distance

$$\rho_M((F \cap S) \oplus (B \cap S), (F \oplus B) \cap S) < 2r(K)$$

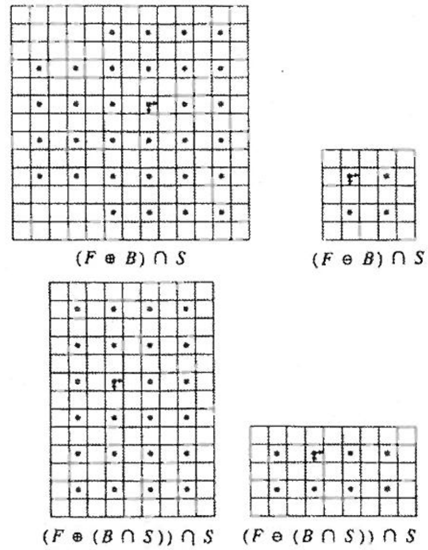


Fig. 17. Some morphological operations in the original domain followed by sampling.

					0		0		0		0	
	4		2		0		0		0		0	
	4		2		0		0		0		0	
	4		2		0		0		0		0	
	4		2		0		0		0		0	
					0		0		0		0	

Fig. 18. The values of $\min_{y \in (F \cap S) \oplus (B \cap S)}$ for all $x \in (F \oplus B) \cap S$.

can be guaranteed only when $F = F \circ K$ and $B = B \circ K$. It can be very big when the set F is not open. The set F of Fig. 14 is an example having a large difference between the pre- and postsampled dilations because the conditions $F = F \circ K$ and $B = B \circ K$ are not satisfied.

We now show the distances between the pre- and postsampled morphological operations. We first check the distance between $(F \cap S) \oplus (B \cap S)$ and $(F \oplus B) \cap S$. The $\min_{y \in (F \cap S) \oplus (B \cap S)} \|x - y\|$ values for all $x \in (F \oplus B) \cap S$ are shown in Fig. 18; their maximum value is $\rho((F \oplus B) \cap S, (F \cap S) \oplus (B \cap S)) = 4$. Since $(F \cap S) \oplus (B \cap S) \subset (F \oplus B) \cap S$, the distance $\rho((F \cap S) \oplus (B \cap S), (F \oplus B) \cap S) = 0$. Thus, $\rho_M((F \cap S) \oplus (B \cap S), (F \oplus B) \cap S) = 4$. Note that

$$\rho_M((F \cap S) \oplus (B \cap S), (F \oplus B) \cap S) = 4 > 2r(K)$$

since $F \neq F \circ K$. Suppose $F' = F \circ K$ and $B' = B \circ K$. Fig. 19 shows the results of $(F' \cap S) \oplus (B' \cap S)$ and

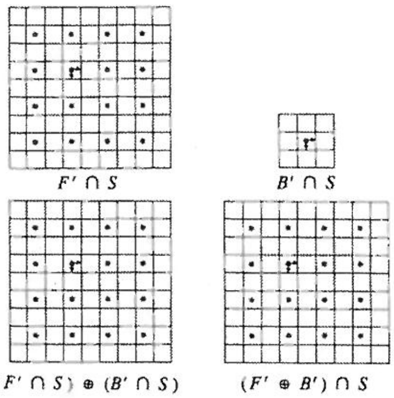


Fig. 19. The results of $(F' \cap S) \oplus (B' \cap S)$ and where $F' = F \circ K$ and $B' = B \circ K$. (See Fig. 14 for the definition of F, B, and K.)

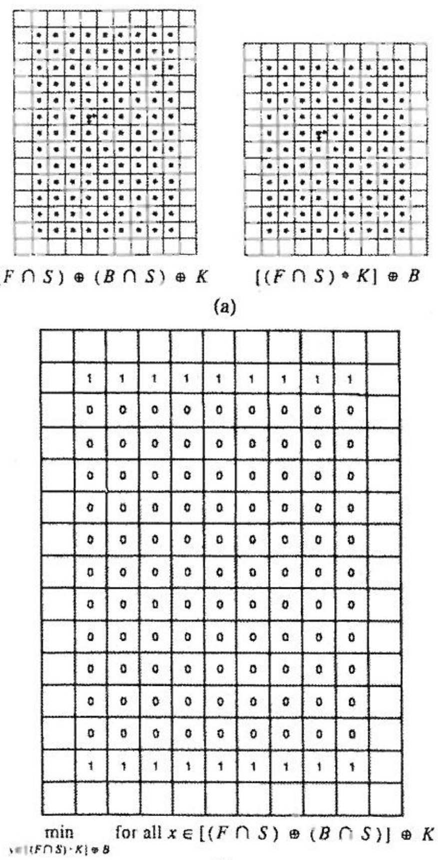


Fig. 20. The distance between the result produced by reconstructing the morphological dilation done in the sampled domain and the dilation of the minimal reconstruction done in the original domain must be less than $r(K) = \sqrt{2}$.

