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Vector-Space Solution for a Morphological Shape-Decomposition Problem

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Abstract. We define a *restricted domain* as the discrete set of points representing any convex, four-connected, filled polygon whose (i) vertices lie on the lattice points, (ii) interior angles are multiples of 45°, and (iii) number of sides are at most eight. We describe the boundary code and discrete half-plane representation and use them for representing restricted domains. Morphological operations of dilation and n-fold dilation on the restricted domains with structuring elements that are also restricted domains are expressed in terms of the above representations. We give algorithms for these operations and prove that they are of $O(1)$ complexity and hence are independent of the size of the objects.

We prove that there is a set of 13 restricted domains $\{K_1, K_2, \ldots, K_{13}\}$ such that any given restricted domain K is expressible as $K = K_0 \oplus \left(\underset{k_1}{\oplus} K_1 \right) \oplus \left(\underset{k_2}{\oplus} K_2 \right) \oplus \cdots \oplus \left(\underset{k_{13}}{\oplus} K_{13} \right)$, where $\left(\underset{k_i}{\oplus} K_i \right)$ represents the k_i-fold dilation of K_i and K_0 is a translation. We show that this entails a linear transformation from a 13-dimensional space in which restricted domains are represented in terms of n-fold dilations of the 13 basis structuring elements, to an eight-dimensional space in which restricted domains are represented in terms of their eight side lengths. Furthermore, we show that *any* particular decomposition forms a particular solution of this transformation and that finding *all* possible dilation decompositions of a restricted domain is equivalent to finding the general solution of this transformation. Finally, we derive a finite-step algorithm for finding a particular decomposition and then give an algorithm for finding all possible decompositions.

Key Words: mathematical morphology, decomposition, vector space

1 Introduction

The concepts of mathematical morphology have been used for shape description. Guibas et al. [1], Guibas [2], Ghosh [3], Pitas and Venetsanopoulos [4], Xu [5], [6], Sinha and Giardina [7], Lozano-Perez [8], and Grunbaum [9] have authored a few of the numerous papers published in this area. Shapes or objects can be described in terms of simpler, better characterized, underlying parts. A morphological description of a shape usually expresses the shape by decomposing it into an equivalent series of dilations of simpler parts. Simpler parts in the case of binary shapes can be disks, lines, rectangles, etc., of various sizes. A shape is expressible as a dilation of two other simpler shapes if the original shape can be described as the area marked out when one of the parts is held fixed and the other is swept over it.

Binary shapes are usually represented as the sets of all the points that constitute them. These shapes are completely characterized by their boundaries, and many efficient representation schemes for representing border information have been presented; see Freeman [10]. Boundary representations make explicit many important features, such as vertices and edge lengths. If these features are used by shape-description algorithms, the use of the boundary representation will make the extraction of the description from the representation much more efficient. Algorithms that perform morphological operations by using object outlines in the two-dimensional continuous domain have been proposed [3], [8].

Morphological operations on machines specialized to perform these operations are limited by the maximum size of the structuring elements that the hardware allows. If a morphological operation has to be performed with a structuring element larger than the maximum allowable size, the structuring element must be decomposed into smaller ones. The new structuring elements have to be such that (i) each of them can be handled by the hardware and (ii) the dilation of all of them is the original structuring element.

From the above discussion we can see that structuring-element decomposition is an important problem from both points of view – shape description and hardware implementation of morphological operations. Several algorithms to find such decompositions have been presented in the literature. Most of these algorithms work on shapes represented either as sets or as their outlines in the continuous domain and have a time complexity of $O(n^2)$. Xu [5] first decomposed chain-coded two-dimensional restricted domains as dilations of a set of 3×3 primitive structuring elements. However, these primitive structuring elements did not form a basis set. Subsequently, we presented a basis set consisting of 13 structuring elements and gave an algorithm for decomposing any two-dimensional restricted domain as dilations of these basis structuring elements [11]. Xu [6] also arrived at the same basis set, although the proofs for the algorithms in the two papers are different. Xu's proof was by induction, whereas our proofs are algebraic and rely on the underlying geometry of the decomposition problem. In this paper we interpret the decomposition problem as a vector-space problem, and show that the solution to the problem is not unique. Next, we show that the decompositions obtained in [6] and [11] are particular solutions of the linear-transformation problem. We show that the general solution of the linear transformation is the sum of a particular solution and the homogeneous solutions, and we give algorithms for finding them. Some of the results presented here have been presented in [11]–[14].

In section 2 we set the stage by giving all the definitions and notations. In section 3 we define B-codes. In section 4 we define restricted domains and give two schemes for representing them, one with B-codes and the other with half planes. Here we also give algorithms for interconversion of representations. B-code dilation and n-fold dilation are discussed in section 5, and morphological algorithms and their computational complexity are given in section 6. The algorithm and proof for structuring-element decomposition are given in section 7. The computational complexity of the algorithms has been considered for each algorithm presented. Finally, a summary of the presented work and directions for future work toward generalizing the algorithms for any discretely convex and nonconvex shape are considered in section 8.

2 Preliminaries

In this section we define all the necessary terms and give the notations used in this paper.

Any $p \in \mathbf{Z}^2$, where $\mathbf{Z}$ is the set of integers, will be referred to as a lattice point. In this paper we are interested in binary images that take on the values 0 or 1 at the lattice points. The terms *structuring element* and *shape* will also refer to binary images.

DEFINITION 2.1. A four- or eight- connected component F is *discretely convex* if and only if all the lattice points lying inside the convex hull of F belong to F. This definition directly implies that a discretely convex connected component has no holes.

Next, we restate the definitions of the basic morphological operations based on the tutorial by Haralick et al. [15] and Haralick and Shapiro [16]. Also see [17] and [18].

DEFINITION 2.2. The *dilation* of A by B is denoted by $A \oplus B$ and is defined as $A \oplus B = \{c \in \mathbf{Z}^2 | c = a + b \text{ for some } a \in A \text{ and } b \in B\}$.

DEFINITION 2.3. The *erosion* of A by B is denoted by $A \ominus B$ and is defined as $A \ominus B = \{x \in \mathbf{Z}^2 | x + b \in A \text{ for every } b \in B\}$.

DEFINITION 2.4. The *opening* of a set B by a structuring element K is denoted by $B \circ K$ and is defined as $B \circ K = (B \ominus K) \oplus K$.

DEFINITION 2.5. The *closing* of a set B by a set K is denoted by $B \bullet K$ and is defined as $B \bullet K = (B \oplus K) \ominus K$.

DEFINITION 2.6. The *n-fold dilation* of a set B by a set A is denoted by $B \oplus \left(\underset{n}{\oplus} A \right)$ and is defined as

$$B \oplus \left(\underset{n}{\oplus} A \right) = B \overbrace{\oplus A \oplus A \oplus \cdots \oplus A}^{n \text{ times}}.$$

DEFINITION 2.7. The *n-fold erosion* of a set B by a set A is denoted by $B \ominus \left(\underset{n}{\oplus} A \right)$ and is defined as

$$B \ominus \left(\underset{n}{\oplus} A \right) = \overbrace{(\cdots(((B \ominus A) \ominus A) \ominus A) \cdots \ominus A)}^{n \text{ times}}.$$

3 Boundary Codes

Line drawings have been commonly used to represent the boundaries of two-dimensional objects. In the case of discrete, binary images these line drawings of the object boundary can be represented in any of the following ways: (i) as a sequence of points, (ii) by a chain-code representation, or (iii) as a sequence of line segments. Descriptions of these methods can be found in [16] and [19].

The chain-code representation as proposed by Freeman [10] does not incorporate the lengths of the edges into its notation. It nevertheless has a provision for a special token in the implementation that allows for the length of the edge to be stored. In this section we discuss a notation for chain codes that requires explicit representation of the boundary edge lengths and directions. This boundary encoding scheme, referred to as B-code, uses a list data structure.

B-code is a representation scheme for connected components in terms of their boundary lattice points. Only one starting boundary point is represented explicitly, and the rest of the boundary points are represented in terms of successive displacements in one of eight possible directions. If the successive displacements happen to be in the same direction, they are encoded as the direction followed by the number of moves in that direction. The formal notation to represent a connected component A is

$$A = \langle (i_A, j_A) | (\mathbf{d_l} : n_l)(\mathbf{d_{l+1}} : n_{l+1}) \cdots (\mathbf{d_m} : n_m) \rangle. \quad (1)$$

Here (i_A, j_A) is the starting boundary lattice point, and the ordered pairs to the right of the vertical bar describe each successive displacement. The number of ordered pairs is equal to the number of changes in the direction of displacement. In the ordered pair $(\mathbf{d_k} : n_k)$, $\mathbf{d_k} \in \{\mathbf{d_0}, \mathbf{d_1}, \ldots, \mathbf{d_7}\}$ represents the direction of the displacement, and the nonnegative integer n_i to the right of the colon represents the number of successive moves in that direction. The directions $\mathbf{d_0}, \ldots, \mathbf{d_7}$ are the same as the chain-code directions $0, \ldots, 7$, which correspond to angles $\{0°, 45°, 90°, 135°, 180°, 225°, 270°, 315°\}$ with respect to the positive x axis: $\mathbf{d_0} = (1, 0)$, $\mathbf{d_1} = (1, 1)$, $\mathbf{d_2} = (0, 1)$, $\mathbf{d_3} = (-1, 1)$, $\mathbf{d_4} = (-1, 0)$, $\mathbf{d_5} = (-1, -1)$, $\mathbf{d_6} = (0, -1)$, and $\mathbf{d_7} = (1, -1)$.

Figure 1 illustrates the relation between B-codes and chain codes. It can be seen that the B-code representation can be thought of as a run-length encoding of the chain code. Also, any simply or multiply connected binary image that can be encoded by using the chain codes can also be encoded by using the B-codes.

4 Restricted Domains

The class of objects we will decompose and work on in this paper are discretely convex, four-connected sets all of whose boundaries are

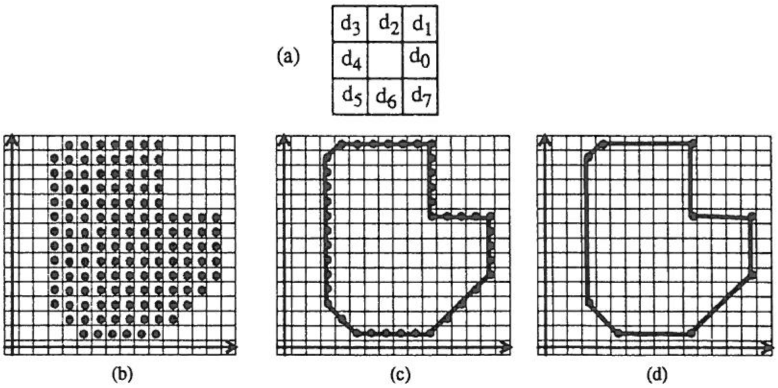

Fig. 1. Example of B-coding of images. (a) Basic directions, (b) a binary image, (c) pixels explicitly represented by its chain code (5, 1): 00000111122224444222224444445666666666677, and (d) pixels explicitly represented by its B-code $\langle (5, 1) | (\mathbf{d_0} : 5)(\mathbf{d_1} : 4)(\mathbf{d_2} : 4)(\mathbf{d_4} : 4)(\mathbf{d_2} : 5)(\mathbf{d_4} : 6)(\mathbf{d_5} : 1)(\mathbf{d_6} : 10)(\mathbf{d_7} : 2) \rangle$.

oriented at angles that are multiples of 45° and whose lengths are multiples of the pixel side lengths for 0° and 90° orientations and multiples of $\sqrt{2}$ times the pixel side lengths for 45° and 135° orientations. We will refer to the set of all objects belonging to this class as *restricted domains*. This class of objects was also studied by Xu [5], [6].

DEFINITION 4.1. A *restricted domain* is a discretely convex, four-connected shape whose convex hull has sides at angles that are multiples of 45° with respect to the positive x axis.

Some examples of restricted domains are given in figure 2. In the following sections we will define the restricted domains in terms of their B-codes and will present an equivalent representation in terms of half planes.

4.1 B-Code Representation

4.1.1 Convention. Given a binary image of a restricted domain A, we will represent it in the B-code form for further processing. The binary image of a restricted domain can be represented in many ways by using a B-code representation, i.e., the B-code representation is not unique, because the only restriction on the starting point of a B-code representation is that it should be a vertex. Thus there are as many B-code representations of a restricted domain as the number of vertices. To avoid ambiguity we will use the following convention:

> The starting point is always the lowest and leftmost vertex of the restricted domain. The other vertices are encoded by traversing around the restricted domain along its boundary points in the counterclockwise direction, encoding the length of the edges that constitute A. The interior points of the set are those to the left of the direction of motion.

The B-code obtained by using this convention represents an equivalence class of B-codes—the class of all B-codes representing the considered restricted domain. Each B-code in the equivalence class is a rotated version of some other (with an appropriately modified starting point), but it represents the same set of lattice points nevertheless.

4.1.2 Properties of B-coded restricted domains. In this section we present some useful properties of B-coded restricted domains, and these properties will be used in later proofs. Discussions of a few of these properties can be found in [6], [12], and [13].

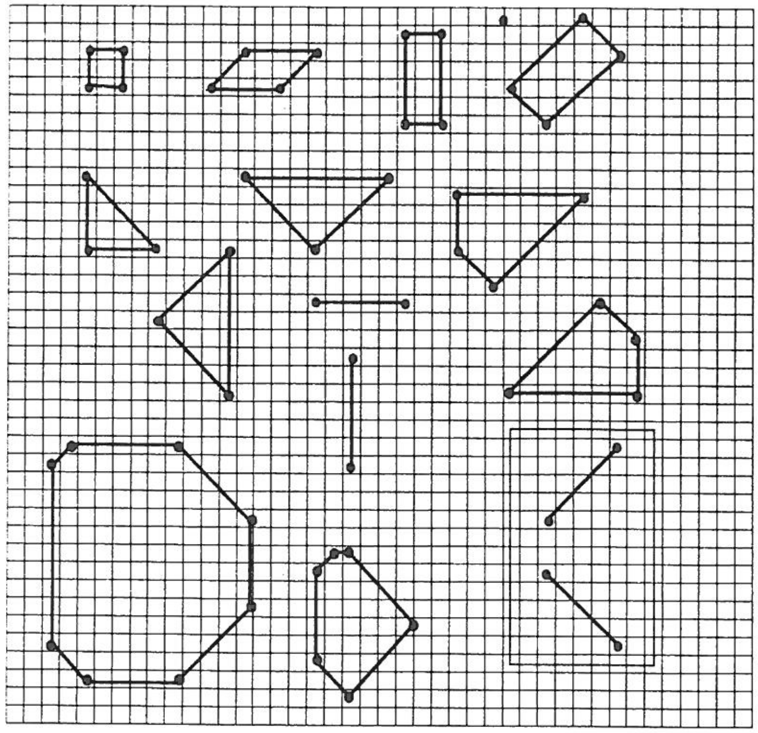

Fig. 2. Examples of restricted domains. Note that the diagonal lines in the box are not strictly restricted domains since they are not four-connected.

PROPERTY 4.1. Any restricted domain can be represented by a general B-code of the form $A = \langle (i, j) | (\mathbf{d_0} : n_0)(\mathbf{d_1} : n_1)(\mathbf{d_2} : n_2)(\mathbf{d_3} : n_3)(\mathbf{d_4} : n_4)(\mathbf{d_5} : n_5)(\mathbf{d_6} : n_6)(\mathbf{d_7} : n_7)\rangle$ by giving appropriate values to the n_i's. Thus in this representation there are always eight vertices and eight displacements and the displacement angles are monotonically increasing from $\mathbf{d_0}$ to $\mathbf{d_7}$. If there is no displacement corresponding to one of the directions, the corresponding pair can be dropped from the B-code and the particular n_i is given a value zero. Note that in this case two vertices become coincident.

Given a closed contour, the net displacement on traversing its complete boundary is zero. Since the B-code of a restricted domain $A = \langle (i, j) | (\mathbf{d_0} : n_0)(\mathbf{d_1} : n_1) \cdots (\mathbf{d_7} : n_7)\rangle$ represents a closed contour, it inherits the following two properties of a closed contour.

