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STUDIES ON PROPERTIES OF DIGITAL OBJECTS USING MATHEMATICAL MORPHOLOGY

BHABATOSH CHANDA

National Centre for Knowledge Based Computing, Electronics and Communication Sciences Unit, Indian Statistical Institute 230, B. T. Road, Calcutta 700 035

AND

ROBERT M. HARALICK

Intelligent Systems Laboratory, Department of Electrical Engineering FT-10, University of Washington, Seattle, WA 98195, U.S.A.

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In this paper we study the geometric properties like connectivity and convexity of digital objects in terms of mathematical morphology. A new definition of digital convexity is suggested. Connectivity and convexity of digital objects obtained through morphological operations are investigated in the light of this new definition. This paper also presents some morphological algorithms for computing topological properties like connectivity number and genus. Operations are simpler, faster and can be implemented on parallel machine.

1. INTRODUCTION

In image analysis and computer vision methodologies, geometrical and topological properties of an object play a major role. Properties like connectivity, connectivity number, convexity, genus carry important information. For example, convex figures are much simpler to deal with than concave figures, and concave figures may be decomposed into smaller convex figures. In a similar note, transforming an object to its skelton that preserves essential structural information should preserve its topological properties also. This transformation is guided by the connectivity number. During last several years considerable amount of effort have been put to determine such properties in the discrete domain. Definitions of the connectivity number and genus of digital objects can be found in[5, 8, 14, 17] using local operator rotated around the candidate pixels. For an understanding of digital convexity, we must translate the notion of convexity from the continuous domain to th discrete domain. Quite a few definitions of the convex digital object have been suggested[1, 3, 7, 16]. However, none of these definitions of digital convexity is totally compatible with the notion of convexity in the continuous domain. For example, in the continuous domain, when a straight line is dilated (a morphological operation) by another straight line the resultant object is always convex. This is not, in general, true in discrete domain if we restrict ourselves to any of the definitions of convexity given in the aforementioned references. In this context we should mention that a different set of convexity, namely orthogonal convexity, block convexity, triangle convexity etc. For explicitly discrete domain have been defined[9]. These definitions of digital convexity are very strict and do not correspond to the concept of convexity in continuous domain. There is, therefore, the necessity of the definition of digital convexity that corresponds better to the convexity in the continuous domain.

Mathematical morphology is becoming more and more a popular tool for image processing and analysis, and has inherent ability in dealing with shapes of the object in the image[6, 15]. It treats an image as an ensemble of sets rather than signal. Its language is that of the Set theory and operations are defined in terms of the interaction between two sets: the first one is the object and the second is called structuring element. Secondly, the morphological operations can be implemented straightway on the parallel machines. So it may be of interest to see whether the topological properties like connectivity number and genus can be computed using purely morphological operations, or to have the definition of digital convexity in terms of mathematical morphology.

In this paper we study the connectivity and convexity of digital objects which are resultant of various morphological operations, and also suggest morphological operations alongwith appropriate structuring elements for computing connectivity number and genus. Section 2 restates definitions which may be required to understand the subsequent discussion.. Algorithms for computing connectivity number and genus are presented in Section 3. In Section 4 we study connectivity and convexity properties of digital objects. Concluding remarks are cited in Section 5.

2. Preliminaries

In a continuous domain $\mathbf{R}^2$ an image is defined as a nonnegative function $\mathbf{f}$, where $f(x, y)$ is the value of function f at the point $(x\ y)$. A digital image is defined by a finite valued function over a discrete domain $\mathbf{Z}^2$. Let us assume the digital image domain $\mathbf{D} \subset \mathbf{Z}^2$ is rectagular array of size $\mathbf{M} \times \mathbf{N}$, i.e.,

$$\mathbf{D} = \{(r, c) \mid r = 0, 1, 2, \ldots\ldots, \mathbf{M} - 1;\ c = 0, 1, 2, \ldots., \mathbf{N} - 1\}$$

obtained by sampling of step size h along two orthogonal directions. The (r, c)'s are called discrete points or pixels of digital image. A frame F of a domain $\mathbf{D}$ may be defined as a subset of $\mathbf{D}$ containing pixels of the first and last l rows, and pixels of the first and last l columns, where $(2l + 1)$ is length and breadth of structuring elements. Morphological operations will be applied only to pixels belonged to the set (D-F). In binary images a point (r, c) in $\mathbf{D}$ is defined as a foreground point or object point if its value is 1, and as a background point if its value is 0. Let E_1 and E_2 denote two sets of integers $\{1, 3, 5\ 7\}$ and $\{1, 2, 3, \ldots\ldots, 8\}$, respectively, as shown in Fig. 1. Pixel p_i denotes its position and value as suitable for the situation. In our discussion we will use the terms 'point' and 'pixel' to mean the same thing in $\mathbf{Z}^2$.

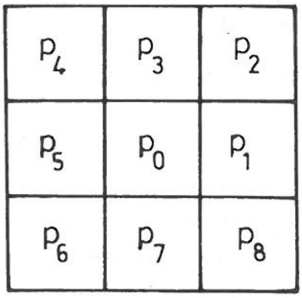


FIG. 1. Pixel p_0 and its neighbourhood.

Definition 1 — The elements of set $\{p_i \mid i \in E_1\}$ are called **4-neighbors** of p_0. Similarly, the elements of set $\{p_i \mid i \in E_2\}$ are called **8-neighbors** of p_0.