$(F' \oplus B') \cap S$. Since $(F' \cap S) \oplus (B' \cap S) = (F' \oplus B') \cap S$ in this example, we find that

$$\rho_M((F' \cap S) \oplus (B' \cap S), (F' \oplus B') \cap S) = 0 < 2r(K).$$

Now we check the distances between the maximally reconstructed dilation (erosion) and the dilation (erosion) of the minimal (maximal) reconstruction, $\rho_M([(F \cap S) \oplus (B \cap S)] \oplus K, [(F \cap S) \bullet K] \oplus B)$ ($\rho_M([(F \cap S) \ominus (B \cap S) \oplus K, [(F \cap S) \oplus K] \ominus B)$). $[(F \cap S) \oplus (B \cap S)] \oplus K$ and $[(F \cap S) \bullet K] \oplus B$ are shown in Fig. 20. The values of $\min_{y \in [(F \cap S) \bullet K] \oplus B} \|x - y\|$ for all $x \in [(F \cap S) \oplus (B \cap S)] \oplus K$ are shown in Fig. 20; their maximum is $\rho_M([(F \cap S) \oplus (B \cap S)] \oplus K, [(F \cap S) \bullet K] \oplus B) = 1$. Since $[(F \cap S) \bullet K] \oplus B \subseteq [(F \cap S) \oplus (B \cap S)] \oplus K$, this implies $\rho([(F \cap S) \bullet K] \oplus B, [(F \cap S) \oplus (B \cap S)] \oplus K) = 0$. Hence, $\rho_M([(F \cap S) \bullet K] \oplus B, [(F \cap S) \oplus (B \cap S)] \oplus K) = 1 \leq r(K)$. $[(F \cap S) \ominus (B \cap S)] \oplus K$ and $[(F \cap S) \oplus K] \ominus B$ are shown in Fig. 21. Note that $[(F \cap S) \oplus K] \ominus B$ is open under K. The values of $\min_{y \in [(F \cap S) \oplus K] \ominus B} \|x - y\|$ for all $x \in [(F \cap S) \ominus (B \cap S)] \oplus K$ are shown in Fig. 21; their maximum is $\rho([(F \cap S) \ominus (B \cap S)] \oplus K, [(F \cap S) \oplus K] \ominus B) = 1$. Since $[(F \cap S) \oplus K] \ominus B \subseteq [(F \cap S) \ominus (B \cap S)] \oplus K$, this implies $\rho([(F \cap S) \oplus K] \ominus B, [(F \cap S) \ominus (B \cap S)] \oplus K) = 0$. Hence, $\rho_M([(F \cap S) \oplus K] \ominus B, [(F \cap S) \ominus (B \cap S)] \oplus K) = 1 \leq r(K)$.

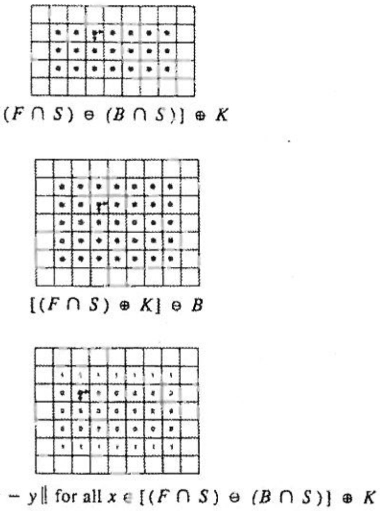


Fig. 21. The distance between the result produced by reconstructing the morphological erosion done in the sample domain and the erosion of the maximal reconstruction done in the original domain must be less than $r(K) = \sqrt{2}$.

V. The Grayscale Morphological Sampling Theorem

In this section we present the extension of the morphological sampling theorem from the binary case to the grayscale case. We begin by reviewing some definitions and results [8].

We adopt the convention that the first $(N - 1)$ coordinates of the N-tuples in a set $A \subseteq E^N$ constitute the spatial domain of A, and the Nth coordinate represents the surface, i.e., $A \subseteq E^{N-1} \times E$. For grayscale images, $N = 3$ and an image is a function $f = E \times E \rightarrow E$. A set $A \subseteq E^{N-1} \times E$ is an *umbra* if and only if $(x, z) \in A$ implies that $(x, z) \in A$ for every $z \leq y$. The *top* of A is a function $T[A]$ mapping the spatial domain of A, $\{x \in E^{N-1} | \text{ for some } y \in E, (x, y) \in A\}$, to the "top surface" of A,

$$T[A](x) = \max\{y \in E | (x, y) \in A\}.$$

If $F \subseteq E^{N-1}$ and $f: F \rightarrow E$, the umbra of f is the set

$$U[f] = \{(x, y) \in F \times E | y \leq f(x)\}.$$

For $F, K \subseteq E^{N-1}$, $f: F \to E$, and $k: K \to E$, the dilation of f by k is the mapping $f \oplus k: F \oplus K \to E$ defined by

$$f \oplus k = T[U[f] \oplus U[k]]$$

from which it follows that

$$(f \oplus k)(x) = \max_{\substack{z \in K \\ x - z \in F}} \{f(x - z) + k(z)\}.$$