PROPERTY 4.2. The sum of displacements contributing to the positive x direction is equal to the sum of displacements contributing to the negative x direction:

$$n_0 + n_1 + n_7 = n_3 + n_4 + n_5. \qquad (2)$$

PROPERTY 4.3. The sum of displacements contributing to the positive y direction is equal to the sum of displacements contributing to the negative y direction:

$$n_1 + n_2 + n_3 = n_5 + n_6 + n_7. \qquad (3)$$

PROPERTY 4.4. Any B-code of the form $A = \langle (i, j) | (\mathbf{d_0} : n_0)(\mathbf{d_1} : n_1)(\mathbf{d_2} : n_2)(\mathbf{d_3} : n_3)(\mathbf{d_4} : n_4)(\mathbf{d_5} : n_5)(\mathbf{d_6} : n_6)(\mathbf{d_7} : n_7)\rangle$ whose n_i's satisfy the properties in equations (2) and (3) is either a restricted domain or a line at 45° or 135°. The lines are special cases and are of the form $A = \langle (i, j) | (\mathbf{d_1} : n_1)(\mathbf{d_5} : n_5)\rangle$ and $A = \langle (i, j) | (\mathbf{d_3} : n_3)\ (\mathbf{d_7} : n_7)\rangle$.

Given a B-code of a restricted domain $A = \langle (i, j) | (\mathbf{d}_0 : n_0)(\mathbf{d}_1 : n_1) \cdots (\mathbf{d}_7 : n_7) \rangle$ all the eight vertices of the polygon are uniquely defined and can be found in the following two ways.

PROPERTY 4.5. Let the vertex v_0 be the starting lattice point (i, j). The other vertices are given recursively. Given the kth vertex v_k, then the x and y coordinates of the $(k + 1)$th vertex v_{k+1} are given by the recursive equations

$$\mathbf{x}[v_{k+1}] = \mathbf{x}[v_k] + n_k \mathbf{x}[d_k], \tag{4}$$

$$\mathbf{y}[v_{k+1}] = \mathbf{y}[v_k] + n_k \mathbf{y}[d_k] \tag{5}$$

for $0 \le k \le 6$. Here $\mathbf{x}[v_0] = i$ and $\mathbf{y}[v_0] = j$ are the x and y coordinates of the starting point of the B-code.

The coordinates of the vertices of A can also be computed relative to the starting point of the restricted domain.

PROPERTY 4.6. The coordinates of the kth vertex v_k can be computed in terms of the starting location (i, j) and the lengths n_l, $0 \le l \le k$. Let $\mathbf{V_x}$, $\mathbf{V_y}$, $\mathbf{V}$, and $\mathbf{N}$ be the matrices

$$\mathbf{V_x} = \begin{bmatrix} \mathbf{x}[v_0] \\ \mathbf{x}[v_1] \\ \vdots \\ \mathbf{x}[v_7] \end{bmatrix}, \quad \mathbf{V_y} = \begin{bmatrix} \mathbf{y}[v_0] \\ \mathbf{y}[v_1] \\ \vdots \\ \mathbf{y}[v_7] \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{V_x} \\ \mathbf{V_y} \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} n_0 \\ n_1 \\ \vdots \\ n_7 \end{bmatrix}. \tag{6}$$

Then

$$\mathbf{V} = \begin{bmatrix} \mathbf{V_x} \\ \mathbf{V_y} \end{bmatrix} = \mathbf{P} \begin{bmatrix} i \\ j \\ \mathbf{N} \\ i \\ j \\ \mathbf{N} \end{bmatrix}, \tag{7}$$

where

$$\mathbf{P} = \begin{bmatrix} \mathbf{P_1} & 0 \\ 0 & \mathbf{P_2} \end{bmatrix}, \tag{8}$$

$$\mathbf{P_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & -1 & -1 & -1 & 0 & 0 \end{bmatrix}, \tag{9}$$

$$\mathbf{P_2} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & 0 \end{bmatrix}. \tag{10}$$

Note that the matrix $\mathbf{P}$ is very sparse and is used here only for notational convenience. Thus in the actual matrix multiplications, only the nonzero entries in the matrix need to be multiplied out.

4.2 *Normalized Half-Plane Representation*

Restricted domains can be represented in terms of the intersections of discrete half planes. Let $A = \langle (i, j) | (\mathbf{d}_0 : n_0)(\mathbf{d}_1 : n_1) \cdots (\mathbf{d}_7 : n_7) \rangle$ be a restricted domain. Then the lattice points belonging to A can be defined in terms of intersections of eight discrete half planes $\mathcal{H}_i$, $0 \le i \le 7$. These half planes $\mathcal{H}_i$ are functions of the basic directions of the displacement d_i and the vertices v_i of the restricted domain. Each discrete half plane $\mathcal{H}_i$ is such that its boundary passes through the vertex v_i and its edge is along the direction d_i. The half plane $\mathcal{H}_i$ represents all the points to the left of and on the boundary for a traverse in the direction d_i along its boundary. Therefore a restricted domain $A = \langle (i, j) | (\mathbf{d}_0 : n_0)(\mathbf{d}_1 : n_1) \cdots (\mathbf{d}_7 : n_7) \rangle$ can be represented as

$$A = \mathcal{H}_0 \cap \mathcal{H}_1 \cap \cdots \cap \mathcal{H}_7, \tag{11}$$

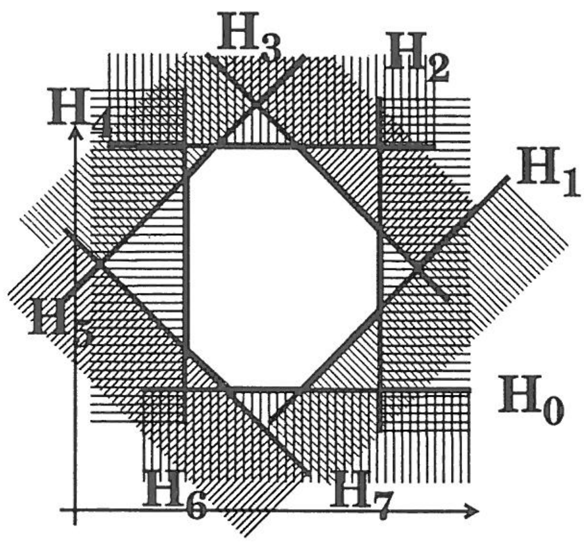

Fig. 3. Restricted domains as intersections of half planes $\mathcal{H}_0 \cdots \mathcal{H}_7$. The unshaded half represents the half plane. Here the intersection set is the unshaded central region.

where $\mathcal{H}_i$ is a discrete half plane given by

$$\mathcal{H}_i = \left\{ p = (x, y) \in \mathbf{Z}^2 \text{ such that} \right.$$

$$\left. \begin{vmatrix} x - \mathbf{x}[v_i] & y - \mathbf{y}[v_i] \\ \mathbf{x}[v_i] + \mathbf{x}[d_i] & \mathbf{y}[v_i] + \mathbf{y}[d_i] \end{vmatrix} \le 0 \right\}. \tag{12}$$

Figure 3 illustrates the half-plane concept. We can expand the above expression for the particular cases of $\mathcal{H}_i$, $0 \le i \le 7$. Substituting the expression for the vertex v_i of the restricted domain given in (7) into inequality (12), one obtains the inequalities for the half planes $\mathcal{H}_0$–$\mathcal{H}_7$:

$$\begin{array}{lrcrcl} \mathcal{H}_0: & (0)x & + & (-1)y & \le & c_0, \\ \mathcal{H}_1: & (1)x & + & (-1)y & \le & c_1, \\ \mathcal{H}_2: & (1)x & + & (0)y & \le & c_2, \\ \mathcal{H}_3: & (1)x & + & (1)y & \le & c_3, \\ \mathcal{H}_4: & (0)x & + & (1)y & \le & c_4, \\ \mathcal{H}_5: & (-1)x & + & (1)y & \le & c_5, \\ \mathcal{H}_6: & (-1)x & + & (0)y & \le & c_6, \\ \mathcal{H}_7: & (-1)x & + & (-1)y & \le & c_7, \end{array} \tag{13}$$

where x, y, $c_i \in \mathbf{Z}$ and the c_i are given by

$$\mathbf{C} = \mathbf{L} \begin{bmatrix} i \\ j \\ \mathbf{N} \end{bmatrix}, \tag{14}$$

where

$$\mathbf{C} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix} \tag{15}$$

and

$$\mathbf{L} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -2 & -1 & 0 & 1 & 2 & 1 & 0 \end{bmatrix}. \tag{16}$$

To make the information more compact, we will use matrices to represent the system of linear inequalities in (13) as

$$\mathbf{M}p' \le \mathbf{C}, \tag{17}$$

where

$$\mathbf{M} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ -1 & 1 \\ -1 & 0 \\ -1 & -1 \end{bmatrix} \tag{18}$$

and $p = (x, y)$ is a lattice point. Note that the inequalities (17) are considered row-wise.

The physical interpretation of the system of inequalities (17) is as follows. Consider eight half planes passing through the origin, each corresponding to a direction d_i, $0 \le i \le 7$. The half planes are translated from the origin up, down, left, and right such that they pass through the

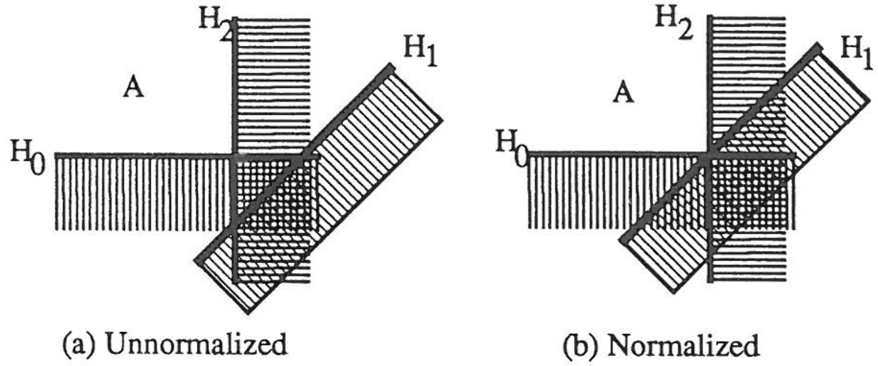

Fig. 4. Unnormalized and normalized half planes. The unshaded half represents the half plane. The half-plane $\mathcal{H}_1$ in (a) is redundant and can be moved unit it passes through the vertex of set A; this situation is shown in (b).

corresponding vertices v_i. The intersections of these half planes give us the lattice points belonging to the restricted domain.

Notice that since the d_i's are fixed, the slopes of the discrete half planes are also fixed and hence the half plane $\mathcal{H}_i$ is uniquely represented by the corresponding c_i's. However, a set of c_i's representing a restricted domain need not be unique. For example, in figure 4 we see that the half planes corresponding to two different sets of c_i's represent the same intersection set, because the half plane $\mathcal{H}_1$ is *redundant* and can be translated to infinitely many locations without having any effect on the intersection set. All the possible sets of c_i's representing a given restricted domain form an equivalence class. This raises a question about a convention that we can follow such that a equivalence class of restricted domains can be represented through a unique c_i set. We notice that the c_i's that are obtained from the B-code representation by using (14) always represent discrete half planes passing through the vertices of the restricted domains. Those that are redundant, that is, those that correspond to a displacement of length zero along the d_i direction, also pass through a vertex, even though they have potentially infinite possibilities. Thus we will follow the convention that if a set of c_i's represent a restricted domain, it should be normalized such that all the half planes pass through the vertices of the intersection set. Such a set of eight c_i's, represented by a vector $\mathbf{C}$, will be called the *normalized half-plane representation* of the restricted domain. The half planes that are not redundant and form the sides of the polygon will be called *primary*.

Before we proceed further, we need to address the following issues:

1. Under what conditions does the set of c_i's represent a nonempty set?
2. Under what conditions is the restricted domain represented by the set of c_i's a normalized representation, and if it is not, how can it be normalized?

The c_i's represent a set of discrete half planes. Hence the set of points belonging to the intersection of these half planes is not empty if and only if the set of points belonging to the intersection of any two of these half planes is not empty. Figure 5(a) is an example in which the half plane $\mathcal{H}_0$ is unnormalized. Since it should be moved so that it touches the intersection set, it is obvious that it should be moved to r, the intersection point of $\mathcal{H}_1$ and $\mathcal{H}_7$. Other possibilities are p, the intersection point of $\mathcal{H}_7$ and $\mathcal{H}_2$, and q, the intersection point of $\mathcal{H}_1$ and $\mathcal{H}_6$. Notice that p and q do not belong to the intersection set and they are below r. Figures 5(b) and 5(c) are examples in which $\mathcal{H}_0$ has to be moved to p and q respectively. Notice that in this case p is above q and r. In figure 5(d) $\mathcal{H}_0$ is a primary half plane and cannot be moved. In this case $\mathcal{H}_0$ is above p, q, and r. Thus the algorithm for normalization of $\mathcal{H}_0$ then becomes (i) find the intersection points p, q, and r, and (ii) update c_0 such that $\mathcal{H}_0$ passes through the one belonging to the set. If $\mathcal{H}_0$ is primary, nothing should be done to c_0. It is convenient that the c_0 found this way forms a bound for the half plane $\mathcal{H}_3$, i.e., the half plane $\mathcal{H}_3$ cannot be below this level. If it is, the intersection of the half planes results in an empty set.

By using the same argument for all other half planes, it can be shown that a set of eight c_i's represents a nonempty set if and only if

$$\mathbf{C} \geq \mathbf{C}_{\text{bound}} = \max[\mathbf{G}_1\mathbf{C}, \mathbf{G}_2\mathbf{C}, \mathbf{G}_3\mathbf{C}, -\lfloor \mathbf{G}_4\mathbf{C} \rfloor] \tag{19}$$

and a set of c_i's is normalized if

$$\mathbf{C} \geq \max[\mathbf{G}_1\mathbf{G}_2\mathbf{C}, \mathbf{G}_1\mathbf{G}_3\mathbf{C}, -\lfloor \mathbf{G}_1\mathbf{G}_4\mathbf{C} \rfloor], \tag{20}$$

where the 8×8 matrices $\mathbf{G}_1, \ldots, \mathbf{G}_4$ used in the algorithm are given by

$$\mathbf{G}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{21}$$

$$\mathbf{G}_2 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ -2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 0 & 0 & 0 \end{bmatrix}, \tag{22}$$

$$\mathbf{G}_3 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ -2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \tag{23}$$

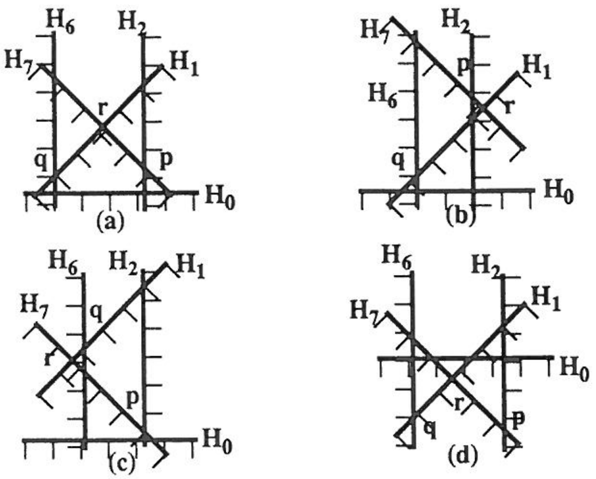

Fig. 5. Normalization of $\mathcal{H}_0$—four cases. In (a), (b), and (c) the half-plane $\mathcal{H}_0$ is redundant and must be normalized, that is, moved up so that it passes through r, p, and q, respectively. In (d) $\mathcal{H}_0$ is primary and cannot be moved.

$$\mathbf{G}_4 = \begin{bmatrix} 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

The lower ceilings come about because the 45° and 135° lines need not intersect at a lattice point. Notice that $\mathbf{G}_1^2 = \mathbf{I}$. Here the matrix multiplications find the intersection points, and the max operation selects the one nearest to the set. Thus both the issues mentioned above have been addressed.