Definition 2 — Let p and q be two discrete points in $\mathbf{Z}^2$ with coordinates (r^p, c^p) and (r^q, c^q), respectively, then the **distance** between p and q may be defined as

$$\text{(i)}\ d_4(p, q) = \mid r^p - r^q \mid + \mid c^p - c^q \mid, \text{ or}$$

$$\text{(ii)}\ d_8(p, q) = \max \{\mid r^p - r^q \mid, \mid c^p - c^q \mid\}.$$

Two pixels p and q are said to be connected by a 4-(8-) connected path if there exists a sequence of pixels $p_1 = p, p_2, p_3, ..., p_n = q$ such that for any i value of p_i is same as that of p and q, and $d_4(p_i, p_i + 1) = 1$ $[d_8(p_i, p_{i+1}) = 1]$.

Definition 3 — A set of pixels **A** is called a **4-(8-) connected component** or **object** if every pair of pixels in **A** is connected by a 4-(8-) connected path.

Although either 4- or 8- connectivity may be assumed for a connected component, care must be taken to avoid contradiction concerning object pixels and background pixels. Unless specially stated we assume that the objects are 8-connected and the background is 4-connected.

Definition 4 — A pixel p is called **surrounded** by an object **A** if every 4-connected path from p to F contains at least one pixel of **A**.

Definition 5 — If the color of p is not the same as that of pixels of **A** and if p is surrounded by **A**, then the set of all pixels connected to p constitutes a **hole** in **A**. The smallest possible hole, i.e., the hole consisting of a single pixel only is called **point hole**.

Definition 6 — The **genus of an object** is defined as "1-(the number of holes within that object)". The **genus of an imge** is defined as "(the number of objects) - (the number of holes)".

Definition 7 — An object is called **simply connected** if genus is 1, and is called **multiply connected** otherwise.

Definition 8 — **Convex hull** of **A**, denoted by H(A), is smallest convex polygon that contains **A**.

In continuous domain convex object may be defined in one of the following ways.

Definition 9a — let $\mathcal{A}$ be an object in $\mathbf{R}^2$. $\mathcal{A}$ is **convex** if for every pair of points p and q, the straight line segment $\overline{pq}$ connecting p and q is contained in $\mathcal{A}$.

This definition can be rewritten as

Defintion 9b — Let $\mathcal{A}$ be an object in $\mathbf{R}^2$. $\mathcal{A}$ is **convex** if for every pair of points p and q belonging to $\mathcal{A}$, and for every real $\gamma \in [0, 1]$, we have

$$\gamma p + (1-\gamma) q \in \mathcal{A}$$

This is again equivalent to: $A \subset \mathbf{R}^2$ is convex if and only if $\mathcal{A} = H(\mathcal{A} \subset A)$.

Translating this concept of convexity to the discrete domain requires the digital straight line segment be defined. An 8- connected finite set L of lattice points is a digital arc if all but two of the points have exactly two neighbors in L, and the exceptional two points [the end points] each have exactly one neighbor in L[11]. Let p, q be any two points in L. The line segment $\overline{pq}$, connecting p and q, is said to lie near L if for any real point (x, y) of $\overline{pq}$ there exists a point (r, c) of L such that $\max\{|r-x|, |c-y|\} < 1$. We say that L has the chord property if, for every p, q in L, the line segment $\overline{pq}$ lies near L[12].

Definition 10 — A digital arc is said to be a **digital straight line segment** (DLS) if it has chord property.

With the chord property, Rosenfeld[12] has strongly established the Freeman's conjectures[2] of DLS. Rosenfeld and Kim[13] have proposed an algorithm to test whether a digital arc is a DLS. A simpler algorithm has been presented by Ronse[10]. Now let us see how we can obtain a DLS from two given end points p and q. We define a straight line parallel to line segment $\overline{pq}$ such that the smallest of the distances, the horizontal and the vertical, between these two straight lines is equal to 1. If slope α of $\overline{pq}$ satisfies $0 < |\alpha| < 1$ then it is the vertical distance which is involved, and in the other case where $1 < |\alpha| < \infty$, if it is the horizontal distance. We shall examine the cases where $\alpha = 0$, 1 or ∞ separately. We can draw two such parallel line segments: one is above and other is below $\overline{pq}$ as shown in Fig. 2. Let us call them m_a and m_b, respectively. By the term 'above' ('below') we mean $m_a(x) > y$ $[m_b(x) < y]$, where (x, y) is any point on $\overline{pq}$. Consider L_a $[L_b]$, the set of all the lattice points that lie between m_a $[m_b]$ and $\overline{pq}$ and on $\overline{pq}$ but not on m_a $[m_b]$, and between p and q[1, 7]. Thus we get two different DLSs for a pair of points. This causes a lot of problems in dealing with DLS. People usually take up L_a or L_b arbitrarily as DLS connecting p and q for their analysis. The problem of this arbitrary selection will be discussed later. In the cases where $\alpha = 0$, 1 or ∞, all the lattice points between p and q lie on the line segment $\overline{pq}$ [Fig. 3]. Hence, in those cases, DLS is unique.