Similarly, the erosion of f by k maps $F \ominus K$ to E by

$$f \ominus k = T[U[f] \ominus U[k]]$$

from which it follows that

$$(f \ominus k)(x) = \min_{z \in K} \{f(x + z) - k(z)\}.$$

The umbra homomorphism theorem states that the operation of taking an umbra is a homomorphism from the grayscale morphology to the binary morphology. That is, if $F, K \subseteq E^{N-1}$, $f: F \to E$, and $k: K \to E$, then

$$U[f \oplus k] = U[f] \oplus U[k]$$
$$U[f \ominus k] = U[f] \ominus U[k].$$

Furthermore, the operations of max and union, and of min and intersection, are homomorphic under the umbra transformation. That is,

$$U[\max\{f, k\}] = U[f] \cup U[k],$$
$$U[\min\{f, k\}] = U[f] \cap U[k].$$

Two more notational conventions are needed before we begin developing the corresponding grayscale morphological sampling results. If $S \subseteq E^{N-1}$ is a sampling set, and $f: F \to E$ is a grayscale image, then the sampled version of f is the restriction of f to $F \cap S$ denoted by $f|_s$. Thus, $f|_s: F \cap S \to E$ defined by $f|_s(x) = f(x)$ for $x \in F \cap S$. If $k: K \to E$, then $\check{k}$ is the reflection of k, defined by $\check{k}: \check{K} \to E$ where $\check{k}(x) = k(-x)$.

A. The Function Bounding Relationships

The set bounding relationships for the binary morphology have a direct correspondence to function bounding relationships in grayscale morphology. In this section we develop the bounding relationships without spending much time or discussion even though the extensions are somewhat more involved. The grayscale analog to the relationship $F \ominus K \subseteq (F \cap S) \oplus K \subseteq F \oplus K$ is $f \ominus k \leq f|_s \oplus k \leq f \oplus k$ and it holds under very much the same conditions that the binary relationship holds. The only new requirement is for $k \geq 0$ which is stronger than the requirement that $0 \in U[k]$.

Proposition 18: Let $F, K, S \subseteq E^{N-1}$, $f: F \to E$, and $k: K \to E$. If $k = \check{k}$, $k \geq 0$, and $K \oplus S = E^{N-1}$, then $f \ominus k \leq f|_s \oplus k \leq f \oplus k$.

Proof: First we show that $f \ominus k \leq f|_s \oplus k$. Let $x \in F \ominus K$. Then $(f \ominus k)(x) = \min_{u \in K} \{f(x + u) - k(u)\}$. But $k = \check{k}$ and $k > 0$ imply $\min_{u \in K} \{f(x + u) - k(u)\} \leq \min_{v \in K} \{f(x - v) + k(v)\}$. Now, $k = \check{k}$ implies $K = \check{K}$, and this with $S \oplus K = E^{N-1}$ implies $F \ominus K \subseteq (F \cap S) \oplus K$. Thus, $x \in F \ominus K$ implies $x \in (F \cap S) \oplus K$. Hence, there exists a $w \in K$ such that $x - w \in F \cap S$. Because $\min_{z \in A} z \leq \max_{z \in B} z$ when $B \cap A \neq \emptyset$, there results

$$\begin{aligned}(f \ominus k)(x) &\leq \min_{v \in K} \{f(x - v) + k(v)\} \\ &\leq \max_{\substack{v \in K \\ x - v \in F \cap S}} \{f(x - v) + k(v)\} \\ &= \max_{\substack{v \in K \\ x - v \in F \cap S}} \{f|_s(x - v) + k(v)\} \\ &= (f|_s \oplus k)(x).\end{aligned}$$

■

Next we show $f|_s \oplus k \leq f \oplus k$. By the umbra homomorphism theorem, $U[f|_s \oplus k] = U[f|_s] \oplus U[k]$. But $U[f|_s] = U[f] \cap (S \times E)$. Hence, $U[f|_s \oplus k] = \{U[f] \cap (S \times E)\} \oplus U[k] \subseteq (U[f] \oplus U[k]) \cap ((S \times E) \oplus U[k])$. But $(S \times E) \oplus U[k] = E^N$ if and only if $S \oplus K = E^{N-1}$. So $U[f|_s \oplus k] \subseteq U[f] \oplus U[k] = U[f \oplus k]$ by the umbra homomorphism theorem. Hence, $T[U[f|_s \oplus k]] \leq T[U[f \oplus k]]$ which by definition of grayscale dilation implies $f|_s \oplus k \leq f \oplus k$.

The analog of $F \cap S = [(F \cap S) \oplus K] \cap S$ is $f|_s = (f|_s \oplus k)|_s$. It holds under the condition that $k(0) = 0$.

Proposition 19: Let $F, K, S \subseteq E^{N-1}$, $f: F \to E$, $k: K \to E$, satisfy $S \oplus S = S$, $S = \check{S}$, $K \cap S = \{0\}$, $k = \check{k}$, and $k(0) = 0$. Then $f|_s = (f|_s \oplus k)|_s$.

Proof: Since $[(F \cap S) \oplus K] \cap S = F \cap S$, we need only to prove that for each $x \in F \cap S$, $(f|_s \oplus k)|_s(x) = f(x)$. But $(f|_s \oplus k)|_s(x) = (f|_s \oplus k)(x)$. Since $x \in S$, $(f|_s \oplus k)|_s(x) = \max_{\substack{z \in K \\ x - z \in F \cap S}} \{f(x - z) + k(z)\}$. Furthermore, $x \in S$ and $x - z \in S$ imply $z = x - (x - z) \in S$. Since $S \oplus S = S$ and $\check{S} = S$. Thus, $z = 0$ as is implied by $z \in K$, $z \in S$ and $K \cap S = \{0\}$. Finally, $(f|_s \oplus k)|_s(x) = f(x) + k(0) = f(x)$, since $k(0) = 0$.