Notice that when $c_1 = -c_5$ or $c_3 = -c_7$, we have diagonal lines at 45° and 135° respectively. These are not strictly restricted domains since they are not four connected (they are eight connected). Thus since restricted domains are four connected, the following constraints should hold:

$$c_1 > -c_5 \quad \text{and} \quad c_3 > -c_7. \tag{25}$$

Function Normalize given in Algorithm 1 takes as input the $\mathbf{C}$ array of a restricted domain and returns the normalized $\mathbf{C}$ array if one exists, or else it returns a null value. Since the algorithm has five multiplications of 8×8 matrices with 8×1 vectors, one lower ceiling of a 8×1 vector, one 8×1 vector comparison, one row-wise max operation of four 8×1 vectors, and no loops it is constant in time.

ALGORITHM 1. Normalization of half planes.

function Normalize(**C**) : **ArrayObject**

Input:
ArrayObject C;

begin
  $\mathbf{C}_{\text{bound}} := \max[\mathbf{G}_1\mathbf{C}, \mathbf{G}_2\mathbf{C}, \mathbf{G}_3\mathbf{C}, -\lfloor \mathbf{G}_4\mathbf{C} \rfloor]$;
  if $(\mathbf{C} < \mathbf{C}_{\text{bound}})$
  then
    return NULL;
  else
    $\mathbf{C} := \max[\mathbf{C}, \mathbf{G}_1\mathbf{G}_2\mathbf{C}, \mathbf{G}_1\mathbf{G}_3\mathbf{C}, -\lfloor \mathbf{G}_1\mathbf{G}_4\mathbf{C} \rfloor]$;
    return C;
end Normalize;

4.3 Conversion from Normalized Half Plane to B-Code

Given the c_i's of a normalized restricted domain, we should be able to (i) find the vertices of a restricted domain in terms of the c_i's, (ii) find the n_i's in terms of the c_i's, and (iii) find the B-code representation of the restricted domain.

The vertices of the restricted domain can be computed by finding the intersections of the consecutive half planes. They can be expressed in terms of the vector **C** as follows:

$$\mathbf{V} = \mathbf{D}\begin{bmatrix} \mathbf{C} \\ \mathbf{C} \end{bmatrix}, \tag{26}$$

where

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_1 & 0 \\ 0 & \mathbf{D}_2 \end{bmatrix}, \tag{27}$$

$$\mathbf{D}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \tag{28}$$

$$\mathbf{D}_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \tag{29}$$

The n_i's can be computed by finding the distance between the two consecutive vertices v_{i+1} and v_i. Thus

$$\mathbf{N} = \mathbf{QC}, \tag{30}$$

where

$$\mathbf{Q} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \tag{31}$$

The B-code representation of the restricted domain is determined by v_0 and **N**.

5 B-Code Morphology for Restricted Domains

In this section we give constant-time algorithms for dilation, erosion, opening, closing, n-fold erosion of restricted domains by using their half-plane and B-code representations. We show that the results obtained by using these algorithms are equivalent to those obtained by using regular morphology. If the input-restricted domains are in their B-code representations or if the output-restricted domains are needed in their B-code representations, the results of section 4 can be used for the interconversion of representations. For the detailed proofs of theorems on erosion, opening, and closing, refer to [12] and [13].

5.1 Dilation of Restricted Domains

Let A and B be two restricted domains given by the B-codes

$$A = \langle (i_A, j_A) | (\mathbf{d}_0 : n_0^A)(\mathbf{d}_1 : n_1^A) \cdots (\mathbf{d}_7 : n_7^A) \rangle, \quad (32)$$

$$B = \langle (i_B, j_B) | (\mathbf{d}_0 : n_0^B)(\mathbf{d}_1 : n_1^B) \cdots (\mathbf{d}_7 : n_7^B) \rangle, \quad (33)$$

and let their normalized half-plane representations be

$$A = \{a \in \mathbf{Z}^2 | \mathbf{M}a' \leq \mathbf{C}^A\}, \quad (34)$$

$$B = \{b \in \mathbf{Z}^2 | \mathbf{M}b' \leq \mathbf{C}^B\}, \quad (35)$$

where $\mathbf{C}^A$ and $\mathbf{C}^B$ are given by

$$\mathbf{C}^A = \mathbf{L}\begin{bmatrix} i^A \\ j^A \\ \mathbf{N}^A \end{bmatrix} \text{ and } \mathbf{C}^B = \mathbf{L}\begin{bmatrix} i^B \\ j^B \\ \mathbf{N}^B \end{bmatrix}. \quad (36)$$

$\mathbf{N}^A$ and $\mathbf{N}^B$ and 8×1 column vectors with the respective edge lengths as their elements, and $\mathbf{M}$ and $\mathbf{L}$ are the matrices defined in (18) and (16), respectively.

LEMMA 5.1. The set C given by

$$C = \{c \in \mathbf{Z}^2 | \mathbf{M}c' \leq \mathbf{C}^C\}, \quad (37)$$

where $\mathbf{C}^C = \mathbf{C}^A + \mathbf{C}^B$, is a restricted domain, and the vector $\mathbf{C}^C$ is a normalized half-plane representation of C.

Proof. From the discussion in subsection 4.2 and from (19), the sufficient condition for $\mathbf{C}^C$ to be a restricted domain is that $\mathbf{C}^C \geq \mathbf{C}^C_{\text{bound}}$. Since A and B are restricted domains and since $\mathbf{C}^A$ and $\mathbf{C}^B$ are normalized half-plane representations,

$$\mathbf{C}^A \geq \mathbf{C}^A_{\text{bound}} = \max[\mathbf{G}_1\mathbf{C}^A, \mathbf{G}_2\mathbf{C}^A, \mathbf{G}_3\mathbf{C}^A, -\lfloor \mathbf{G}_4\mathbf{C}^A \rfloor], \quad (38)$$

$$\mathbf{C}^B \geq \mathbf{C}^B_{\text{bound}} = \max[\mathbf{G}_1\mathbf{C}^B, \mathbf{G}_2\mathbf{C}^B, \mathbf{G}_3\mathbf{C}^B, -\lfloor \mathbf{G}_4\mathbf{C}^B \rfloor]. \quad (39)$$

Thus by adding the above equations we get

$$\begin{aligned} \mathbf{C}^C &= \mathbf{C}^A + \mathbf{C}^B \\ &\geq \max[\mathbf{G}_1\mathbf{C}^A, \mathbf{G}_2\mathbf{C}^A, \mathbf{G}_3\mathbf{C}^A, \\ &\quad -\lfloor \mathbf{G}_4\mathbf{C}^A \rfloor] + \max[\mathbf{G}_1\mathbf{C}^B, \mathbf{G}_2\mathbf{C}^B, \\ &\quad \mathbf{G}_3\mathbf{C}^B, -\lfloor \mathbf{G}_4\mathbf{C}^B \rfloor]. \end{aligned} \quad (40)$$

However, we know that $\max[a, b] + \max[c, d] \geq \max[(a + c), (b + d)]$. Hence

$$\begin{aligned} \mathbf{C}^C \geq\ & \max[\mathbf{G}_1(\mathbf{C}^A + \mathbf{C}^B), \mathbf{G}_2(\mathbf{C}^A + \mathbf{C}^B), \\ & \mathbf{G}_3(\mathbf{C}^A + \mathbf{C}^B), \\ & -\lfloor \mathbf{G}_4(\mathbf{C}^A + \mathbf{C}^B) \rfloor]. \end{aligned} \quad (41)$$

Therefore $\mathbf{C}^C$ represents a restricted domain. Similarly, we now show that $\mathbf{C}^C$ is a normalized half-plane representation. Since $\mathbf{C}^A$ and $\mathbf{C}^B$ are normalized representation, we have

$$\mathbf{C}^A \geq \max[\mathbf{G}_1\mathbf{G}_2\mathbf{C}^A, \mathbf{G}_1\mathbf{G}_3\mathbf{C}^A, -\lfloor \mathbf{G}_1\mathbf{G}_4\mathbf{C}^A \rfloor], \quad (42)$$

$$\mathbf{C}^B \geq \max[\mathbf{G}_1\mathbf{G}_2\mathbf{C}^B, \mathbf{G}_1\mathbf{G}_3\mathbf{C}^B, -\lfloor \mathbf{G}_1\mathbf{G}_4\mathbf{C}^B \rfloor]. \quad (43)$$

As before, from the above equations we get

$$\begin{aligned} \mathbf{C}^A + \mathbf{C}^B \geq\ & \max[\mathbf{G}_1\mathbf{G}_2(\mathbf{C}^A + \mathbf{C}^B), \\ & \mathbf{G}_1\mathbf{G}_3(\mathbf{C}^A + \mathbf{C}^B), \\ & -\lfloor \mathbf{G}_1\mathbf{G}_4(\mathbf{C}^A + \mathbf{C}^B) \rfloor]. \end{aligned} \quad (44)$$

Thus $\mathbf{C}^C$ is normalized. Furthermore, since A and B are four-connected restricted domains, we have from (25) that $c_1^A > -c_5^A$, $c_3^A > -c_7^A$, $c_1^B > -c_5^B$, and $c_3^B > -c_7^B$. Manipulating, we get $c_1^A + c_1^B > c_5^A + c_5^B$ and $c_3^A + c_3^B > c_7^A + c_7^B$. Thus C is four-connected and $\mathbf{C}^C$ is a normalized half-plane representation of a restricted domain. Note that even if either A or B is a diagonal line, e.g., $c_1^A = c_5^A$ or $c_1^B = c_5^B$ for the case of 45° diagonal lines, the resultant shape is still be a restricted domain (because the four-connectivity constraints are still satisfied).

LEMMA 5.2. The eight vertices $\mathbf{V}^C$ of C are the vector sums of the respective vertices $\mathbf{V}^A$ and $\mathbf{V}^B$ of A and B.

Proof. The vertices of C are given by

$$\mathbf{V}^C = \mathbf{D}\begin{bmatrix} \mathbf{C}^C \\ \mathbf{C}^C \end{bmatrix}, \quad (45)$$

where $\mathbf{D}$ is the matrix (27). Since $\mathbf{C}^C = \mathbf{C}^A + \mathbf{C}^B$, we have

$$\begin{aligned} \mathbf{V}^C &= \mathbf{D}\begin{bmatrix} \mathbf{C}^A + \mathbf{C}^B \\ \mathbf{C}^A + \mathbf{C}^B \end{bmatrix} \\ &= \mathbf{D}\begin{bmatrix} \mathbf{C}^A \\ \mathbf{C}^A \end{bmatrix} + \mathbf{D}\begin{bmatrix} \mathbf{C}^B \\ \mathbf{C}^B \end{bmatrix}. \end{aligned} \tag{46}$$

Hence

$$\mathbf{V}^C = \mathbf{V}^A + \mathbf{V}^B, \tag{47}$$
$$v_i^C = v_i^A + v_i^B, \quad 0 \le i \le 7. \tag{48}$$

The lemma is thus proved.

Now we show that the dilation of two restricted domains can be performed by adding their respective $\mathbf{C}$ vectors.

PROPOSITION 5.1. $A \oplus B$ is given by the restricted domain C whose normalized half-plane representation is given by $\mathbf{C}^C = \mathbf{C}^A + \mathbf{C}^B$.

Proof. We will proceed by proving (i) that $A \oplus B \subseteq C$ and then (ii) that $C \subseteq A \oplus B$.

(i) Let $c \in A \oplus B$. Then by the definition of dilation there exist $a \in A$ and $b \in B$ such that $c = a + b$. Since $a \in A$ and $b \in B$, from (34) and (35) we have

$$\begin{aligned} \mathbf{M}a' &\le \mathbf{C}^A, \\ \mathbf{M}b' &\le \mathbf{C}^B. \end{aligned}$$

Adding the above equations, we get

$$\begin{aligned} \mathbf{M}a' + \mathbf{M}b' &\le \mathbf{C}^A + \mathbf{C}^B, \\ \mathbf{M}(a+b)' &\le \mathbf{C}^A + \mathbf{C}^B. \end{aligned}$$

Hence $c \in C$, and therefore $A \oplus B \subseteq C$.

(ii) Let $c \in C$. We have to prove that there exist $a \in A$ and $b \in B$ such that $a + b = c$. Since $c \in C$, it satisfies the relation

$$\mathbf{M}c' \le \mathbf{C}^A + \mathbf{C}^B. \tag{49}$$

From Lemma 5.1 we know that C is a restricted domain and thus by definition it is discretly convex. Hence c belongs to the convex hull of C and it can be expressed as the convex combination of the vertices of C. Thus

$$c = \sum_{0 \le i \le 7} \alpha_i v_i^C, \tag{50}$$

where $\alpha_i \in \mathbf{R}$, $0 \le \alpha_i \le 1$, and $\sum \alpha_i = 1$. From Lemma 5.2 we get

$$c = \sum_{0 \le i \le 7} \alpha_i \left(v_i^A + v_i^B\right), \tag{51}$$
$$c = c_A + c_B, \tag{52}$$

where $c_A = \sum_{0 \le i \le 7} \alpha_i v_i^A$ and $c_B = \sum_{0 \le i \le 7} \alpha_i v_i^B$. The first term, c_A, on the right-hand side of (52) belongs to the convex hull of the restricted domain A, whereas the second term, c_B, belongs to the convex hull of the restricted domain B. Notice that (51) does not guarantee that c_A and c_B are lattice points, i.e., they need not belong to $\mathbf{Z}^2$. It merely represents the fact that the vector sum of two points c_A and c_B in $\mathbf{R}^2$ is the lattice point c in $\mathbf{Z}^2$. We will now show that we can always find a lattice point belonging to A in the neighborhood of c_A and another belonging to B in the neighborhood of c_B such that their vector sum is the lattice point c. This is illustrated in figure 6.

Let $c = (l, m)$, $c_A = (p + \delta, q + \gamma)$, and $c_B = (r + 1 - \delta, s + 1 - \gamma)$ such that $l, m, p, q, r, s \in \mathbf{Z}$ and $0 \le \delta, \gamma \le 1$. Going back to (52) and replacing c, c^A, and c^B by their values, we get

$$\begin{aligned} c &= (l, m) \\ &= (p + \delta, q + \gamma) \\ &\qquad + (r + 1 - \delta, s + 1 - \gamma) \\ &= (p + r + 1, q + s + 1). \end{aligned} \tag{53}$$

It can be seen from the above equations that the point c_A lies between the four lattice points (p, q), $(p + 1, q)$, $(p, q + 1)$, and $(p + 1, q + 1)$. Similarly, the point c_B lies between the four lattice points (r, s), $(r + 1, s)$, $(r, s + 1)$, and $(r + 1, s + 1)$. We will prove that the vector sum of two of these eight points, one belonging to A and the other belonging to B, is the lattice point $(l, m) = (p + r + 1, q + s + 1)$.

We can determine which of the four surrounding lattice points necessarily belongs to the restricted domain A, given c_A (and hence δ and γ). Depending on the values of δ and γ, the area between the lattice points surrounding c_A and c_B can be divided into several regions. The inclusion of a particular neighbor in the set A depends on where the point c_A falls. The attack has to be on a case-by-case basis.

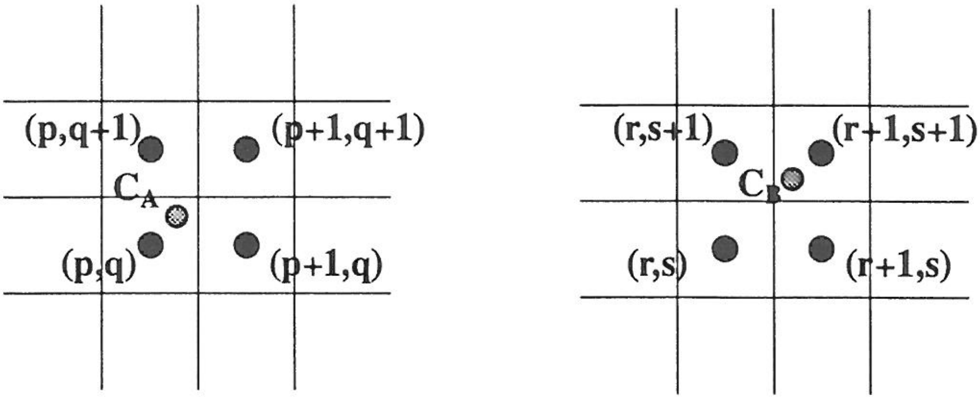

Fig. 6. Relations among p, q, r, s, δ, and γ. We have to find one neighboring lattice point of c_A and another of c_B such that their vector sum is $c = (p + r + 1, q + s + 1)$.