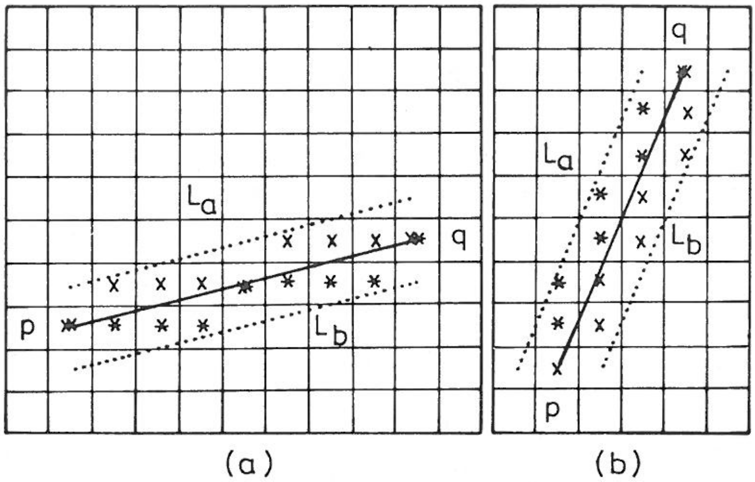


FIG. 2. Digital straight line segment connecting p and q. L_a consists of pixels marked with 'x' and L_b consists of pixels marked with '*'.

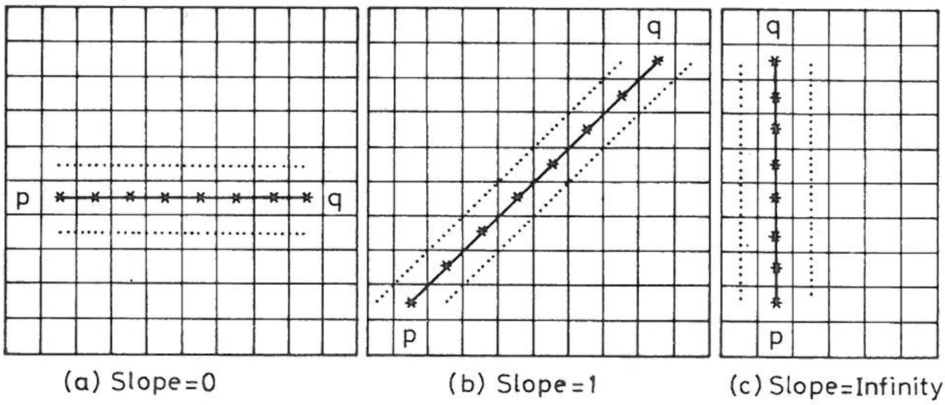


FIG. 3. Digital straight line segment for slope 0, 1 and ∞. In all three cases $L_a = L_b$.

In addition to the above definitions we also need to define morphological operations for completeness and clarity. Let $\mathbf{A}$ and $\mathbf{B}$ are subsets of $\mathbf{Z}^2$ and t be a point of $\mathbf{Z}^2$. Then

Translation : $A_t = \{p \in \mathbf{Z}^2 \mid p = a + t \text{ for some } a \in A\}$

Dilation : $A \oplus B = \{p \in \mathbf{Z}^2 \mid p = a + b \text{ for some } a \in A \text{ and } b \in B\}$ or, alternately, can be represented as a union of translates:

$$A \oplus B = \bigcup_{a \in A} B_a = \bigcup_{b \in B} A_b$$

Erosion : $A - B = \{p \in \mathbf{Z}^2 \notin p + b \in \mathbf{A} \text{ for every } b \in B\}$

or, alternately, can be represented as an intersection of the negative translates:

$$A \ominus B = \bigcap_{b \in B} A_{-b}$$

Opening : $A \circ B = (A \ominus B) \oplus B$

Closing: $A \bullet B = (A \oplus B) \ominus B)$

B is a **structuring element**. From now on the objects we deal with, unless otherwise specified, are digital objects and the properties are defined on discrete domain.

3. Connectivity Number and Genus

In this section we will describe some algorithms for computing topological properties like connectivity number and genus using morphological operations, namely erosion. Suppose a binary image of digital object **A** is denoted by a **Indicator function** $I_A(r, c) \in \{0, 1\}$. In other words, **A** is called **Domain of support** defined as $\{(r, c) \mid I_A(r, c) = 1\}$.

3.1 Connectivity Number

Connectivity numbers are assigned to the pixels which belong to the domain of support. Purpose of this number is to show how the pixels of domain of support is connected to its like neighbors. Though we call it 'connectivity number', it is actually a label and has no arithmetic property.

Definition 11 — The **connectivity number** is a label that indicates how the pixel of interest is connected to it neighbors.

The pixels of an object can be grouped as border pixel and interior pixel.

Definition 12 — **A border pixel** or **boundary pixel** of **A** is a pixel of **A** such that atleast one of its 4-neighbors is not in **A**. The **border** or **boundary A** of an object **A** is characterized by the set of its border pixels.

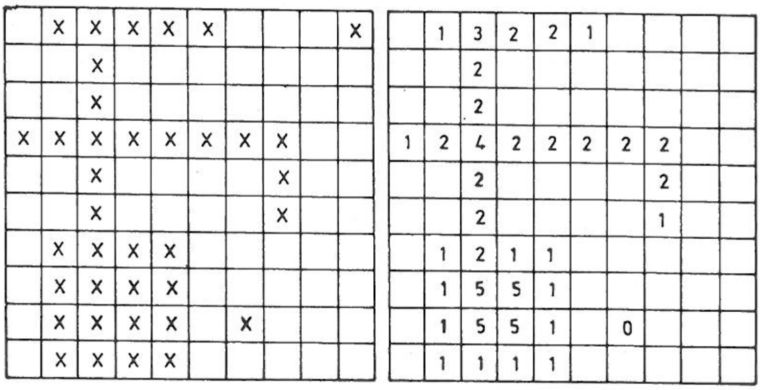
The border $\bar{A}$ may be viewed as a polygon with finite number of vertices. So the set of vertices of $H(A)$ is a subset of that of $\bar{A}$.