In order to continue with the parallel development $f \circ k \leq f|_s \oplus k$, we first prove the stronger relation that for every $x \in F \circ K$, there exists an $s \in S \cap F$ such that $x \in K_s$ and $(f \circ k)(x) \leq f(s) + k(x - s)$. This result follows from the sampling condition $u \in K_v$ which implies $S \cap K_u \cap K_v \neq \emptyset$ and a constraint on the structuring element $k: k(a) \leq k(a - b) + k(b)$ for every a, b satisfying $a \in K$, $b \in K$ and $a - b \in K$. This latter constraint is a new concept essential for the reconstruction structuring element in the grayscale morphology.

Before developing the proof for the inequality $(f \circ k)(x) \leq f(s) + k(x - s)$, it will be useful to explore the meaning of the inequality $k(a) \leq k(a - b) + k(b)$ since this is a constraint we have not had to deal with until now. The inequality $k(a) \leq k(a - b) + k(b)$ together with $k = \check{k}$ implies that $k(y) \geq 0$ for every $y \in K$. This can easily be seen by letting $a = x + y$ and $b = x$. This leads to $k(x + y) \leq k(x) + k(y)$. Then let $a = x$ and $b = x + y$. This leads to $k(x) \leq k(y) + k(x + y)$. The two inequalities imply $k(x) \leq k(y) + k(x + y) \leq k(x) + 2k(y)$ from which $k(y) \geq 0$ quickly follows.

The inequality $k(a) \le k(a-b) + k(b)$ also implies that for any integer $n \ge 2$, $k(nx) \le nk(x)$ for every $x \in K$ satisfying $mx \in K$ for every m, $2 \le m \le n$. The proof is by induction. Taking $n = 2$, $a = 2x$, and $b = x$ establishes the base case $k(2x) \le 2k(x)$. Suppose that for every m, $2 \le m \le n$, $k(ma) \le mk(a)$ and $ma \in K$ and $a \in K$. Taking $a = (n+1)x$ and $b = x$ produces $k(n+1) \le k(nx) + k(x)$. But $k(nx) \le nk(x)$. Hence, $k((n+1)x) \le nk(x) + k(x) = (n+1)k(x)$. Now by induction $k(nx) \le nk(x)$ for every integer $n \ge 2$ satisfying $mx \in K$ for every m, $2 \le m \le n$.

Finally, notice that $k(a) \le k(a-b) + k(b)$ for every $a, b \in K$ satisfying $a - b \in K$ and $K = \check{K}$ imply, as well, $k(b) \le k(b-a) + k(a)$ for every $a, b \in K$ satisfying $a - b \in K$. Since $k = \check{k}$, we obtain $k(a-b) \ge \max\{k(a) - k(b), k(b) - k(a)\}$.

Constructing functions which satisfy the inequality is easy with the following procedure. Define $k(0) = 0$ and $k(1)$ to be any positive number. Suppose $k(m)$, $m = 0, \cdots, n$ have been defined. Take $k(n+1)$ to be any number satisfying

$$\max_{\substack{u \\ 1 \le u \le n}} \{k(u) - k(n+1-u)\} \le k(n+1) \le \min_{\substack{v \\ 1 \le v \le n}} \{k(v) + k(n+1-v)\}.$$

After k is defined for all nonnegative numbers in its domain, define $k(-n) = k(n)$ for $n \ge 0$.

The generating procedure works because $k = \check{k}$ and the inequality implies $k(x) - k(y-x) \le k(y) \le k(x) + k(y-x)$. Hence, $\max_{u, 1 \le u \le y} \{k(u) - k(y-u) \le k(y)\} \le \min_{v, 1 \le v \le y} \{k(v) + k(y-v)\}$.

Proposition 20: Let $F, K, S \subseteq E^{N-1}$, $f: F \to E$, $k: K \to E$. Suppose $u \in K_v$ implies $S \cap K_u \cap K_v \ne \varnothing$ and $k(a) \le k(a-b) + k(b)$. Then for every $x \in F \circ K$, there exists an $s \in S \cap F$ such that $x \in K_s$ and $(f \circ k)(x) \le f(s) + k(x-s)$.

Proof: Let $x \in F \circ K$. Then $(x, (f \circ k)(x)) \in U[f \circ k]$. But $U[f \circ k] = U[f] \circ U[k]$. Hence, there exists $(u, v) \in E^{N-1} \times E$ such that $(x, (f \circ k)(x)) \in U[k]_{(u,v)} \subseteq U[f]$. Now $(x, (f \circ k)(x)) \in U[k]_{(u,v)}$ implies $(x, (f \circ k)(x)) - (u, v) \in U[k]$. So $(x - u, (f \circ k)(x) - v) \in U[k]$ which implies $(f \circ k)(x) - v \le k(x-u)$. Thus, $v \ge (f \circ k)(x) - k(x-u)$. But $U[k]_{(u,v)} \subseteq U[f]$ implies for every $a \in K$, $(a, k(a)) + (u, v) \in U[f]$. Hence, for every $a \in K$, $a + u \in F$ and $k(a) + v \le f(a+u)$. Now $x \in K_u$ implies there exists $s \in K_x \cap K_u \cap S$ so that $s - u \in K$. And since $s - u \in K$, $k(s-u) + v \le f(s-u+u) = f(s)$. Now $(f \circ k)(x) - k(x-u) \le v$ and $v \le f(s) - k(s-u)$ imply $(f \circ k)(x) - k(x-u) \le f(s) - k(s-u)$. But $k(a) \le k(a-b) + k(b)$ for every $a, b \in K$ satisfying $a - b \in K$. Letting $a = x - u$ and $b = s - u$, $a - b = x - s$, it is obvious that $x - u \in K$, $s - u \in K$, and $x - s \in K$. Hence, $k(x-u) - k(s-u) \le k((x-u) - (s-u)) = k(x-s)$. Therefore, $(f \circ k)(x) \le f(s) + k(x-s)$. ∎