Consider the case for which c_A lies in the region defined by $\gamma > \delta$. We can see that for such a case the neighboring lattice point $(p, q + 1)$ necessarily belongs to A, because if $(p, q + 1)$ did not belong to A, no convex combination of any subset of the other three neighboring lattice points could produce a c_A in the region defined by $\gamma > \delta$. From symmetry we can see that for $\gamma > \delta$, $(r + 1, s)$ necessarily belongs to the set B. Thus the desired lattice points are $(p, q + 1)$ and $(r + 1, s)$, since their vector sum is the lattice point $(p + 1, q + 1) = (l, m) = c$.

Similarly, for the cases for which $\gamma < \delta$, $\gamma > 1 - \delta$, and $\gamma < 1 - \delta$, we can find lattice points belonging to A and B such that their vector sum is the lattice point $(l, m) = c$.

The only region inside the square not yet considered is $\delta = \gamma = 0.5$, which is the center of the square. When $\delta = \gamma = 0.5$, we fall into neither of the above categories and hence we must treat this case separately. We notice that the lattice point $(p + 0.5, q + 0.5)$ can result from the convex combination if all four neighboring lattice points belong to the set, only three of the neighboring lattice points belong to the set, or any two diagonally opposite lattice points belong to the set. We can eliminate the last case, since it implies the A is a diagonal line and hence is not four connected, thereby contradicting our assumptions. Thus for the case for which $\delta = \gamma = 0.5$, either or four of the lattice points neighboring c_A necessarily belong to A. The same is true for the set B. Note that the lattice points belonging to A do not in any way constraint the ones belonging to B. It is easy to verify that given any three lattice points surrounding c_A and three lattice points surrounding c_B, we can always find two lattice points, one neighboring c_A and one neighboring c_B, such that their sum is $c = (l, m) = (p + r + 1, q + s + 1)$. In fact, there are many such pairs.

Thus we have proved that $C \subset A \oplus B$. Hence $C = A \oplus B$.

We now prove an important lemma that states that a dilation of A by B is just the addition of the respective side lengths and the starting points.

LEMMA 5.3. If $C = A \oplus B$, then $(i_C, j_C) = (i_A, j_A) + (i_B, j_B)$ and $\mathbf{N}^C = \mathbf{N}^A + \mathbf{N}^B$.

Proof. Since $\mathbf{C}^C = \mathbf{C}^A + \mathbf{C}^B$, we have $\mathbf{N}^C = \mathbf{Q}\mathbf{C}^C = \mathbf{Q}(\mathbf{C}^A + \mathbf{C}^B) = \mathbf{N}^A + \mathbf{N}^B$. The rest of the lemma follows from the fact that $\mathbf{V}^C = \mathbf{V}^A + \mathbf{V}^B$.

5.2 *Erosion of Restricted Domains*

Let A and B be two restricted domains with normalized half-plane representations:

$$A = \{a \in \mathbf{Z}^2 | \mathbf{M}a' \leq \mathbf{C}^A\}, \quad (54)$$

$$B = \{b \in \mathbf{Z}^2 | \mathbf{M}b' \leq \mathbf{C}^B\}, \quad (55)$$

where $\mathbf{C}^A$ and $\mathbf{C}^B$ are given by

$$\mathbf{C}^A = \mathbf{L}\begin{bmatrix} i^A \\ j^A \\ \mathbf{N}^A \end{bmatrix}, \quad (56)$$

$$\mathbf{C}^B = \mathbf{L}\begin{bmatrix} i^B \\ j^B \\ \mathbf{N}^B \end{bmatrix}. \quad (57)$$

$\mathbf{N}^A$ and $\mathbf{N}^B$ are 8×1 column vectors with the respective edge lengths as their elements, and $\mathbf{M}$ and $\mathbf{L}$ are the matrices defined in (18) and (16), respectively.

The erosion of A and B can be performed by subtracting the $\mathbf{C}$ matrix of B from that of A. The resulting $\mathbf{C}$ matrix need not be a normalized half-plane representation—it must be normalized by using Algorithm 1. Furthermore, the erosion of a restricted domain with another need not produce a restricted domain. Consider, for example, the erosion of a rectangle by a rhombus, where the sides of the rectangle and the rhombus are oriented at 45° and 135° and the sides of the rhombus equal the smaller of the two sides of the rectangle. It can be easily seen that the result of the erosion is a line oriented along the longer side of the rectangle, i.e., it is a line at 45° or 135°. Since lines at 45° and 135° are not four-connected (but are eight-connected), they are not restricted domains. These special cases must be considered separately.

Now we state a lemma, the proof of which can be found in [12].

PROPOSITION 5.2. $A \ominus B$ is given by C, whose half-plane representation is given by

$$C = \{c \in \mathbf{Z}^2 | \mathbf{M}c' \leq \mathbf{C}^C\},$$

where $\mathbf{C}^C = \mathbf{C}^A - \mathbf{C}^B$. C can be either a restricted domain or a diagonal line.

5.3 *Opening*

Morphological opening of a binary set A by another binary set B is denoted by $A \circ B$ and is defined as

$$A \circ B = (A \ominus B) \oplus B. \quad (58)$$

Since dilations and erosions of restricted domains have been defined, the above definition of opening is also valid for restricted domains. The definition is also valid for the following more general cases when either A or B or both are not restricted domains: (i) $A \ominus B$ is a restricted domain and B is a line at 45°, (ii) $A \ominus B$ is a restricted domain and B is a line at 135°, (iii) $A \ominus B$ and B are lines at 135°, and (iv) $A \ominus B$ and B are lines at 45°. Note that lines at 45° and 135° are not restricted domains since they are not four-connected. The algorithm cannot be used if $A \ominus B$ and B are lines at 45° and 135°, respectively. This constraint is due to the fact that the dilation of a 45° line with a 135° line results in a rhombuslike shape with one-pixel holes, i.e., the shape is not filled. Thus set-theory dilation results in a shape that not filled, but the half-plane and B-code dilation algorithms produce a shape that is filled.

5.4 *Closing*

Morphological closing of a binary set A by another binary set B is denoted by $A \bullet B$ and is defined as

$$A \bullet B = (A \oplus B) \ominus B. \quad (59)$$

Since dilations and erosions of restricted domains have been defined, the above definition of opening is also valid for restricted domains. The definition is also valid for the following more general cases when either A or B or both are not restricted domains: (i) A is a line at 45° and B is a restricted domain, (ii) B is a line at 45° and A is a restricted domain, (iii) A and B are lines at 45°, and (iv) A and B are lines at 135°. Note that lines at 45° and 135° are not restricted domains since they are not four-connected. The algorithm cannot be used if A and B are lines at 45° and 135°, respectively. This constraint is due to the fact that the set-theory dilation of a 45° line with a 135° line results in a rhombus-like shape with one-pixel holes, i.e., the shape is not filled. Thus set-theory dilation results in a shape that is not filled, but the half-plane and B-code dilation algorithms produce a shape that is filled.

6 Algorithms and Their Complexity

In this section we give the algorithms for computing the dilation and erosion of restricted domains represented by half planes. The algorithms for opening and closing can be easily obtained by applying the dilation and erosion algorithms in the appropriate order. The algorithm for n-fold dilation and n-fold erosion need one multiplication step, which we explain at the end of this section. The following data structures are used in the algorithms:

ArrayObject is a data structure containing an array and its dimensions. In the algorithms the vectors associated with B-codes, half planes, etc., are stored by using this data-structure type.

RDObject is a data structure used to represent restricted domains. It contains the three matrices, **N**, **V**, and **C**, associated with the restricted domain.

Procedure DilateRDObject in Algorithm 2 takes as input two **RDObject** and outputs **RDObject**, which is the dilation of the two input **RDObject**.

ALGORITHM 2. Dilation of restricted domains.

procedure DilateRDObject(A, B, C)

Input:
RDObject A, B;
Output:
RDObject C;

begin
$\mathbf{N}^C := \mathbf{N}^A + \mathbf{N}^B$;
$(i_C, j_C) := (i_A, j_A) + (i_B, j_B)$;
end DilateRDObjects;

Procedure ErodeRDObject [12] takes as input two **RDObject** and outputs **RDObject**, which is the erosion of the two-input **RDObject**. This procedure calls the normalization function that is given in Algorithm 1. Function Normalize takes as input an **ArrayObject** containing the **C** array of a restricted domain and returns the normalized **C** array if one exists; otherwise it returns a null value.

The n-fold dilation of a restricted domain B by a restricted domain A is $B \oplus \left(\underset{n}{\oplus} A \right)$, the dilation of B by the n-fold dilation of A. In the B-code domain this amounts to multiplying the side lengths of A by n and adding them to the side lengths of B and then multiplying the starting point of A by n and adding it to the starting point of B. If A and B have the side lengths given by the vectors $\mathbf{N}^A$ and $\mathbf{N}^B$ and starting points (i_A, j_A) and (i_B, j_B), then $\left(\underset{n}{\oplus} A \right)$ has side lengths given by the vector $n\mathbf{N}^A$ and starting point $n(i_A, j_A)$. It follows that $B \oplus \left(\underset{n}{\oplus} A \right)$ has side lengths given by the vector $\mathbf{N} = \mathbf{N}^B + n\mathbf{N}^A$ and the starting point $(i_B, j_B) + n(i_A, j_A)$. Dilation can also be performed by adding the **C** vectors associated with A and B in the discrete half-plane representation. Thus the **C** vector associated with $B \oplus \left(\underset{n}{\oplus} A \right)$ is $\mathbf{C} = \mathbf{C}^B + n\mathbf{C}^A$.

The n-fold erosion of a restricted domain B by a restricted domain A is $B \ominus \left(\underset{n}{\oplus} A \right)$, the erosion of B by the n-fold dilation of A. Let $\mathbf{C}^A$ and $\mathbf{C}^B$ be the vectors associated with A and B. Then $n\mathbf{C}^A$ is the vector associated with $\left(\underset{n}{\oplus} A \right)$. Thus in the half-plane domain the n-fold erosion of B by A amounts to $\mathbf{C}^B - n\mathbf{C}^A$.

Algorithm 2 for the dilation of restricted domains consists of only 10 additions. Hence it is a constant-time algorithm. Note that the time complexity is independent of the size of the structuring element. In conventional morphology this is not the case—the time complexity is $O(n^2)$, where n is the number of elements in each set.

The algorithm for an n-fold dilation of restricted domains consists of eight multiplications. Hence it is also a constant-time algorithm.

The erosion algorithm given in [12] consists of eight subtractions followed by the process of normalization. The normalization in Algorithm 1 was shown to be constant time. Thus the erosion algorithm is a constant-time algorithm.

The n-fold erosion is represented in terms of n-fold dilation. Since the n-fold dilation algorithm is constant time, the n-fold erosion algorithm is constant time.

The algorithm for opening consists of two stages—an erosion stage followed by a dilation stage. Since erosion and dilation algorithms are constant time, the algorithm for opening is also constant time. Similarly, the algorithm for closing consists of two stages—a dilation stage followed by an erosion stage. Since erosion and dilation both are constant time, the algorithm for closing is also constant time.

We now apply the algorithms on some typical restricted domains. Figure 7 is an example of a dilation. Here $C = A \oplus B$, where

$$\begin{aligned} A &= \langle (0,2) | (\mathbf{d}_1 : 3)(\mathbf{d}_3 : 2)(\mathbf{d}_4 : 1)(\mathbf{d}_6 : 5) \rangle, \\ B &= \langle (1,1) | (\mathbf{d}_1 : 1)(\mathbf{d}_3 : 1)(\mathbf{d}_5 : 1)(\mathbf{d}_7 : 1) \rangle. \end{aligned}$$

Thus

$$\begin{aligned} (i_A, j_A) &= (0, 2) \text{ and} \\ \mathbf{N}^A &= [0\,3\,0\,2\,1\,0\,5\,0]', \\ (i_B, j_B) &= (1, 1) \text{ and} \\ \mathbf{N}^B &= [0\,1\,0\,1\,0\,1\,0\,1]', \\ \mathbf{N}^C &= \mathbf{N}^A + \mathbf{N}^B \\ &= [0\,4\,0\,3\,1\,1\,5\,1]', \\ (i_C, j_C) &= (i_A, j_A) + (i_B, j_B) \\ &= (1, 3). \end{aligned}$$

Therefore

$$\begin{aligned} C = \langle (1,3) | &(\mathbf{d}_1 : 4)(\mathbf{d}_3 : 3)(\mathbf{d}_4 : 1)(\mathbf{d}_5 : 1) \\ &(\mathbf{d}_6 : 5)(\mathbf{d}_7 : 1) \rangle. \end{aligned}$$

Thus we started from the B-code representations of A and B, and then we added the respective side lengths and starting locations to get the B-code for C.

For a worked-out example of erosion of restricted domain, see [12] and [13].

7 Decomposition of B-Coded Restricted Domains

In this section we state the decomposition problem in terms of restricted domains. We formally express the problem as a theorem and prove it by giving a constructive solution. Furthermore, we interpret the problem of finding the decomposition as a vector-space mapping problem and show that the constructive solution is just a particular solution of a linear system of equations. The null space of the mapping is then found by finding the homogeneous solution. Finally, the general solution is expressed as the sum of a particular solution and the homogeneous solution of the system. Since the decomposition problem may not have a unique solution, the particular solution is only *one possible* decomposition, whereas the general solution is the set of *all* possible decompositions.

7.1 Statement of the Problem

We will consider the following problems: (i) If we are given a restricted domain, is it possible to decompose it and express it as the dilation of simpler restricted domains? (ii) In particular, is it possible to represent any restricted domain as dilations of members of a finite-basis set of restricted domains? (iii) Is the decomposition unique? If not, how do we find all the possible decompositions?