Definition 13 — The set of pixels $(\mathbf{A} - \bar{A})$ is called the **interior** of **A**.

Border pixels may again be subgrouped as: isolated pixel, edge pixel, connecting pixel, branching pixel and crossing pixel. Connectivity number operator associates with each pixel belonging to domain of support one of the six different values: five values for border pixels (0-4) and one value (5) for interior pixels. Accordingly labels are shown in Fig. 4 with an example. Two different definitions of connectivity are considered. Here we adopt the definition of connectivity number as was given in [5]. The definitions are slight modificaiton of the definitions suggested by Yokoi *et al.*[17] and Rutovitz[14].

3.1.1 Yokoi Connectivity Number

Definitions are slightly different depending on whether the object is a 4-connected component or an 8-connected component of image. The Yokoi connectivity number



Key Number	Class	Meaning
0	Border pixel	Isolated
1	Border pixel	Edge
2	Border pixel	Connecting
3	Border pixel	Branching
4	Border pixel	Crossing
5	Interior pixel	Interior

Fig. 4. Illustration of a connectivity number labeling of a binary image[5].

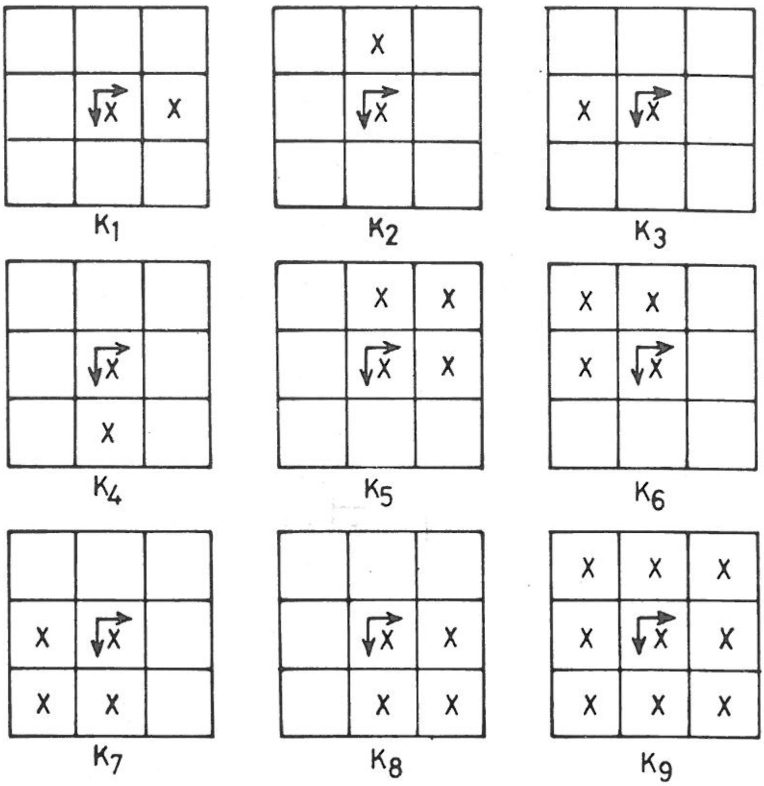


FIG. 5. Structuring elements for Yokoi connectivity number for 4-connectivity.

is defined as follows. For n-onnectivity, a 1-pixel is an interior pixel if value of each of its n-neighbors is 1. In this case the index value 5 is assigned to the pixel. Otherwise the n-connectivity of the pixel is given by the number of times a n-neighbor has value 1 but the corresponding three-pixel corner neighbourhood does not. The value of n can be 4 or 8.

4-connectivity case :

Connectivity number $C(r, c)$ at pixel (r, c) is

$$C(r, c) = \max\left\{\sum_{i=1}^{4} (A \ominus K_i)(r, c) - \sum_{i=5}^{8} (A \ominus K_i)(r, c), 5 * (A \ominus K_9)(r, c)\right\}$$

where, K_i's are structuring elements as shown in Fig. 5.

8 - connectivity case :

Connectivity number $C(r, c)$ at pixel (r, c) is

$$C(r, c) = \max\left\{\sum_{i=1}^{8} (A \ominus K_i)(r, c) - \sum_{i=9}^{16} (A \ominus K_i)(r, c), 5 * (A \ominus K_{17})(r, c)\right\}$$