Corollary: Let $F, K, S \subseteq E^{N-1}$, $f: F \to E$, and $k: K \to E$. Suppose $u \in K_v$ implies $S \cap K_u \cap K_v \ne \varnothing$ and $k(a) \le k(a-b) + k(b)$. Then $(f \circ k)(x) \le (f|_s \oplus k)(x)$.

Having determined that $f \circ k \le f|_s \oplus k$, the maximality of $f|_s \oplus k$ comes easily.

Proposition 21: Let $G, F, K, S \subseteq E^{N-1}$, $g: G \to E$, $k: K \to E$, $f: F \to E$. If $k(a) \le k(a-b) + k(b)$ for every $a \in K$ and $b \in K$ satisfying $a - b \in K$, and $x \in K_y$ implies $K_x \cap K_y \cap S \ne \varnothing$, then $g = g \circ k$, $g|_s = f|_s$, and $g \ge f|_s \oplus k$ implies $g = f|_s \oplus k$.

We continue our development with the bounding relations for the minimal reconstruction.

Proposition 22: Let $F, K, S \subseteq E^{N-1}$, $f: F \to E$, and $k: K \to E$. If $k = \check{k}$, $k(a) \le k(a-b) + k(b)$ for every $a, b \in K$ satisfying $a - b \in K$, and $x \in K_y$ implies $K_x \cap K_y \cap S \ne \varnothing$, then for every $u \in K$, $f(x+z) - k(z) \le (f|_s \oplus k)(x+u) - k(u)$ for each $z \in K$.

Proof: Since $k(a) \le k(a-b) + k(b)$, for every $a, b \in K$ satisfying $a - b \in K$, $-k(z) \le -k(u) + k(u-z)$ for every $u, z \in K$ satisfying $u - z \in K$. Hence, $f(x+z) - k(z) \le f(x+z) + k(u-z) - k(u)$ for every $u, z \in K$ satisfying $u - z \in K$. Making a change of variables $t = u - z$, there results

$$f(x+z) - k(z) \le [f(x+u-t) + k(t)] - k(u)$$

then

$$f(x+z) - k(z) \le \left[\max_{\substack{t \in K \\ x+u-t \in F \cap S \\ u-t \in K}} f|_s(x+u-t) + k(t)\right] - k(u) \le (f|_s \oplus k)(x+u) - k(u)$$

for every $u \in K$. ∎

Proposition 23: Let $F, K, S \subseteq E^{N-1}$, $f: F \to E$, and $k: K \to E$. Suppose $u \in K_v$ implies $K_u \cap K_v \cap S \ne \varnothing$, $k = \check{k}$, and $k(a) \le k(a-b) + k(b)$ for every $a, b \in K$ satisfying $a - b \in K$. Then for every $x \in F \ominus K$, $(f \ominus k)(x) \le (f|_s \bullet k)(x)$.

Proof: Let $x \in F \ominus K$. Then $(f \ominus k)(x) = \min_{z \in K}\{f(x+z) - k(z)\} \le (f|_s \oplus k)(x+u) - k(u)$ for every $u \in K$. There results $(f \ominus k)(x) \le \min_{u \in K}\{(f|_s \oplus k)(x+u) - k(u)\} = (f|_s \bullet k)(x)$. ∎

Proposition 24: Let $F, K, S \subseteq E^{N-1}$ and $f: F \to E$, $k: K \to E$. Then

1) $f|_s = (f|_s \bullet k)|_s$,
2) $f|_s \bullet k \le f \bullet k$.

Proof:

1) Since closing is an increasing operation, $f|_s \le f|_s \bullet k$. Since dilation is an increasing operation, $f|_s \bullet k \le (f|_s \bullet k) \oplus k = f|_s \oplus k$. Hence, $f|_s \le (f|_s \bullet k)|_s \le (f|_s \oplus k)|_s$. But $(f|_s \oplus k)|_s = f|_s$ and this proves $f|_s = (f|_s \bullet k)|_s$.