7.2 Basis Set

We show here that any restricted domain A can be decomposed as the n-fold dilations of 13 basis structuring elements. That is, any restricted domain is a point in a 13-dimensional space whose basis directions are 13 shapes. These basis structuring elements are shown in figure 8. Their B-code representations are given by

$$\begin{aligned} K_1 &= \langle (0,0) | (\mathbf{d}_0 : 1)(\mathbf{d}_4 : 1) \rangle, \\ K_2 &= \langle (0,0) | (\mathbf{d}_1 : 1)(\mathbf{d}_5 : 1) \rangle, \\ K_3 &= \langle (0,0) | (\mathbf{d}_2 : 1)(\mathbf{d}_6 : 1) \rangle, \\ K_4 &= \langle (0,0) | (\mathbf{d}_3 : 1)(\mathbf{d}_7 : 1) \rangle, \\ K_5 &= \langle (0,0) | (\mathbf{d}_0 : 1)(\mathbf{d}_2 : 1)(\mathbf{d}_5 : 1) \rangle, \\ K_6 &= \langle (0,0) | (\mathbf{d}_0 : 1)(\mathbf{d}_3 : 1)(\mathbf{d}_6 : 1) \rangle, \\ K_7 &= \langle (0,0) | (\mathbf{d}_1 : 1)(\mathbf{d}_4 : 1)(\mathbf{d}_6 : 1) \rangle, \\ K_8 &= \langle (0,0) | (\mathbf{d}_2 : 1)(\mathbf{d}_4 : 1)(\mathbf{d}_7 : 1) \rangle, \\ K_9 &= \langle (0,0) | (\mathbf{d}_0 : 3)(\mathbf{d}_3 : 1)(\mathbf{d}_5 : 1) \rangle, \\ K_{10} &= \langle (0,0) | (\mathbf{d}_1 : 1)(\mathbf{d}_4 : 2)(\mathbf{d}_7 : 1) \rangle, \end{aligned}$$

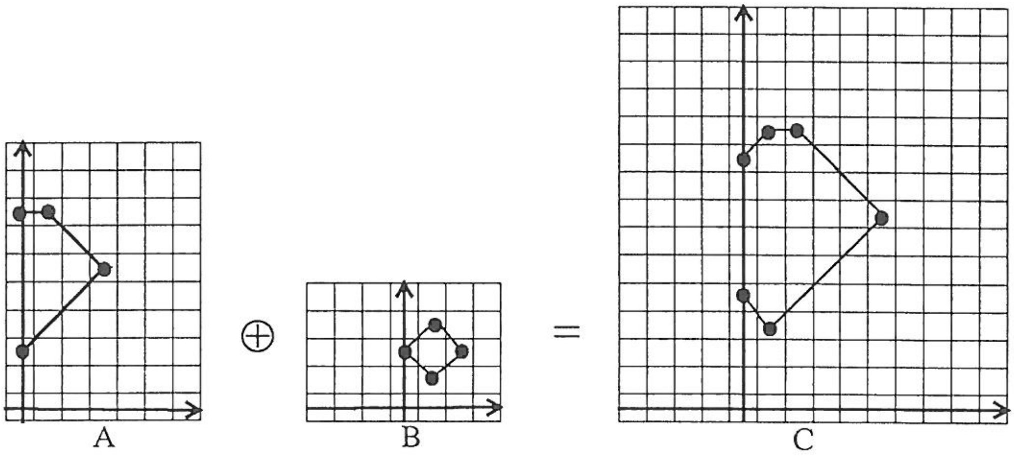

Fig. 7. Example of dilation. C is obtained by dilating A by B. Here $A = \langle(0, 2)|(\mathbf{d}_1 : 3)(\mathbf{d}_3 : 2)(\mathbf{d}_4 : 1)(\mathbf{d}_6 : 5)\rangle$, $B = \langle(1, 1)|(\mathbf{d}_1 : 1)(\mathbf{d}_3 : 1)(\mathbf{d}_5 : 1)(\mathbf{d}_7 : 1)\rangle$, and $C = \langle(1, 3)|(\mathbf{d}_1 : 4)(\mathbf{d}_3 : 3)(\mathbf{d}_4 : 1)(\mathbf{d}_5 : 1)(\mathbf{d}_6 : 5)(\mathbf{d}_7 : 1)\rangle$.

$$\begin{aligned} K_{11} &= \langle(0, 0)|(\mathbf{d}_1 : 1)(\mathbf{d}_3 : 1)(\mathbf{d}_6 : 2)\rangle, \\ K_{12} &= \langle(0, 0)|(\mathbf{d}_2 : 2)(\mathbf{d}_5 : 1)(\mathbf{d}_7 : 1)\rangle, \\ K_{13} &= \langle(0, 0)|(\mathbf{d}_1 : 1)(\mathbf{d}_3 : 1)(\mathbf{d}_5 : 1) \\ &\quad (\mathbf{d}_7 : 1)\rangle. \end{aligned} \tag{60}$$

That is, any restricted domain A can be decomposed as the k_ith-fold dilations of the 13 basis structuring elements K_i shown in figure 8:

$$\begin{aligned} A &= K_0 \oplus \left(\underset{k_1}{\oplus} K_1 \right) \oplus \left(\underset{k_2}{\oplus} K_2 \right) \oplus \\ &\quad \cdots \oplus \left(\underset{k_{13}}{\oplus} K_{13} \right), \end{aligned} \tag{61}$$

where K_i is a member of $\mathcal{K}$, the basis set of structuring elements, and k_i are nonnegative integers representing the number of times K_i is dilated. Notice that the K_i are triangles, lines, or a rhombus.

Comparing the left-hand side of (61) to its right-hand side, we see that for the above proposition to be true, the lengths of the sides of the restricted domains on the left-hand side and the right-hand side should be the same. We can compute the dilations on the right-hand side by using the B-code dilation and finding the lengths of the sides of the resulting restricted domain in terms of the k_i, $1 \le i \le 13$. Then we need to find a set of k_i such that the lengths of the sides of the resulting restricted domain are the same as those of A. In subsection 7.3 we will show that we can find a set of nonnegative integers k_i that satisfy the following relations:

$$\begin{aligned} n_0 &= k_1 + k_5 + k_6 + 2k_9, \\ n_1 &= k_2 + k_7 + k_{10} + k_{11} + k_{13}, \\ n_2 &= k_3 + k_5 + k_8 + 2k_{12}, \\ n_3 &= k_4 + k_6 + k_9 + k_{11} + k_{13}, \\ n_4 &= k_1 + k_7 + k_8 + 2k_{10}, \\ n_5 &= k_2 + k_5 + k_9 + k_{12} + k_{13}, \\ n_6 &= k_3 + k_6 + k_7 + 2k_{11}, \\ n_7 &= k_4 + k_8 + k_{10} + k_{12} + k_{13}, \end{aligned} \tag{62}$$

where n_i are the lengths of the sides of the restricted domain A.

7.3 *Linear-Space Interpretation of the Problem*

From subsection 7.2 we see that solving the decomposition problem is equivalent to finding all the solutions of the set of equations (62). The set of linear equations in (62) can be rewritten by using a matrix representation as follows:

$$\mathbf{T} \cdot \mathbf{K} = \mathbf{N}, \tag{63}$$

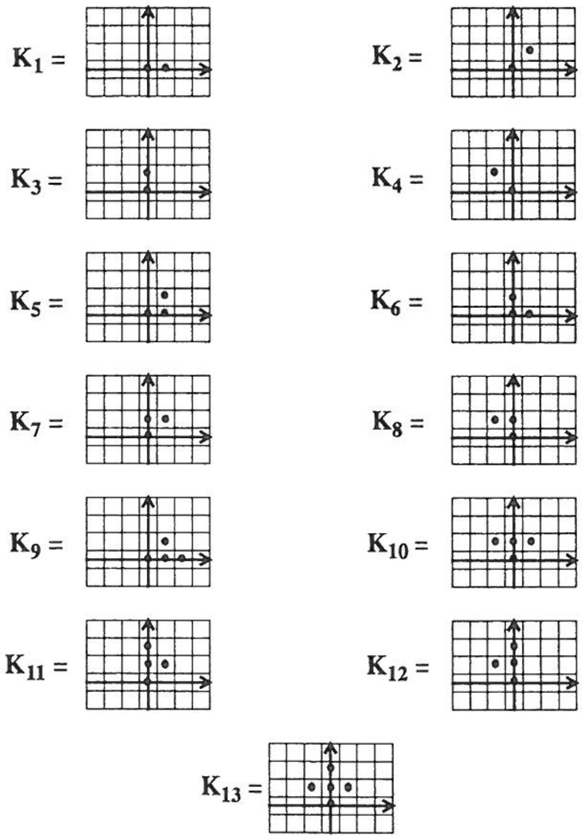

Fig. 8. Thirteen basis structuring elements $K_1, \ldots, K_{13}$. The respective B-codes are give in equation (60).

where $\mathbf{T}$ is the 8×13 transformation matrix

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \quad (64)$$

$\mathbf{N}$ is a 8×1 vector with nonnegative integer entries representing the restricted domain A in terms of its side lengths, and $\mathbf{K}$ is a 13×1 vector representing A in terms of its dilation decomposition.

Thus the decomposition problem can be restated as follows: given the matrix $\mathbf{T}$ and the vector $\mathbf{N}$, find a vector $\mathbf{K}$ such that its elements are nonnegative integers and it satisfies (63).

The matrix $\mathbf{T}$ is a linear transformation from a 13-dimensional linear space into an eight-dimensional linear space that transforms the vector $\mathbf{K}$ into the vector $\mathbf{N}$. Therefore to find the vector $\mathbf{K}$ we must find the preimage of the vector $\mathbf{N}$ in the transformation $\mathbf{T}$.

Since the dimension of the null-space $\mathbf{K}^h$ of the transformation $\mathbf{T}$ is not zero, the vector $\mathbf{N}$ has more than one preimage in $\mathbf{T}$ and there is more than one possible decomposition of the restricted domain. In general, we can express any preimage of $\mathbf{N}^A$, the $\mathbf{N}$ vector of a restricted domain A, as the sum of one particular preimage $\mathbf{K}^p$ and a vector from the null-space $\mathbf{K}^h$. Finally, all the solutions $\mathbf{K}^g$ can be expressed as $\mathbf{K}^g = \mathbf{K}^p + \mathbf{K}^h$. In subsection 7.3.1 we find a particular solution $\mathbf{K}^p$. In subsection 7.3.2 we compute the homogeneous solution $\mathbf{K}^h$. The general solution $\mathbf{K}^g$ is finally computed in subsection 7.3.3. The vector-space interpretation is summarized in figure 9.

7.3.1 Particular solution. We will find a particular solution of the system of equations (62) $\mathbf{K}^p$ by looking at the underlying geometry of the problem. A different proof by induction can be found in [6]. Note that finding a particular solution to (63) simultaneously proves the existence of the decomposition of a restricted domain. In this section we will give a constructive algorithm for finding such a decomposition. For each step we will give the geometry and follow it up with a lemma that proves that the geometrical construction is valid.

We start by noticing that a restricted domain A can be represented as the dilation of A translated to the origin with a point that translates it back to its original position. This is stated formally in Lemma 7.1, and an example is shown in figure 10.

LEMMA 7.1. Any restricted domain $A = \langle (i_A, j_A) | (\mathbf{d_0} : n_0)(\mathbf{d_1} : n_1) \cdots (\mathbf{d_7} : n_7) \rangle$ can be decomposed as

$$A = A^{(0)} \oplus K_0, \quad (65)$$

where $A^{(0)}$ is a restricted domain given by the

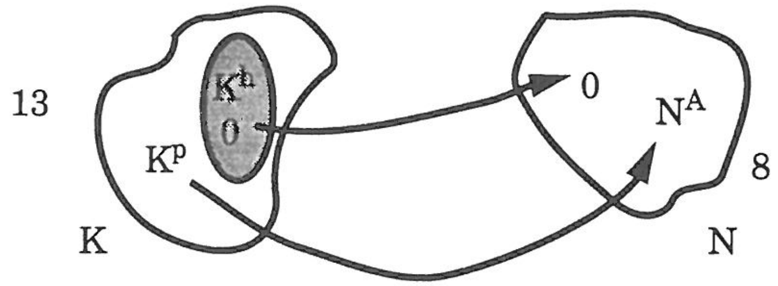

Fig. 9. T is a linear transformation from a 13-dimensional space, in which the restricted domains are represented in terms of dilation decompositions (**K**), to an eight-dimensional space, in which the restricted domains are represented in terms of their side lengths (**N**).

B-code $\langle(0, 0)|(\mathbf{d}_0 : n_0)(\mathbf{d}_1 : n_1)\cdots(\mathbf{d}_7 : n_7)\rangle$ and $K_0 = \langle(i_A, j_A)|\rangle$.

Proof. The proof follows immediately from the rule for B-code dilations:

$$\begin{aligned}(i_{A^{(0)}}, j_{A^{(0)}}) + (i_{K_0}, j_{K_0}) &= (0, 0) + (i_A, j_A) \\ &= (i_A, j_A) \qquad (66)\end{aligned}$$

and

$$n_i^{(0)} + 0 = n_i.$$

Since $n_i^{(0)} = n_i$, $A^{(0)}$ is just a translated version of A. Hence it is a restricted domain.

Next consider $A^{(0)}$, defined in Lemma 7.1. It can be represented as the dilation of a horizontal line, a vertical line, and a residual restricted domain $A^{(1)}$. The horizontal line is of length k_1 equal to the smaller of the two horizontal sides of $A^{(0)}$ (i.e., the minimum of $n_0^{(0)}$ and $n_4^{(0)}$), and the vertical line is of length k_3 equal to the smaller of the two vertical sides of $A^{(0)}$ (i.e., the minimum of $n_2^{(0)}$ and $n_6^{(0)}$). The restricted domain $A^{(1)}$ is a structuring element whose horizontal and vertical sides are smaller than those of $A^{(0)}$ by amounts k_1 and k_3, respectively, and whose other sides are equal. The horizontal and vertical lines can be represented as the k_1-fold dilation of structuring element K_1 and the k_3-fold dilation of structuring element K_3, respectively. An example is depicted in figure 11, and a formal statement is in Lemma 7.2.

LEMMA 7.2. The restricted domain $A^{(0)}$ defined in Lemma 7.1 can be further decomposed as

$$A^{(0)} = A^{(1)} \oplus \left(\underset{k_1}{\oplus} K_1\right) \oplus \left(\underset{k_3}{\oplus} K_3\right), \qquad (67)$$

where

$$\begin{aligned}K_1 &= \langle(0, 0)|(\mathbf{d}_0 : 1)(\mathbf{d}_4 : 1)\rangle, && (68)\\ K_3 &= \langle(0, 0)|(\mathbf{d}_2 : 1)(\mathbf{d}_6 : 1)\rangle, && (69)\\ k_1 &= \min[n_0^{(0)}, n_4^{(0)}], && (70)\\ k_3 &= \min[n_2^{(0)}, n_6^{(0)}], && (71)\end{aligned}$$

and $A^{(1)}$ is the restricted domain or diagonal line with

$$(i_{A^{(1)}}, j_{A^{(1)}}) = (0, 0), \qquad (72)$$

$$n_i^{(1)} = \begin{cases} n_i^{(0)} - k_1 & \text{if } i = 0 \text{ or } 4, \\ n_i^{(0)} - k_3 & \text{if } i = 2 \text{ or } 6, \\ n_i^{(0)} & \text{otherwise.} \end{cases} \qquad (73)$$

Proof. We will first prove that any B-coded shape with the n_i's as defined by (73) is a restricted domain or a diagonal line and hence $A^{(1)}$ is a restricted domain or a diagonal line. Next, we will show that the restricted domain obtained by computing the dilation on the right-hand side of (67) is in fact $A^{(0)}$.

From section 6 on restricted domains, for $A^{(1)}$ to be a restricted domain the $n_i^{(1)}$, $0 \le i \le 7$, must satisfy the following equations:

$$n_0^{(1)} + n_1^{(1)} + n_7^{(1)} = n_3^{(1)} + n_4^{(1)} + n_5^{(1)}, \qquad (74)$$

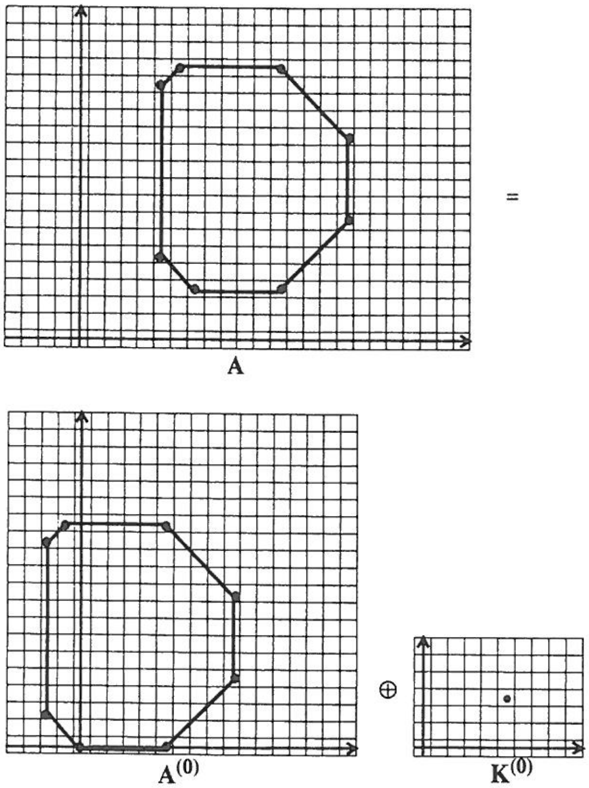

Fig. 10. Example of A decomposed as $K^{(0)} \oplus A^{(0)}$, where $A^{(0)}$ is A translated to the origin. Here $A = \langle(7, 3)|(\mathbf{d_0} : 5)(\mathbf{d_1} : 4)(\mathbf{d_2} : 5)(\mathbf{d_3} : 4)(\mathbf{d_4} : 6)(\mathbf{d_5} : 1)(\mathbf{d_6} : 10)(\mathbf{d_7} : 2)\rangle$, and $K_0 = \langle(7, 3)|\rangle$.

$$n_1^{(1)} + n_2^{(1)} + n_3^{(1)} = n_5^{(1)} + n_6^{(1)} + n_7^{(1)}. \quad (75)$$

By using the definition of $n_i^{(1)}$ given by equation (73), (74) can be rewritten as follows:

$$\left[n_0^{(0)} - k_1\right] + n_1^{(0)} + n_7^{(0)} = n_3^{(0)} + \left[n_4^{(0)} - k_1\right] + n_5^{(0)}.$$

Simplifying, we get

$$n_0^{(0)} + n_1^{(0)} + n_7^{(0)} = n_3^{(0)} + n_4^{(0)} + n_5^{(0)}. \quad (76)$$

However, this equation holds, since $A^{(0)}$ is a restricted domain. Similarly, we can show that (75) also holds. Note that $A^{(1)}$ is a diagonal line if $n_1^{(1)} = n_5^{(1)} \neq 0$ and $n_i^{(1)} = 0$ for $i \neq 1, 5$ or else if $n_3^{(1)} = n_7^{(1)} \neq 0$ and $n_i^{(1)} = 0$ for $i \neq 3, 7$.