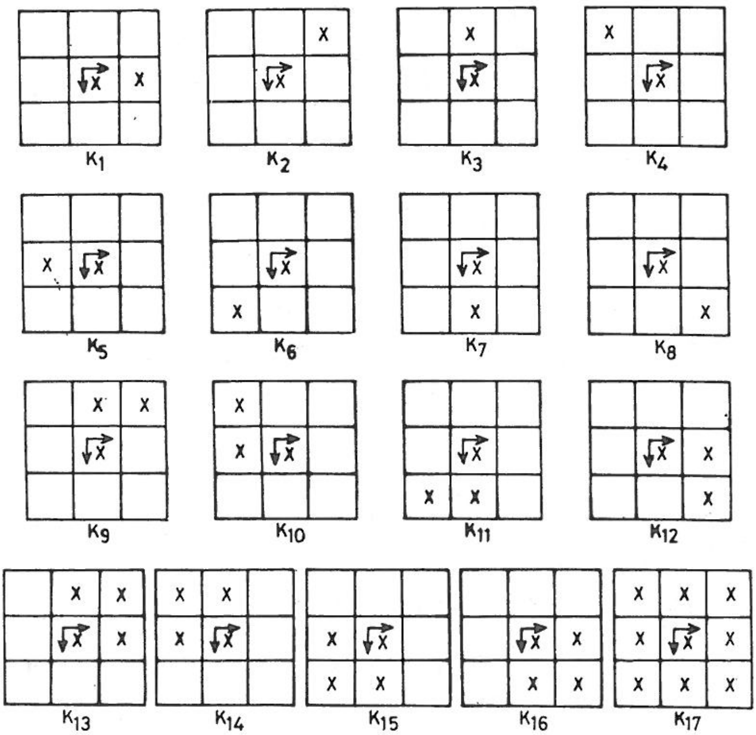


FIG. 6. Structuring elements for Yokoi connectivity number for 8-connectivity.

where, K_i's are structuring elements as shown in Fig. 6.

In both cases 'max' may be replaced by 'sum' of arguments.

3.1.2 Rutovitz Connectivity Number

The Rutovitz connectivity number is defined for 8-connectivity only and is defined as follows. A 1-pixel is an interior pixel if value of each of its 8-neighbors is 1. In this case the index value 5 is assigned to the pixel. Otherwise the connectivity number of the pixel is the number of transitions from 1 to 0 occur as one travels around the 8-neighborhood of the pixel. This number is also called 'crossing number'. So connectivity number $C(r, c)$ at pixel (r, c) is

$$C(r,c) = \sum_{i=1}^{8} (A \ominus K_i)(r,c) - \sum_{i=9}^{16} (A \ominus K_i)(r,c)$$

where, K_i are structuring elements as shown in Fig. 7. The connectivity numbers reveal the structural characteristics of an object in a certain way. Secondly, it may

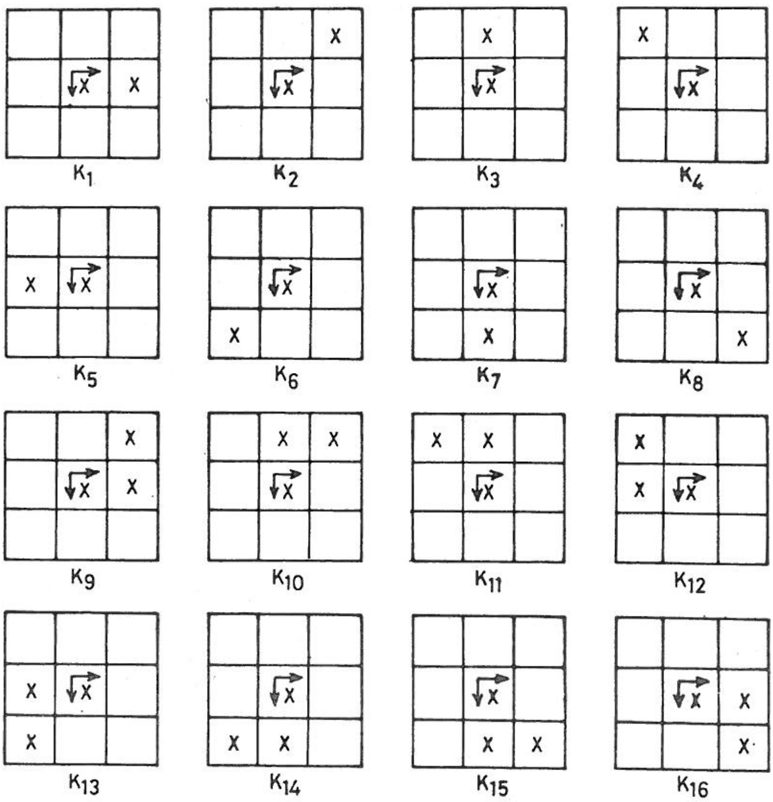


FIG. 7. Structuring elements for Rutovitz connectivity number.

be noted that removal of the pixels having connectivity number 1 does not change the genus; whereas the removal of pixels having connectivity number 0 or 2 or 4 or 5 does. Hence, the information is useful for designing structure/topology preserving transformations, such as thining.

3.2 Genus of Image

Genus of an image is defined as the 'number of connected components (or objects) minus number of holes in the entire image'. Situations are different if objects are considered to be 4-connected from that if they are considered as 8-connected[8,17]. Minsky and Papert[8] defined the genus $G(I_A)$ of the image as

$$G(I_A) = \Sigma\, \alpha_1 I_A(r,c) + \Sigma\, \alpha_2 I_A(r,c)\, I_A(r-1,c) + \Sigma\, \alpha_3 I_A(r,c)\, I_A(r,c+1) + \Sigma\, \alpha_4 I_A(r,c)\, I_A(r-1,c)\, I_A(r,c+1)\, I_A(r-1,c+1)$$

where summations are taken over all $(r, c) \in (D-F)$, and $\alpha_1 = \alpha_4 = 1$ and $\alpha_2 = \alpha_3 = -1$. The first term of the right-hand side gives sum of value of all pixels or the number of object [foreground] pixels since the value of foreground pixel is 1. The second and the third terms give the number of pairs of object pixels that are adjacent horizontally and vertically, respectively. Finally, the fourth term gives the number of 2×2 blocks of object pixels. Instead of taking sum of products as suggested by the above definition, we can compute these numbers by counting the elements of the sets, $\mathbf{A}, \mathbf{A} \ominus \mathbf{K}_1$, $\mathbf{A} \ominus \mathbf{K}_2$ and $\mathbf{A} \ominus \mathbf{K}_3$, respectively, where the structuring elements $\mathbf{K}_1, \mathbf{K}_2$ **and** $\mathbf{K}_3$ are shown in Fig. 8. Adjacency of pair of pixels and the 2×2 block of this approach reminds us the 4-connectedness of object. So we can write in