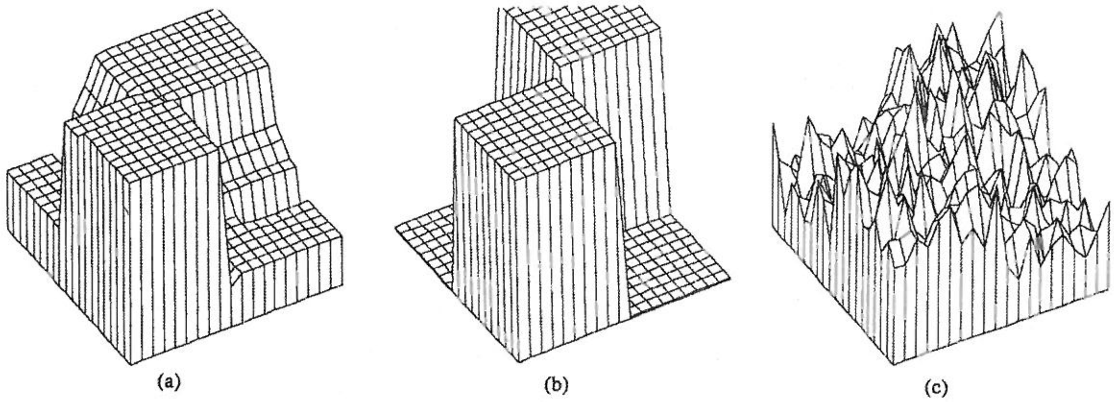


Fig. 22. A 3-D plot illustrates the grayscale opening operation. (a) A 20 × 20 checkerboard image with checker size 10 × 10. The gray value of the bright checker is 100 and the gray value of the dark checker is 50. (b) The noisy image by adding a zero mean Gaussian noise of standard deviation 20 to (a). (c) The result of opening (b) by a brick of size 10 × 10.

2) $U[f|_s \bullet k] = U[f|_s] \bullet U[k]$ by the umbra homomorphism theorem. Also, $U[f|_s] = U[f] \cap (S \times E)$ so that $U[f|_s] \subseteq U[f]$. Hence, $U[f|_s \bullet k] \subseteq U[f] \bullet U[k] = U[f \bullet k]$ by the umbra homomorphism theorem. This implies $f|_s \bullet k \leq f \bullet k$. ■

As before, the maximality comes easy.

Proposition 25: Let $G, F, K, S \subseteq E^{N-1}$, $g: G \to E$, $f: F \to E$, and $k: K \to E$. Suppose $g|_s = f|_s$ and $g = g \bullet k$. Then $g \leq f|_s \bullet k$ implies $g = f|_s \bullet k$.

B. The Grayscale Digital Morphologic Sampling Theorem

This section summarizes the results developed in the previous sections. These results constitute the grayscale digital morphological sampling theorem.

Theorem 6—Grayscale Digital Morphological Sampling Theorem:

Let $F, K, S \subseteq E^{N-1}$. Suppose K and S satisfy the following sampling conditions.

1) $S \oplus S = S$.
2) $S = \check{S}$.
3) $K \cap S = \{0\}$.
4) $K = \check{K}$.
5) $x \in K_y$ implies $K_x \cap K_y \cap S \neq \emptyset$. Let $f: F \to E$ and $k: K \to E$. Suppose further that k satisfies.
6) $k = \check{k}$.
7) $k(a) \leq k(a - b) + k(b)$ for every $a, b \in K$ satisfying $a - b \in K$.
8) $k(0) = 0$.

Then

1) $f|_s = (f|_s \bullet k)|_s$.
2) $f|_s = (f|_s \oplus k)|_s$.
3) $f|_s \bullet k \leq f \bullet k$.
4) $f|_s \circ k \leq f \circ k$.
5) If $f = f \circ k = f \bullet k$, then $f|_s \bullet k \leq f \leq f|_s \oplus k$.
6) If $g = g \bullet k$ and $g|_s = f|_s$, then $g \leq f|_s \bullet k$ implies $g = f|_s \bullet k$.
7) If $g = g \circ k$ and $g|_s = f|_s$, then $g \geq f|_s \oplus k$ implies $g = f|_s \oplus k$.

C. Examples

Fig. 22 illustrates an artificial example of opening a noisy 20 × 20 checkerboard image having 10 × 10 checks with a brick structuring element whose size is also 10 × 10. The gray level difference between the bright and dark checks before the noise was added was 50. The noise is Gaussian noise with standard deviation 20. The example illustrates how when the structuring element is matched to the structure of the signal, a surprising amount of noise reduction can take place.

The remaining examples illustrate how the morphological sampling theorem can lead to multiresolution processing techniques. The resolution hierarchy, called a pyramid, is produced typically by low-pass filtering and then sampled to generate the next lower resolution level. The purpose of the low-pass filter is to remove from the higher resolution image those spatial frequencies which are higher than the Nyquist frequency corresponding to the sample spacing. Fig. 23 shows a 5-level pyramid produced by pure sampling from a laser radar range map. The highest resolution image size is 256 × 256. A 2-pixel wide line and a 4 × 4 box are placed intentionally at the upper right and upper left, respectively. Fig. 24 shows the 5-level pyramid produced by a 3 × 3 box filtering followed by sampling to generate the next pyramid level.

Because multiresolution pyramids are used for detection and identification of objects or features of at least a specified size, it is natural to ask if mathematical morphology [14], [17], [8] might provide a better basis than low-pass filters, for constructing pyramids. This question is suggested by the fact that mathematical morphology deals directly with shape, whereas low-pass filtering techniques are based on linear combinations of sinusoidal waveforms, a representation far removed from shape.