Next, we show that the dilation on the right-hand side of (67) results in the restricted domain $A^{(0)}$. From the dilation rule the following relations between $A^{(0)}$ and $A^{(1)}$ must hold:

$$(i_{A^{(0)}}, j_{A^{(0)}}) = (i_{A^{(1)}}, j_{A^{(1)}}) + k_1(i_{K_1}, j_{K_1}) + k_3(i_{K_3}, j_{K_3}), \quad (77)$$

$$n_i^{(0)} = n_i^{(1)} + k_1(n_i^{K_1}) + k_3(n_i^{K_3}), \quad 0 \leq i \leq 7. \quad (78)$$

Since $(i_{K_1}, j_{K_1}) = (i_{K_3}, j_{K_3}) = (i_{A^{(1)}}, j_{A^{(1)}}) = (0, 0)$ and $(i_{A^{(0)}}, j_{A^{(0)}}) = (0, 0)$, equation (77) holds. Expanding the right-hand side of (78) for $0 \leq i \leq 7$, we get the following:

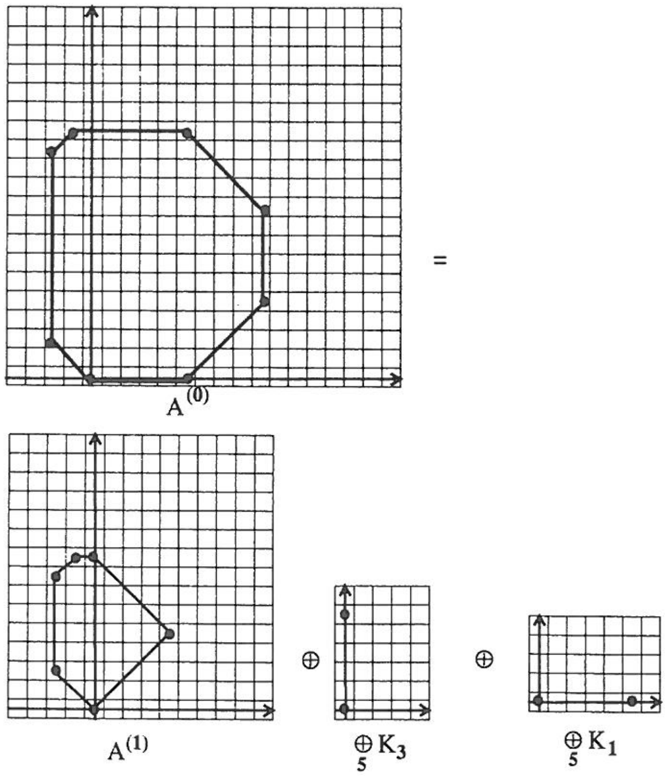

Fig. 11. Example of $A^{(0)}$ decomposed as $A^{(1)} \oplus \left(\underset{k_1}{\oplus} K_1 \right) \oplus \left(\underset{k_3}{\oplus} K_3 \right)$, where K_1 and K_3 are horizontal- and vertical-line structuring elements (see figure 8). Here $A^{(0)} = \langle (0, 0) | (\mathbf{d_0} : 5)(\mathbf{d_1} : 4)(\mathbf{d_2} : 5)(\mathbf{d_3} : 4)(\mathbf{d_4} : 6)(\mathbf{d_5} : 1)(\mathbf{d_6} : 10)(\mathbf{d_7} : 2) \rangle$, $k_1 = 5$, $k_3 = 5$, and $A^{(1)} = \langle (0, 0) | (\mathbf{d_0} : 5)(\mathbf{d_1} : 4)(\mathbf{d_3} : 4)(\mathbf{d_4} : 1)(\mathbf{d_5} : 1)(\mathbf{d_6} : 5)(\mathbf{d_7} : 2) \rangle$.

Case $i = 0, 4$:

$$
\begin{aligned}
n_i^{(1)} &+ k_1(n_i^{K_1}) + k_3(n_i^{K_3}) \\
&= \left[n_i^{(0)} - k_1 \right] + k_1(1) + k_3(0) \\
&= n_i^{(0)}.
\end{aligned}
$$

Case $i = 2, 6$:

$$
\begin{aligned}
n_i^{(1)} &+ k_1(n_i^{K_1}) + k_3(n_i^{K_3}) \\
&= \left[n_i^{(0)} - k_3 \right] + k_1(0) + k_3(1) \\
&= n_i^{(0)}.
\end{aligned}
$$

Case $i = 1, 3, 5, 7$:

$$
\begin{aligned}
n_i^{(1)} &+ k_1(n_i^{K_1}) + k_3(n_i^{K_3}) \\
&= n_i^{(0)} + k_1(0) + k_3(0) \\
&= n_i^{(0)}.
\end{aligned}
$$

Hence $A^{(0)}$ is the dilation of $A^{(1)}$, the k_1-fold dilation of K_1, and the k_3-fold dilation of K_3. Thus Lemma 7.2 holds.

Similarly, we represent the restricted domain $A^{(1)}$ as the dilation of two 45° diagonal lines, a rhombus, and a restricted domain $A^{(2)}$. An example of such a decomposition is illustrated in figure 12, and formal statement is in Lemma 7.3.

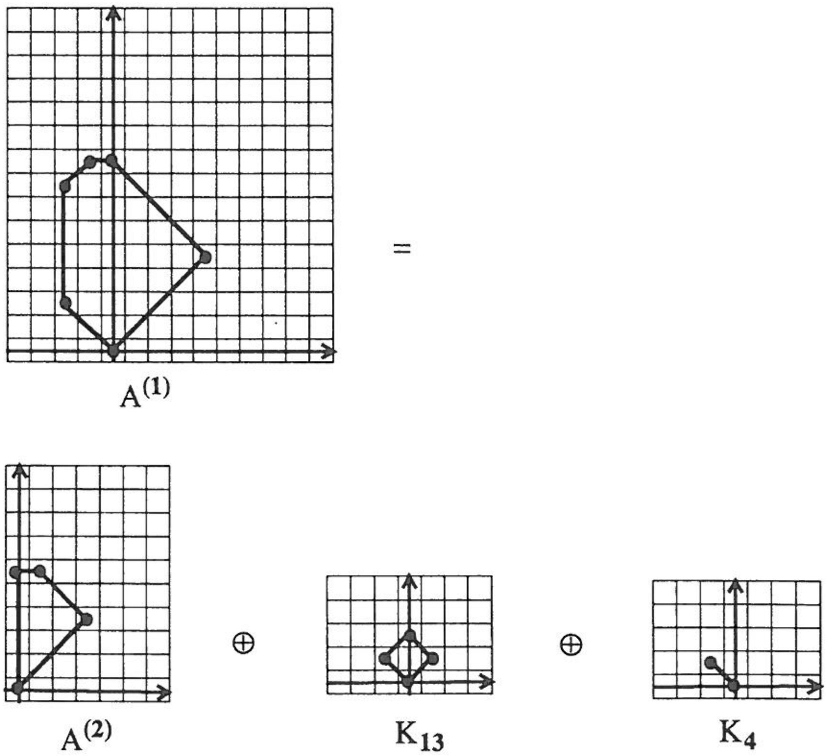

Fig. 12. Example of $A^{(1)}$ decomposed as $A^{(2)} \oplus \left(\underset{k_{13}}{\oplus} K_{13} \right) \oplus \left(\underset{k_2}{\oplus} K_2 \right) \oplus \left(\underset{k_4}{\oplus} K_4 \right)$, where K_2 and K_4 are diagonal-line structuring elements and K_{13} is a rhombus (see figure 8). Here $A^{(1)} = \langle (0, 0) | (\mathbf{d}_0 : 5)(\mathbf{d}_1 : 4)(\mathbf{d}_3 : 4)(\mathbf{d}_4 : 1)(\mathbf{d}_5 : 1)(\mathbf{d}_6 : 5)(\mathbf{d}_7 : 2) \rangle$, $A^{(2)} = \langle (0, 0) | (\mathbf{d}_1 : 3)(\mathbf{d}_3 : 2)(\mathbf{d}_4 : 1)(\mathbf{d}_6 : 5) \rangle$, $k_{13} = 1$, $k_2 = 0$, and $k_4 = 1$.

LEMMA 7.3. The restricted domain $A^{(1)}$ defined in Lemma 7.2 can be further decomposed as

$$A^{(1)} = A^{(2)} \oplus \left(\underset{k_{13}}{\oplus} K_{13} \right) \oplus \left(\underset{k_2}{\oplus} K_2 \right) \oplus \left(\underset{k_4}{\oplus} K_4 \right), \quad (79)$$

where

$$K_{13} = \langle (0, 0) | (\mathbf{d}_1 : 1)(\mathbf{d}_3 : 1)(\mathbf{d}_5 : 1)(\mathbf{d}_7 : 1) \rangle, \quad (80)$$

$$K_2 = \langle (0, 0) | (\mathbf{d}_1 : 1)(\mathbf{d}_5 : 1) \rangle, \quad (81)$$

$$K_4 = \langle (0, 0) | (\mathbf{d}_3 : 1)(\mathbf{d}_7 : 1) \rangle, \quad (82)$$

$$k_{13} = \min[n_1^{(1)}, n_3^{(1)}, n_5^{(1)}, n_7^{(1)}], \quad (83)$$

$$k_2 = \min[n_1^{(1)}, n_5^{(1)}] - k_{13}, \quad (84)$$

$$k_4 = \min[n_3^{(1)}, n_7^{(1)}] - k_{13}, \quad (85)$$

and $A^{(2)}$ is a restricted domain with

$$(i_{A^{(2)}}, j_{A^{(2)}}) = (0, 0), \quad (86)$$

$$n_i^{(2)} = \begin{cases} n_i^{(1)} - k_2 - k_{13} & \text{if } i = 1 \text{ or } 5, \\ n_i^{(1)} - k_4 - k_{13} & \text{if } i = 3 \text{ or } 7, \\ n_i^{(1)} & \text{otherwise.} \end{cases} \quad (87)$$

Proof. The proof is similar to that of Lemma 7.2. Notice that from the definition of k_2, k_4, and k_{13} either k_2 or k_4 is equal to zero. Thus the dilation on the right-hand side of (79) is defined, since only one diagonal line is involved (the other one reduces to a point). As in Lemma 7.2, we will proceed by first proving that the B-coded shape with the n_i's as defined by (87) is a restricted domain and hence $A^{(2)}$ is a restricted domain. Next, we will show that the restricted domain obtained by computing the dilation on the right-hand side of (79) is in fact $A^{(1)}$.

For $A^{(2)}$ to be a restricted domain the $n_i^{(2)}$, $0 \le i \le 7$, must satisfy the following equations:

$$n_0^{(2)} + n_1^{(2)} + n_7^{(2)} = n_3^{(2)} + n_4^{(2)} + n_5^{(2)}, \quad (88)$$
$$n_1^{(2)} + n_2^{(2)} + n_3^{(2)} = n_5^{(2)} + n_6^{(2)} + n_7^{(2)}. \quad (89)$$

By using the definition of $n_i^{(2)}$ given by (87), (88) can be rewritten as follows:

$$n_0^{(1)} + \left[n_1^{(1)} - k_2 - k_{13}\right] + \left[n_7^{(1)} - k_4 - k_{13}\right] = \left[n_3^{(1)} - k_4 - k_{13}\right] + n_4^{(1)} + \left[n_5^{(1)} - k_2 - k_{13}\right].$$

Simplifying, we get

$$n_0^{(1)} + n_1^{(1)} + n_7^{(1)} = n_3^{(1)} + n_4^{(1)} + n_5^{(1)}. \quad (90)$$

However, this equation holds, since $A^{(1)}$ is a restricted domain. Similarly, we can show that (89) also holds.

Next, we show that the dilation on the right-hand side of (79) results in the restricted domain $A^{(1)}$. From the dilation rule the following relations between $A^{(1)}$ and $A^{(2)}$ must hold:

$$(i_{A^{(1)}}, j_{A^{(1)}}) = (i_{A^{(2)}}, j_{A^{(2)}}) + k_2(i_{K_2}, j_{K_2}) + k_4(i_{K_4}, j_{K_4}) + k_{13}(i_{K_{13}}, j_{K_{13}}), \quad (91)$$
$$n_i^{(1)} = n_i^{(2)} + k_2(n_i^{K_2}) + k_4(n_i^{K_4}) + k_{13}(n_i^{K_{13}}), \quad 0 \le i \le 7. \quad (92)$$

Since $(i_{K_2}, j_{K_2}) = (i_{K_4}, j_{K_4}) = (i_{K_{13}}, j_{K_{13}}) = (i_{A^{(2)}}, j_{A^{(2)}}) = (0, 0)$ and from Lemma 7.2 $(i_{A^{(1)}}, j_{A^{(1)}}) = (0, 0)$, equation (91) holds. Expanding the right-hand side of (92) for $0 \le i \le 7$, we get the following:

Case $i = 1, 5$:

$$n_i^{(2)} + k_2(n_i^{K_2}) + k_4(n_i^{K_4}) + k_{13}(n_i^{K_{13}}) = \left[n_i^{(1)} - k_2 - k_{13}\right] + k_2(1) + k_3(0) + k_{13}(1) = n_i^{(1)}.$$

Case $i = 3, 7$:

$$n_i^{(2)} + k_2(n_i^{K_2}) + k_4(n_i^{K_4}) + k_{13}(n_i^{K_{13}}) = \left[n_i^{(1)} - k_4 - k_{13}\right] + k_2(0) + k_4(1) + k_{13}(1) = n_i^{(1)}.$$

Case $i = 0, 2, 4, 6$:

$$n_i^{(2)} + k_2(n_i^{K_2}) + k_4(n_i^{K_4}) + k_{13}(n_i^{K_{13}}) = n_i^{(1)} + k_2(0) + k_4(0) + k_{13}(0) = n_i^{(1)}.$$

Hence $A^{(1)}$ is the dilation of $A^{(2)}$, the k_2-fold dilation of K_2, the k_4-fold dilation of K_4, and the k_{13}-fold dilation of K_{13}. Thus Lemma 7.3 holds.