4-connectivity case :

$$G(I_A) = \#(A) - \#(A \ominus K_1) - \#(A \ominus K_2) + \#(A \ominus K_3)$$

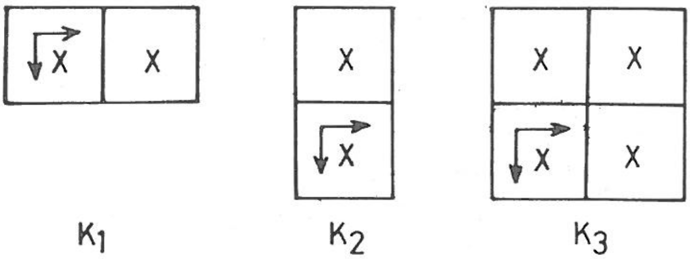


FIG. 8. Structuring elements for computing genus for 4-connectivity.

where, (A) **means number of elements in set** A. The structing elements K_i's are shown in Fig. 8.

Now if we imagine a 4-connected object as a web type collection of polygons formed by most closely situated pixels, then each pixel of domain of support may be viewed as a vertex of these polygons. So each pair of horizontally or vertically adjacent pixels represents an edge, and a 2×2 block represents a face. Hence, the right-hand side of above definition of genus may be written as

Number of vertices − Number of edges + Number of faces

This is, again, well known *Euler Polygon Formula.* Minsky and Papert[8] has shown that genus and the number computed by this formula exactly agree to each other in 4-connectivity case.

When domain of support or object is considered as 8-connected, along with horizontally and vertically adjacent pair of pixels, diagonally adjacent pair of pixels, also constitute edge. The number of edges can be obtained by counting the elements of the sets $\mathbf{A} \ominus \mathbf{K}_1$, $\mathbf{A} \ominus \mathbf{K}_2$, $\mathbf{A} \ominus \mathbf{K}_3$, $\mathbf{A} \ominus \mathbf{K}_4$, where the structuring elements $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ and $\mathbf{K}_4$ are shown in Fig. 9. Secondly, the polygons formed by the most closely situated pixels, in this case, are right-angles triangles consisting of three pixels and 2 × 2 squares. Since we are dealing with objects in two-dimensional space, sides of polygons should not cross each other. So during counting faces, we count only the nonoverlapping triangular faces and the square faces. Now it can be readily seen that the number of nonoverlapping triangular faces plus the number of squares faces is equal to the number of all triangular faces minus the number of square faces. The

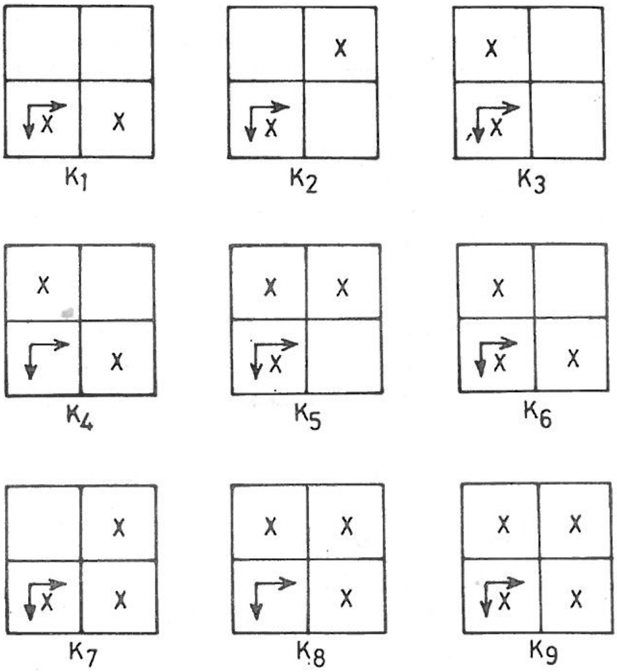


FIG. 9. Structuring elements for computing genus for 8-connectivity.

number of all triangular faces can be obtained by counting the elemetns of the sets $\mathbf{A} \ominus \mathbf{K}_5$, $\mathbf{A} \ominus \mathbf{K}_6$, $\mathbf{A} \ominus \mathbf{K}_7$ and $\mathbf{A} \ominus \mathbf{K}_8$, and the number of square faces can be obtained by counting the elements of the sets $\mathbf{A} \ominus \mathbf{K}_9$, where the structing elements $\mathbf{K}_5, \mathbf{K}_6, \mathbf{K}_7, \mathbf{K}_8$ and $\mathbf{K}_9$ are given in Fig. 9. Hence, the polygonal formula for 8-connected domain of support looks like :

number of vertices − number of edges + number of triangular faces − number of square faces.