Fig. 25 shows a 5-level morphological pyramid. At each level, the image is opened by a brick of size 3 × 3, and

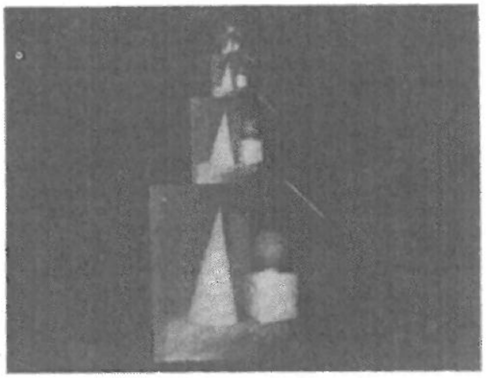


Fig. 23. A 5-level pyramid of laser radar range image produced by pure sampling. The highest resolution image size is 256 by 256. A 2-pixel-wide line segment and a 4 × 4 box are intentionally placed at the upper right and upper left of the image, respectively.

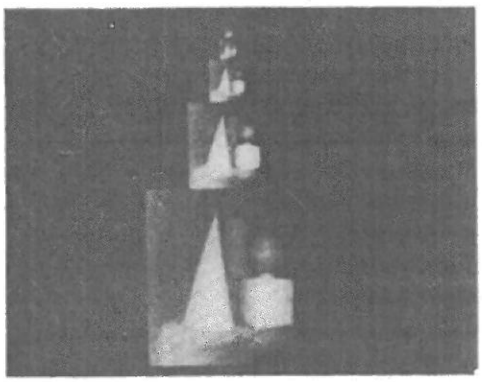


Fig. 24. A 5-level pyramid of the same image as Fig. 23, produced by 3 × 3 box filtering and then sampling.

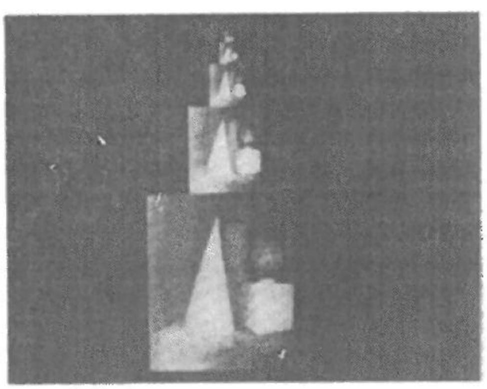


Fig. 25. A 5-level pyramid of the same image as Fig. 23, produced by, opened by a brick of 3 × 3 and then sampling to generate the next layer.

then sampled to generate the next lower resolution layer. Notice how the line in the upper right part of the image has been eliminated. Fig. 26 shows a similar 5-level morphological pyramid. In this pyramid, the image at each level has been opened by a 3 × 3 brick, sampled, and then reconstructed, using the maximal reconstruction. The next lower resolution layer is generated by sampling as before. The greater smoothness of the images in Fig. 25 over the corresponding images of Fig. 24 is precisely due to the smoothness introduced by sampling and reconstructing.

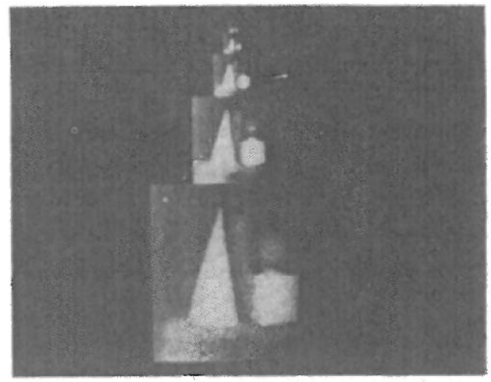


Fig. 26. A 5-level pyramid of the same image as Fig. 23. In each level, the image is opened by a brick of 3 × 3 and then is sampled and reconstructed before it is downsampled to generate the next level.

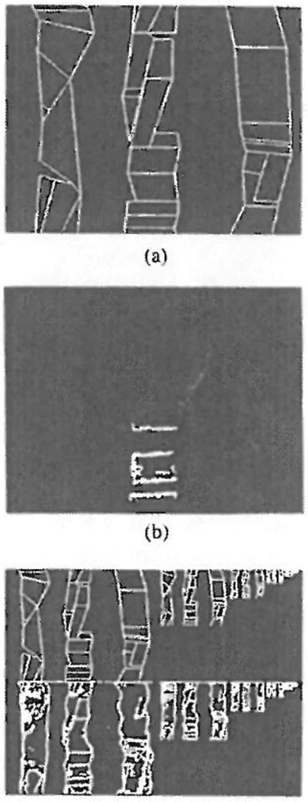
This illustrates that when images have been properly morphologically smoothed before sampling, the reconstructed images are indeed representative of the unsampled images.

Next we describe a multiresolution matching scenario to integrate ground truth maps with terrain region boundaries detected from registered millimeter wave (MMW) radar polarimetric imagery acquired from the TABILS 5 section of the Air Force TABILS (Target and Background Information Library System) database to estimate position along a planned flight path.

The ground truth database stores vertices of terrain region boundaries, along with region descriptions such as pine forest or wheat field. We estimate flight position by matching detected boundaries in the incoming data to patterns in the ground truth. Fig. 27(a) shows the boundary patterns for scene 2051A derived from the ground truth vertices. The pattern in Fig. 27(b) corresponds to the current flight position, to match against ground truth. Large-scale boundary pattern structures can estimate the displacement between detected and stored patterns over a large range at low accuracy and sampling density; smaller structures can be used over short ranges at increased accuracy and sampling. Therefore, we use a multiresolution coarse-to-fine matching strategy to meet the response-time needs of flight path estimation.