The resulting restricted domain $A^{(2)}$ is a restricted domain with four or fewer sides that satisfy the following properties:

$$n_0^{(2)} + n_1^{(2)} + n_7^{(2)} = n_3^{(2)} + n_4^{(2)} + n_5^{(2)}, \quad (93)$$
$$n_1^{(2)} + n_2^{(2)} + n_3^{(2)} = n_5^{(2)} + n_6^{(2)} + n_7^{(2)}, \quad (94)$$
$$n_0^{(2)} \text{ or } n_4^{(2)} \text{ or both} = 0, \quad (95)$$
$$n_1^{(2)} \text{ or } n_5^{(2)} \text{ or both} = 0, \quad (96)$$
$$n_2^{(2)} \text{ or } n_6^{(2)} \text{ or both} = 0, \quad (97)$$
$$n_3^{(2)} \text{ or } n_7^{(2)} \text{ or both} = 0, \quad (98)$$

and

$$n_i^{(2)} \ge 0 \quad \text{for } 0 \le i \le 7. \quad (99)$$

The restricted domain $A^{(2)}$ can be decomposed further. In the case for which $A^{(2)}$ is a four-sided restricted domain, it can be decomposed as the n-fold dilations of two triangles from the basis set. If $A^{(2)}$ is a triangle, then it can be expressed as the n-fold dilation of one of the triangles in the basis set. Otherwise, $A^{(2)}$ is just one point, the origin, and does not have to be decomposed any further. This is stated in the following lemma.

LEMMA 7.4. If the restricted domain $A^{(2)}$ in Lemma 7.3 has some nonzero $n_i^{(2)}$, it can be further decomposed as either the n-fold dilation of one of the triangular basis structuring elements or the n-fold dilation of two different triangular structuring elements from the basis set $\mathcal{K}$.

Proof. It can be easily verified from the properties of $A^{(2)}$ that there are only eight four-sided restricted domains that can satisfy the constraint equations (93)–(99). They are

$$\langle (0,0) \mid (\mathbf{d_0} : n_0^{(2)})(\mathbf{d_1} : n_1^{(2)}) (\mathbf{d_3} : n_3^{(2)})(\mathbf{d_6} : n_6^{(2)})\rangle, \quad (100)$$

$$\langle (0,0) \mid (\mathbf{d_1} : n_1^{(2)})(\mathbf{d_2} : n_2^{(2)}) (\mathbf{d_4} : n_4^{(2)})(\mathbf{d_7} : n_7^{(2)})\rangle, \quad (101)$$

$$\langle (0,0) \mid (\mathbf{d_0} : n_0^{(2)})(\mathbf{d_2} : n_2^{(2)}) (\mathbf{d_3} : n_3^{(2)})(\mathbf{d_5} : n_5^{(2)})\rangle, \quad (102)$$

$$\langle (0,0) \mid (\mathbf{d_1} : n_1^{(2)})(\mathbf{d_3} : n_3^{(2)}) (\mathbf{d_4} : n_4^{(2)})(\mathbf{d_6} : n_6^{(2)})\rangle, \quad (103)$$

$$\langle (0,0) \mid (\mathbf{d_2} : n_2^{(2)})(\mathbf{d_4} : n_4^{(2)}) (\mathbf{d_5} : n_5^{(2)})(\mathbf{d_7} : n_7^{(2)})\rangle, \quad (104)$$

$$\langle (0,0) \mid (\mathbf{d_0} : n_0^{(2)})(\mathbf{d_3} : n_3^{(2)}) (\mathbf{d_5} : n_5^{(2)})(\mathbf{d_6} : n_6^{(2)})\rangle, \quad (105)$$

$$\langle (0,0) \mid (\mathbf{d_1} : n_1^{(2)})(\mathbf{d_4} : n_4^{(2)}) (\mathbf{d_6} : n_6^{(2)})(\mathbf{d_7} : n_7^{(2)})\rangle, \quad (106)$$

$$\langle (0,0) \mid (\mathbf{d_0} : n_0^{(2)})(\mathbf{d_2} : n_2^{(2)}) (\mathbf{d_5} : n_5^{(2)})(\mathbf{d_7} : n_7^{(2)})\rangle. \quad (107)$$

All the eight possibilities for $A^{(2)}$ are shown in figure 13. Notice that all the restricted domains are the rotated versions of one another. Furthermore, some of the $n_i^{(2)}$ can be zero and thus make $A^{(2)}$ a triangle or a point (the origin), depending on whether one or all four n_i are zero. It is not possible for only two $n_i^{(2)}$ to be nonzero, since in that case (93) and (94) will not hold. If all the $n_i^{(2)}$ are zero, the $A^{(2)}$ is a single point, the origin, and does not have to be decomposed further. The case of only three nonzero $n_i^{(2)}$ can be treated as a special case of the four nonzero $n_i^{(2)}$. Thus we need to prove that $A^{(2)}$ with four

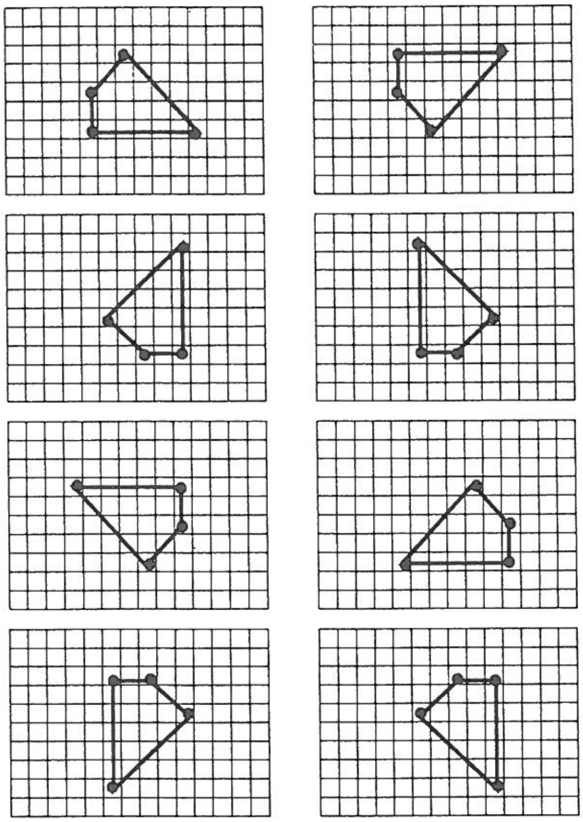

Fig. 13. The eight possibilities for $A^{(2)}$. Each is a dilation of two triangles.

nonzero $n_i^{(2)}$ can be decomposed as the n-fold dilations of two triangles.

We will solve the decomposition problem for the eight possible $A^{(2)}$ on a case-by-case basis:

Case (i).

$$A^{(2)} = \langle (0,0) | (\mathbf{d_0} : n_0^{(2)})(\mathbf{d_1} : n_1^{(2)}) (\mathbf{d_3} : n_3^{(2)})(\mathbf{d_6} : n_6^{(2)}) \quad (108)$$

For this case, the restricted domains obtained after decomposition should necessarily have $n_2 = n_4 = n_5 = n_7 = 0$, because from the dilation rule the dilation of a restricted domain having a nonzero n_i with any restricted domain will have a nonzero n_i. This is also evident from (62). The only basis structuring elements that satisfy the above conditions are K_6 and K_1.

Thus we should now find k_6 and k_{11} such that

$$A^{(2)} = \left(\underset{k_6}{\oplus} K_6 \right) \oplus \left(\underset{k_{11}}{\oplus} K_{11} \right).$$

If we compute the dilation on the right-hand side, we see that k_6 and k_{11} must satisfy the following relations:

$$n_0^{(2)} = k_6, \quad (109)$$
$$n_1^{(2)} = k_{11}, \quad (110)$$
$$n_3^{(2)} = k_6 + k_{11}, \quad (111)$$
$$n_6^{(2)} = k_6 + 2k_{11}. \quad (112)$$

If we let $k_6 = n_0^{(2)}$ and $k_{11} = n_1^{(2)}$, we see from (93) and (94) that the above equations (111) and (112) hold. Thus we have found a decomposition for $A^{(2)}$.

All the other seven cases can be approached in a similar fashion. We give the results below:

Case (ii).

$$\begin{aligned} A^{(2)} &= \langle (0,0) | (\mathbf{d_1} : n_1^{(2)})(\mathbf{d_2} : n_2^{(2)}) \\ &\quad (\mathbf{d_4} : n_4^{(2)})(\mathbf{d_7} : n_7^{(2)}) \rangle \\ &= \left(\underset{k_8}{\oplus} K_8 \right) \oplus \left(\underset{k_{10}}{\oplus} K_{10} \right), \quad (113) \end{aligned}$$

where $k_8 = n_2^{(2)}$ and $k_{10} = n_1^{(2)}$.

Case (iii).

$$\begin{aligned} A^{(2)} &= \langle (0,0) | (\mathbf{d_0} : n_0^{(2)})(\mathbf{d_2} : n_2^{(2)}) \\ &\quad (\mathbf{d_3} : n_3^{(2)})(\mathbf{d_5} : n_5^{(2)}) \rangle \\ &= \left(\underset{k_5}{\oplus} K_5 \right) \oplus \left(\underset{k_9}{\oplus} K_9 \right), \quad (114) \end{aligned}$$

where $k_5 = n_2^{(2)}$ and $k_9 = n_3^{(2)}$.

Case (iv).

$$\begin{aligned} A^{(2)} &= \langle (0,0) | (\mathbf{d_1} : n_1^{(2)})(\mathbf{d_3} : n_3^{(2)}) \\ &\quad (\mathbf{d_4} : n_4^{(2)})(\mathbf{d_6} : n_6^{(2)}) \rangle \\ &= \left(\underset{k_7}{\oplus} K_7 \right) \oplus \left(\underset{k_{11}}{\oplus} K_{11} \right), \quad (115) \end{aligned}$$

where $k_{11} = n_3^{(2)}$ and $k_7 = n_4^{(2)}$.

Case (v).

$$\begin{aligned} A^{(2)} &= \langle (0,0) | (\mathbf{d_2} : n_2^{(2)})(\mathbf{d_4} : n_4^{(2)}) \\ &\quad (\mathbf{d_5} : n_5^{(2)})(\mathbf{d_7} : n_7^{(2)}) \rangle \\ &= \left(\underset{k_8}{\oplus} K_8 \right) \oplus \left(\underset{k_{12}}{\oplus} K_{12} \right), \quad (116) \end{aligned}$$

where $k_8 = n_4^{(2)}$ and $k_{12} = n_5^{(2)}$.

Case (vi).

$$\begin{aligned} A^{(2)} &= \langle (0,0) | (\mathbf{d_0} : n_0^{(2)})(\mathbf{d_3} : n_3^{(2)}) \\ &\quad (\mathbf{d_5} : n_5^{(2)})(\mathbf{d_6} : n_6^{(2)}) \rangle \quad (117) \\ &= \left(\underset{k_6}{\oplus} K_6 \right) \oplus \left(\underset{k_9}{\oplus} K_9 \right), \quad (118) \end{aligned}$$

where $k_9 = n_5^{(2)}$ and $k_6 = n_6^{(2)}$.

Case (vii).

$$\begin{aligned} A^{(2)} &= \langle (0,0) | (\mathbf{d_1} : n_1^{(2)})(\mathbf{d_4} : n_4^{(2)}) \\ &\quad (\mathbf{d_6} : n_6^{(2)})(\mathbf{d_7} : n_7^{(2)}) \rangle \quad (119) \\ &= \left(\underset{k_7}{\oplus} K_7 \right) \oplus \left(\underset{k_{10}}{\oplus} K_{10} \right), \quad (120) \end{aligned}$$

where $k_7 = n_6^{(2)}$ and $k_{10} = n_7^{(2)}$.

Case (viii).

$$\begin{aligned} A^{(2)} &= \langle (0,0) | (\mathbf{d_0} : n_0^{(2)})(\mathbf{d_2} : n_2^{(2)}) \\ &\quad (\mathbf{d_5} : n_5^{(2)})(\mathbf{d_7} : n_7^{(2)}) \rangle \quad (121) \\ &= \left(\underset{k_5}{\oplus} K_5 \right) \oplus \left(\underset{k_{12}}{\oplus} K_{12} \right), \quad (122) \end{aligned}$$

where $k_5 = n_0^{(2)}$ and $k_{12} = n_7^{(2)}$. An example is shown in figure 14.

The remaining case is the case for which $A^{(2)}$ is a triangle, that is, three of the n_i's are nonzero. We can still approach the decomposition problem by using the solution for the four-sided case. The only difference in this case is that one of the two k_i will turn out to be zero. We have thus proved lemma 7.4

Now we can formally state the decomposition problem as the following theorem:

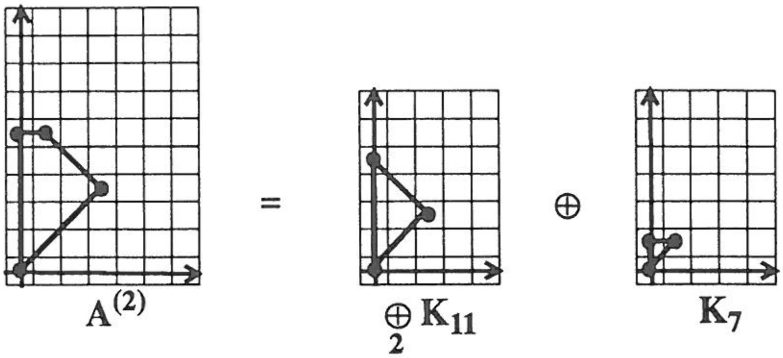

Fig. 14. Example of decomposition of a four-sided $A^{(2)}$ as dilation of two triangles, $A^{(2)} = \left(\underset{k_{11}}{\oplus} K_{11}\right) \oplus \left(\underset{k_7}{\oplus} K_7\right)$, where K_{11} and K_7 are triangles (see figure 8). Here $A^{(2)} = \langle(0, 0)|(\mathbf{d_1} : 3)(\mathbf{d_3} : 2)(\mathbf{d_4} : 1)(\mathbf{d_6} : 5)\rangle$, $k_{11} = 2$, and $k_7 = 1$.

THEOREM 7.1. Any restricted domain A can be decomposed as the k_ith-fold dilations of 13 basis structuring elements K_i:

$$A = K_0 \oplus \left(\underset{k_1}{\oplus} K_1\right) \oplus \left(\underset{k_2}{\oplus} K_2\right) \oplus \cdots \oplus \left(\underset{k_{13}}{\oplus} K_{13}\right), \tag{123}$$

where $K_i \in \mathcal{K}$, the basis set of structuring elements, are given by the B-codes in figure 8 and k_i are nonnegative integers representing the number of times K_i is dilated.

Proof. From Lemmas 7.1, 7.2, and 7.3 we see that there is a sequence of constructive steps by which we can find a decomposition of any restricted domain A as the n-fold dilations of dilations of 13 basis structuring elements $K_i \in \mathcal{K}$.