Thus we can write in

8-connectivity case :

$$G(I_A) = \#(A) - \sum_{i=1}^{4} \#(A \ominus K_i) + \sum_{i=5}^{8} \#(A \ominus K_i) - \#(A \ominus K_9)$$

The structing elements K_i's are shown in Fig. 9. Genus can also be computed using morphological hit-and-miss transforms[4,15]

4. CONNECTIVITY AND CONVEXITY

In this section we study the properties like connectivity and convexity of digital objects which has undergone some morphological transformation. We begin with connectivity property since it is simpler to define and easier to conceive than convexity.

Connectivity is translation invariant property of object, i.e. if A is simply connected, so is A_t. Similarly, A_t contains a hole if and only if A contains a hole and *vice versa*.

Proposition 1 : If A and B are convex and are simply connected regions, then $A \oplus B$ is simply connected.

PROOF : Suppose $A \oplus B$ is not simply connected. Then there exists at least hole in $A \oplus B$. Suppose A is simply connected. If we define $A \oplus B$ as $\bigcup_{a \in A} B_a$, then A is simple connected and $A \oplus B$ contains a hole imply that there exists an $a \in A$ such that B_a contains a hole. Now B_a can contain a hole if and only if B contains a hole. That means B is not simply connected.

Similarly if B is simply connected, then $A \oplus B$ contains a hole implies that A is not simply connected.

Hence, $A \oplus B$ is simply connected. Q.E.D.

The convexity constraint in the above proposition is important, and cannot be, in general, relaxed. Fig. 10 shows an example where the gulfs of an object are merged by dilaton operation as a result of which a lake (hole) is created from the bay.

We have already said that DLSs cannot be treated simply as a digital object because.of its digitization techniques. For example, we have shown that if A and B are imply connected convex objects, then $A \oplus B$ will also be a simply connected object. However, this is not always true for DLSs as we will see now.

Proposition 2 : If L_1 and L_2 are two DLSs then $L_1 \oplus L_2$ can have at most point holes.

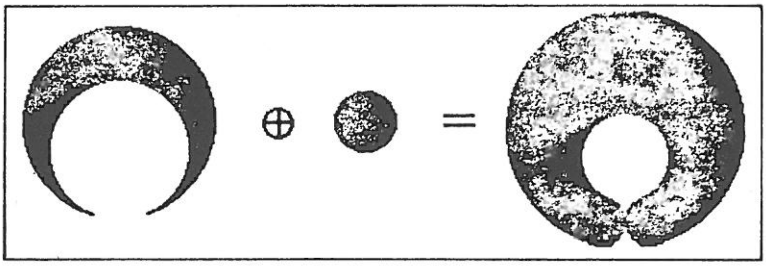


FIG. 10. An example of dilation.

PROOF : Suppose L_1 is a sequence of lattice points $p_0, p_1, p_2, ..., p_m$, and L_2 is another sequence of lattice points $q_0, q_1, q_2, ..., q_n$. Coordinates of p_is and q_is are given by (r_i^p, c_i^p) and (r_i^q, c_i^q), respectively. Since L_1 is a DLS then

$$r_{i+1}^p - r_i^p = 1 \text{ and } c_{i+1}^p - c_i^p \in \{-1, 0, 1\},$$

or

$$c_{i+1}^p - c_i^p = 1 \text{ and } r_{i+1}^p - r_i^p \in \{-1, 0, 1\}.$$

The same is true for L_2 also. Now $L_1 \oplus L_2$

1. will form a DLS with end points $(p_0 + q_0)$ and $(p_m + q_n)$,

if $\dfrac{c_m^p - c_0^p}{r_m^p - r_0^p} = \dfrac{c_n^q - c_0^q}{r_n^q - r_0^q}$, or

2. will form a parallelogram with vertices $(p_0 + q_0)$, $(p_m + q_0)$, $(p_m + q_n)$ and $(p_0 + q_n)$,

if $\dfrac{c_m^p - c_0^p}{r_m^p - r_0^p} \neq \dfrac{c_n^q - c_0^q}{r_n^q - r_0^q}$, and is bounded by L_1, L_2 a DLS parallel to L_1 and another DLS parallel to L_2.

In the first case, $L_1 \oplus L_2$ is simply connected. In the second case, let us take any two sets S_1 and S_2 of pair of successive pixels one pair from L_1 and other pair from L_2, i.e.,

$$S_1 = \{ (r_i^p, c_i^p), (r_{i+1}^p, c_{i+1}^p) \}$$

$$S_2 = \{ (r_i^q, c_i^q), (r_{i+1}^q, c_{i+1}^q) \}.$$

There are many different combinations of relations between r_i and r_{i+1}, and c_i and c_{i+1} possible. But the results of S_1 dilated by S_2 can be grouped into only three distinct classes. Here we will consider only three different inputs whose output correspond to those distinct classes.

Case 1 : $r_{i+1}^p = r_i^p + 1, \;\; c_{i+1}^p = c_i^p + 1$

$$r_{j+1}^q = r_j^q + 1, \;\; c_{j+1}^q = c_j^q + 1$$

Hence, $S_1 \oplus S_2 = \{ (r_i^p + r_j^q, c_i^p + c_j^q), (r_i^p + r_j^q + 1, c_i^p + c_j^q + 1),$

$(r_i^p + r_j^q + 2, c_i^p + c_j^q + 2) \}.$

So $S_1 \oplus S_2$ contains three points, each in 8-connected to others and there are no holes.