Fig. 27 illustrates morphological dilation pyramids for both the ground truth boundaries and the detected edge patterns. Dilating the binary patterns by a 2 × 2 kernel before downsampling (2 to 1 in both directions) retains the pattern points at coarse image resolution. Coarse-to-fine binary correlation on these pyramids [10] determines the best match: the flight position estimate is refined progressively, with estimates at one level restricting the search area at the next higher resolution.

The overall hierarchical correlation scheme outlined in Fig. 28 begins each stage with one pattern preshifted by its input offset. The pattern at the current resolution is shifted by various displacements, for which (binary) global correlation error counts are determined. Each count sums over both the detected pattern points which are not ground truth boundary points, and the ground truth



(a)

(b)

(c)

Fig. 27. (a) Terrain boundary patterns for scene 2051 from the ground truth vertex database; (b) detected region boundary image corresponding to the current flight position; and (c) morphological dilation pyramids for the ground truth boundary and detected edge patterns.

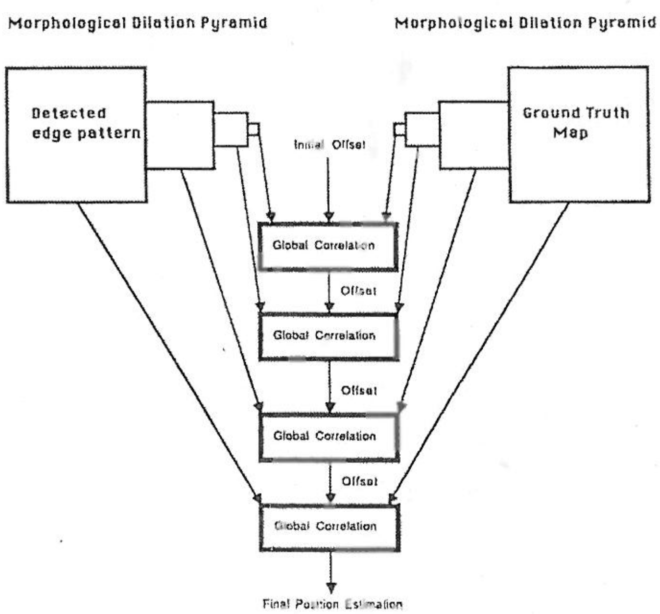
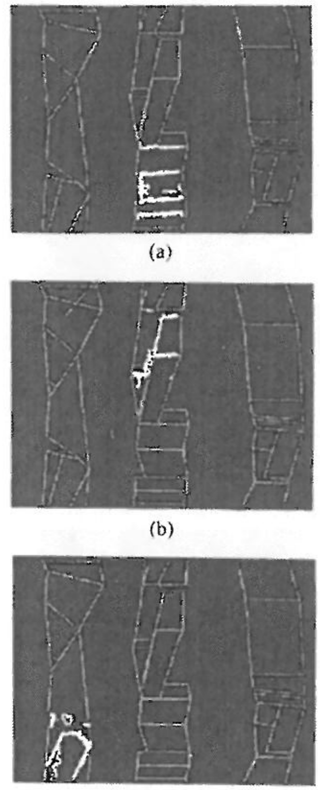


Fig. 28. An illustration of multiresolution coarse-to-fine hierarchical correlation strategy for current flight position estimation.

boundary points (within the region enclosing the detected patterns) not corresponding to detected pattern points. The output offset of this stage, input to the next stage, equals the input offset plus the offset with the smallest error. We repeat to the finest image resolution, where the final offset



(a)

(b)

(c)

Fig. 29. (a)–(c) Estimated positions overlaid on the ground truth map for three sections of the detected edge patterns.

specifies the estimated position of the current detected pattern in the ground truth map. Fig. 29 shows estimated positions overlaid in white on the ground truth for three detected pattern sections.

VI. Conclusion

We have shown that before an image can be sampled, it must be morphologically simplified by an opening or a closing with the reconstruction structuring element. A sampled image has a minimal and maximal reconstruction. The minimal reconstruction is generated by closing the sampled image with the reconstruction structuring element, and is a valid reconstruction when the morphological simplification done before sampling is a closing. The maximal reconstruction is generated by dilating the sampled image with the reconstructed structuring element, and is a valid reconstruction when the morphological simplification done before sampling is an opening. The spatial meaning of the minimal and maximal reconstruction is direct. The minimal and maximal reconstruction delineate the spatial bounds within which the image event on the unsampled morphologically simplified image actually occurs. That is, the uncertainty due to sampling is precisely specified by the bounds given by the minimal and maximal reconstruction.

We developed the relationships between the sampling interval and the reconstruction structuring element, and in the binary morphology we fully developed the relationships between operating in the sampled domain and reconstructing versus performing the equivalent operations in the unsampled domain. Likewise, we developed the relationship between operating in the unsampled domain and then sampling versus sampling and performing the equivalent operation in the sampled domain. In both cases, the uncertainty introduced is closely related to the sampling interval.

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