7.3.2 Homogenous solution. The homogeneous equation associated with (63) is given as

$$\mathbf{T} \cdot \mathbf{K} = 0. \tag{124}$$

The solution of (124) is the null subspace of the space spanned by the rows of the matrix $\mathbf{T}$ or, in other words, the kernel of the transformation $\mathbf{T}$. To determine the dimension of the kernel and a set of basis vectors spanning it, we can transform the matrix $\mathbf{T}$ to the echelon form [20]. On doing so we get the equation

$$\mathbf{T}^{(1)} \cdot \mathbf{K} = 0, \tag{125}$$

where

$$\mathbf{T}^{(1)} = \begin{bmatrix} 1&0&0&0&0&0&1&1&0&2&0&0&0 \\ 0&1&0&0&0&0&1&0&0&1&1&0&1 \\ 0&0&1&0&0&0&1&1&-1&1&1&1&0 \\ 0&0&0&1&0&0&0&1&0&1&0&1&1 \\ 0&0&0&0&1&0&-1&0&1&-1&-1&1&0 \\ 0&0&0&0&0&1&0&-1&1&-1&1&-1&0 \end{bmatrix}. \tag{126}$$

Since the number of rows in the matrix $\mathbf{T}^{(1)}$ is six, the transformation $\mathbf{T}$ is a mapping of a 13-dimensional space onto a six-dimensional space with a seven-dimensional null space. This space is, of course, a subset of the 13-dimensional domain space. A set of basis vectors $\mathbf{b}_j$, $1 \leq j \leq 7$, that span the null space can be found by assigning $k_{6+j} = 1$ and the other $k_7, \ldots, k_{13}$ to zero and then solving the linear system (126) to determine the corresponding $k_1, \ldots, k_6$. The set of basis vectors obtained on doing so is

$$\mathbf{B} = [\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3\ \mathbf{b}_4\ \mathbf{b}_5\ \mathbf{b}_6\ \mathbf{b}_7] \tag{127}$$

$$
= \begin{bmatrix}
-1 & -1 & 0 & -2 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & -1 & 0 & -1 \\
-1 & -1 & 1 & -1 & -1 & -1 & 0 \\
0 & -1 & 0 & -1 & 0 & -1 & -1 \\
1 & 0 & -1 & 1 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 & -1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}. \quad (128)
$$

The homogeneous solution $\mathbf{K}^h$ of $\mathbf{T}$ are, then, all the vectors expressible as the linear combination of the basis vectors of the null space. That is,

$$
\mathbf{K}^h = \sum_{i=1}^{7} \alpha_i \mathbf{b}_i, \quad \alpha_i \in \mathbf{R}. \qquad (129)
$$

7.3.3 General solution. From linear spaces we know that the general solution to (63) is the sum of the particular solution and the homogeneous solution. That is,

$$
\begin{aligned}
\mathbf{K}^g &= \mathbf{K}^p + \mathbf{K}^h && (130) \\
&= \mathbf{K}^p + \sum_{i=1}^{7} \alpha_i \mathbf{b}_i, \quad \alpha_i \in \mathbf{R}. && (131)
\end{aligned}
$$

However, we know that the general solution can have only nonnegative entries, since they represent the n-fold dilation of the structuring elements. Furthermore, the entries of the particular solution $\mathbf{K}^p$ are nonnegative integers. Thus the α_i's must belong to $\mathbf{Z}$ instead of to $\mathbf{R}$. Thus the equation becomes

$$
\mathbf{K}^g = \mathbf{K}^p + \sum_{i=1}^{7} \alpha_i \mathbf{b}_i, \quad \alpha_i \in \mathbf{Z}. \qquad (132)
$$

We notice that (62) provides upper and lower bounds for the k_i's, since the right-hand sides are additions of nonnegative integers k_i, which cannot exceed the nonnegative integer constants on the left-hand sides. For example, two equations involving k_1 in (62) are

$$
\begin{aligned}
n_0 &= k_1 + k_5 + k_6 + 2k_9, && (133) \\
n_4 &= k_1 + k_7 + k_8 + 2k_{10}, && (134)
\end{aligned}
$$

where the n_i are the side lengths of the restricted domains and the k_i are nonnegative integers representing the k_ith-fold dilation of the K_i basis structuring element. Thus both n_i and k_i are nonnegative integers, and it follows that k_1 attains the maximum value when all other k_i are zero. Thus the maximum value that k_1 can attain with the above two constraints is $\min[n_0, n_4]$. Furthermore, since we know that the lower bound of n_i is zero, the lower bound of the k_i's is zero. We can similarly compute the upper and lower bounds for all k_i's. All the bounds obtained are

$$
\begin{aligned}
0 \le\ & k_1 \le \min[n_0, n_4], && (135) \\
0 \le\ & k_2 \le \min[n_1, n_5], && (136) \\
0 \le\ & k_3 \le \min[n_2, n_6], && (137) \\
0 \le\ & k_4 \le \min[n_3, n_7], && (138) \\
0 \le\ & k_5 \le \min[n_0, n_2, n_5], && (139) \\
0 \le\ & k_6 \le \min[n_0, n_3, n_6], && (140) \\
0 \le\ & k_7 \le \min[n_1, n_4, n_6], && (141) \\
0 \le\ & k_8 \le \min[n_2, n_4, n_7], && (142) \\
0 \le\ & k_9 \le \min[\lfloor n_0/2 \rfloor, n_3, n_5], && (143) \\
0 \le\ & k_{10} \le \min[n_1, \lfloor n_4/2 \rfloor, n_7], && (144) \\
0 \le\ & k_{11} \le \min[n_1, n_3, \lfloor n_6/2 \rfloor], && (145) \\
0 \le\ & k_{12} \le \min[\lfloor n_2/2 \rfloor, n_5, n_7], && (146) \\
0 \le\ & k_{13} \le \min[n_1, n_3, n_5, n_7]. && (147)
\end{aligned}
$$

We will represent these bounds compactly by using matrices:

$$
\mathbf{0} \le \mathbf{K}^g \le \mathbf{K}^{\max}, \qquad (148)
$$

where $\mathbf{K}^{\max}$ is a vector having the 13 upper bounds as its elements. Using (132), we get

$$
\mathbf{0} \le \mathbf{K}^p + \sum_{i=1}^{7} \alpha_i \mathbf{b}_i \le \mathbf{K}^{\max}, \quad \alpha_i \in \mathbf{Z}. \qquad (149)
$$

Rearranging, we obtain

$$
-\mathbf{K}^p \le \sum_{i=1}^{7} \alpha_i \mathbf{b}_i \le \mathbf{K}^{\max} - \mathbf{K}^p, \quad \alpha_i \in \mathbf{Z}. \qquad (150)
$$

From the bottom seven rows of the inequality, we get upper and lower bounds on α_i's. Thus the bounds on α_i are

$$
-\mathbf{K}^p_{6+i} \le \alpha_i \le \mathbf{K}^{\max}_{6+i} - \mathbf{K}^p_{6+i}, \quad \text{for } 1 \le i \le 7. \qquad (151)
$$

Thus the general solution $\mathbf{K}^g$ can be found by substituting the α_i values from the valid domain defined in (151) in (132). It must be verified that the $\mathbf{K}^g$ thus obtained is within the bounds given in (148), in which case it is a solution. Thus must be done for all points in the α space.

7.4 Algorithms

This section reviews the algorithm for finding the particular and general decompositions of the restricted domains. The algorithm for finding the particular solution gives us one possible decomposition. The algorithm for finding the general solution gives us all the possible decompositions of the restricted domain. Algorithm 3 is for finding the particular solution and is based on subsection 7.3.1. Algorithm 4 is for generating all the solutions of the decomposition problem. It is explained in detail in subsections 7.3.2 and 7.3.3.

ALGORITHM 3. Particular solution of the decomposition.

procedure DecomposeGeneral(A, $\mathbf{K}^p$)

Input:
RDObject A;
Output:
ArrayObject $\mathbf{K}^p$;

begin

STEP 1:

Initialize all k_i's, $1 \leq i \leq 13$, and K_0 to zero;

STEP 2:

$K_0 = \langle(i_A, j_A)|\rangle$;

$n_i^{(0)} = n_i,\ 0 \leq i \leq 7$;

STEP 3:

$k_1 = \min[n_0^{(0)}, n_4^{(0)}]$;

$k_3 = \min[n_2^{(0)}, n_6^{(0)}]$;

$$n_i^{(1)} = \begin{cases} n_i^{(0)} - k_1 & \text{if } i = 0 \text{ or } 4, \\ n_i^{(0)} - k_3 & \text{if } i = 2 \text{ or } 6, \\ n_i^{(0)} & \text{otherwise.} \end{cases}$$

STEP 4:

$k_{13} = \min[n_1^{(1)}, n_3^{(1)}, n_5^{(1)}, n_7^{(1)}]$;

$k_2 = \min[n_1^{(1)}, n_5^{(1)}] - k_{13}$;

$k_4 = \min[n_3^{(1)}, n_7^{(1)}] - k_{13}$;

$$n_i^{(2)} = \begin{cases} n_i^{(1)} - k_2 - k_{13} & \text{if } i = 1 \text{ or } 5, \\ n_i^{(1)} - k_4 - k_{13} & \text{if } i = 3 \text{ or } 7, \\ n_i^{(1)} & \text{otherwise.} \end{cases}$$

STEP 5:

;Count the number of $n_i^{A^{(2)}}$ that are nonzero.

;The count can be 0, 3, or 4.

case count equals 4:

if $(n_0^{(2)} \neq 0$ and $n_1^{(2)} \neq 0)$
  $k_6 = n_0^{(2)}$ and $k_{11} = n_1^{(2)}$;
if $(n_1^{(2)} \neq 0$ and $n_2^{(2)} \neq 0)$
  $k_8 = n_2^{(2)}$ and $k_{10} = n_1^{(2)}$;
if $(n_2^{(2)} \neq 0$ and $n_3^{(2)} \neq 0)$
  $k_5 = n_2^{(2)}$ and $k_9 = n_3^{(2)}$;
if $(n_3^{(2)} \neq 0$ and $n_4^{(2)} \neq 0)$
  $k_{11} = n_3^{(2)}$ and $k_7 = n_4^{(2)}$;
if $(n_4^{(2)} \neq 0$ and $n_5^{(2)} \neq 0)$
  $k_8 = n_4^{(2)}$ and $k_{12} = n_5^{(2)}$;
if $(n_5^{(2)} \neq 0$ and $n_6^{(2)} \neq 0)$
  $k_9 = n_5^{(2)}$ and $k_6 = n_6^{(2)}$;
if $(n_6^{(2)} \neq 0$ and $n_7^{(2)} \neq 0)$
  $k_7 = n_6^{(2)}$ and $k_{10} = n_7^{(2)}$;
if $(n_0^{(2)} \neq 0$ and $n_7^{(2)} \neq 0)$
  $k_5 = n_0^{(2)}$ and $k_{12} = n_7^{(2)}$;

case count equals 3:

if $(n_0^{(2)} \neq 0$ and $n_1^{(2)} \neq 0$ and $n_5^{(2)} \neq 0)$
  $k_5 = n_0^{(2)}$;
if $(n_0^{(2)} \neq 0$ and $n_3^{(2)} \neq 0$ and $n_6^{(2)} \neq 0)$
  $k_6 = n_0^{(2)}$;
if $(n_1^{(2)} \neq 0$ and $n_4^{(2)} \neq 0$ and $n_6^{(2)} \neq 0)$
  $k_7 = n_1^{(2)}$;
if $(n_2^{(2)} \neq 0$ and $n_4^{(2)} \neq 0$ and $n_7^{(2)} \neq 0)$
  $k_8 = n_2^{(2)}$;
if $(n_0^{(2)} \neq 0$ and $n_3^{(2)} \neq 0$ and $n_5^{(2)} \neq 0)$
  $k_9 = n_3^{(2)}$;
if $(n_1^{(2)} \neq 0$ and $n_4^{(2)} \neq 0$ and $n_7^{(2)} \neq 0)$
  $k_{10} = n_1^{(2)}$;
if $(n_1^{(2)} \neq 0$ and $n_3^{(2)} \neq 0$ and $n_6^{(2)} \neq 0)$

$k_{11} = n_1^{(2)}$;
if $(n_2^{(2)} \neq 0$ and $n_5^{(2)} \neq 0$ and $n_7^{(2)} \neq 0)$
$k_{12} = n_5^{(2)}$;
case count equals 0:
break;
end DecomposeParticular;

ALGORITHM 4. General solution of the decomposition.

procedure DecomposeGeneral(A, $\mathbf{K}^p$, $\mathbf{K}^g$)

Input:
RDObject A;
ArrayObject $\mathbf{K}^p$;
Output:
ArrayObject $\mathbf{K}^g$;

begin
Initialize $\mathbf{K}^{\max}$ by using the set of equations in (147);
Initialize the bounds on $\boldsymbol{\alpha}$: $-\mathbf{K}^p_{6+i} \leq \alpha_i \leq \mathbf{K}^{\max}_{6+i} - \mathbf{K}^p_{6+i}$ for $1 \leq i \leq 7$;
for each $\boldsymbol{\alpha}$ within bound **do**
Construct $\mathbf{K}^g = \mathbf{K}^p + \mathbf{B} \cdot \boldsymbol{\alpha}$;
if $\mathbf{0} \leq \mathbf{K}^g \leq \mathbf{K}^{\max}$
Store $\mathbf{K}^g$;
end For;
end DecomposeGeneral;

7.5 Complexity of the Algorithms

Algorithm 3 for the particular solution of the decomposition of a restricted domain consists of only assignment statements and comparisons and contains no loops. Thus the algorithm is finite time.

The complexity of Algorithm 4 for generating all the possible decompositions of a restricted domain is a function of the size of the restricted domain. The complete $\boldsymbol{\alpha}$ domain must be searched for all the legal $\mathbf{K}^g$'s. Thus the complexity of the algorithm is of the order of the number of $\boldsymbol{\alpha}$ vectors:

$$\text{Complexity} = O((\mathbf{K}_7^{\max}) \times (\mathbf{K}_8^{\max}) \times \cdots \times (\mathbf{K}_{13}^{\max})). \tag{152}$$

Thus we have proved that any restricted domain can be decomposed as the n-fold dilations of 13 basis structuring elements. We have framed the decomposition problem as a linear-space problem and have shown that the solution is not unique. Furthermore, we have shown that all the decompositions of a restricted domain can be expressed as the sum of a particular solution and the homogeneous solution of a vector-space problem. We have provided an algorithm for decomposing any restricted domain as n-fold dilations of 13 basis structuring elements. Finally, we have provided a second algorithm for constructing all possible decompositions of the restricted domain.

8 Future Work

Many extensions to the work presented here are being tried out. Here we list a few of them.

The algorithms presented in this paper can be generalized for the case of any discretely convex shape. In that case the polygon edges can be at any angle. These angles can be defined in terms of the basic angles that can be formed by a vector starting from the origin and ending on any pixel (m, n) such that m and n are coprime.

The problem of decomposing nonconvex shapes is difficult. One way to attack this problem is to first represent the nonconvex shape as a union of restricted domains and then to decompose each of the restricted domains of the union. Another approach is to represent a shape A as a union of disjoint sets A^1 and K^1, where K^1 is the largest restricted domain that is a subset of A and $A = A^1 - K^1$. This process can be repeated, and the shape A can be represented as $A = K^1 \cup K^2 \cup \cdots \cup K^n$, where each restricted domain K^i can then be decomposed further. The proposed approach is related to the approach in [4].

Morphological dilation on nonconvex shapes will have to be carried out by first representing the shape as a union of restricted domains. Morphological erosion of nonconvex shapes can be accomplished by representing the shapes as the intersection of restricted domains and complements of restricted domains. How to decompose a nonconvex shape as a union of restricted domains and the intersection of restricted domains remains a problem, and an algorithm for doing this has yet to be developed. Furthermore, a representation scheme of nonconvex shapes in terms of half planes and B-codes is necessary.

9 Conclusion

We defined restricted domains—a restricted class of two-dimensional shapes. Two boundary schemes for representing restricted domains, the B-code and the discrete half-plane representation, were introduced. Morphological dilation, erosion, n-fold dilation, n-fold erosion, openings, and closings of restricted-domains structuring elements, which are also restricted domains, were expressed in terms of B-codes and half planes. Algorithms for performing these operations were provided and were proved to have constant-time complexity.

We proved that any restricted domain can be decomposed as n-fold dilations of 13 basis structuring elements and hence can be represented in a 13-dimensional space. This 13-dimensional space is spanned by the 13-basis structuring elements consisting of lines, triangles, and a rhombus. A constant-time algorithm was provided for finding a decomposition of any restricted domain as the n-fold dilations of the basis structuring elements. We showed that there is a linear transformation from this 13-dimensional space to an eight-dimensional space wherein a restricted domain is represented in terms of its side lengths. Furthermore, we showed that the decomposition in general is not unique and that all the decompositions can be constructed by finding the homogeneous solutions of the transformation and adding them to the particular solution. An algorithm for finding all possible decompositions was provided.

Suggesting have been made as to how the algorithms can be generalized to any arbitrary two-dimensional discretely convex shape. Pointers also have been given on how to approach the more difficult problem of decomposing nonconvex shapes.

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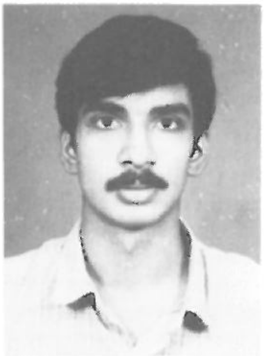

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