Case 2 : $r_{i+1}^p = r_i^p + 1, \ c_{i+1}^p = c_i^p$

$r_{j+1}^q = r_j^q, \ c_{j+1}^q = c_j^q + 1$

Hence, $S_1 \oplus S_2 = \{ (r_i^p + r_j^q, c_i^p + c_j^q), (r_i^p + r_j^q, c_i^p + c_j^q + 1), (r_i^p + r_j^q + 1, c_i^p + c_j^q),$

$(r_i^p + r_j^q + 1, c_i^p + c_j^q + 1) \}.$

So $S_1 \oplus S_2$ contains four points, each is 4-connected [8-connected also] to others and there are no holes.

Case 3 : $r_{i+1}^p = r_i^p + 1, \ c_{i+1}^p = c_i^p + 1$

$r_{j+1}^q = r_j^q + 1, \ c_{j+1}^q = c_j^q - 1.$

Hence, $S_1 \oplus S_2 = \{ (r_i^p + r_j^q, c_i^p + c_j^q), (r_i^p + r_j^q + 1, c_i^p + c_j^q - 1),$

$(r_i^p + r_j^q + 1, c_i^p + c_j^q + 1), (r_i^p + r_j^q + 2, c_i^p + c_j^q) \}.$

So $S_1 \oplus S_2$ contains four points, each is 8-connected to others, and there is one hole at $(r_i^p + r_j^q + 1, c_i^p + c_j^q)$. No other point from L_1 and L_2 will contribute to this point hole.

Now for all other input combinations, the results of $S_1 \oplus S_2$ will fall in one of these three cases. Since above discussion is valid for any two pairs of successive pixels from L_1 and L_2, then $L_1 \oplus L_2$ can have at most point holes. Q.E.D.

We have said earlier that quite a few definitions of digital convexity have been suggested in the literatures[1,3,7,16]. One of the problems with these definitions of digital convexity is that they cannot gaurantee, in general, digital convexity of objects resulting from dilation of a DLS with another DLS. So we define the convexity in the discrete domain in the following way to accommodate such kind of objects in the class of digitally convex objects. In the following discussion, if not specially mentioned for continuous domain, the term *convex* means *digitally convex.*

Suppose A is a 8-connected component and $K = \{(-1, 0), (1, 0), (0, -1), (0, 1)\}$ is the structuring element with origin at (0, 0) as shown in Fig. 11. Unless otherwise stated, the structuring element K always represents this in the rest of our discussion. Then $A \ominus K$ is the union of the set of interior pixels and the set of point holes of A.

So the set of points holes and interior pixels may be collectively called the set of 'closely surrounded pixels', where

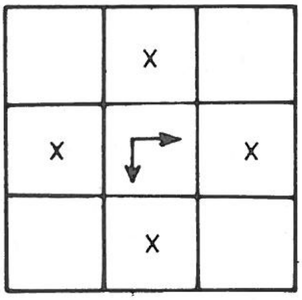


FIG. 11. The structuring element to extract the closely surrounded pixels.

Definition 14 — A pixel p is closely surrounded by an object A if all 4-neighbours of p are in A.

Since by definition, boundary pixels have at least one of the 4-neighbours in A^c, eroding with K deletes all boundary pixels of A. However, union of $A \ominus K$ with A fills up point holes in A. Therefore, if A contains at most point holes, then $A \cup (A \ominus K)$ is simply connected. Pixels closely surrounded by an object play an important role in convexity analysis. A digital object is said to be digitally convex if the union of its all pixels and pixels closely surrounded by it conver its convex hull. Thus

Definition 15 — An object A is **digitally convex** if and only if

$$A \cup (A \ominus K) = H(A) \cap \mathbf{Z}^2$$

where, $K = \{(-1, 0), (1, 0), (0, -1), (0, 1)\}$.

This definition implies that the condition for an object being convex is that it can have at most point holes. This notion of convexity implicitly assumes that no hole can have an area of one pixel only. Any such hole will be regarded as noise including digitization error. Secondly, let us denoted the intersection of $\mathbf{Z}^2$ with a half-plane that may have point holes in it by **Hph**, and also denote a **Hph** that contains the object A by **Hph**(A). If A is *convex* then A must be obtained from the intersection of all possible **Hph**(A). Thirdly, suppose a pixel x does not belong to A, then there must exist a *DLS* that separates x from A. Finally, in Euclidean plane geometry the convexity of a region is defined in terms of the straight line segment between every pair of points in the region. In continuous domain, an object is said to be convex if for every pair of points in the object, the line segment connecting them lies entirely in it [Definition 9]. It is of natural interest to check if this property is valid for present definiton of digital convexity also. The line containment property of a convex digital region is already proved by Kim and Rosenfeld[7] :

Proposition 3 — *A* simply connected object A is convex if and only if any two points of A are connected by a DLS in A.

PROOF : Proof can be found in the above mentioned literature. O.E.D.

Proposition 4 — If A is convex then any two points in $A \cup (A \ominus K)$ are connected by a DLS in $\mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K})$.

PROOF : Suppose A is an object with at most point holes. Then $\mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K})$ is a simply connected object. By definition of convex hull, $H(A \bigcup (A \ominus K)) \bigcap \mathbf{Z}^2$ is digital convex region. According to Proposition 3, any two points of $H(\mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K})) \bigcap \mathbf{Z}^2$ are connected by a DLS in $H(\mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K})) \bigcap \mathbf{Z}^2$. Since, the boundary pixels and, in turn, vertices of $\mathbf{A}$ and $H(A \bigcup (A \ominus K))$ are same, so $H(\mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K})) - H(\mathbf{A})$. From the definition of digital convexity, we can write, a $\mathbf{A}$ is convex then

$$H(\mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K})) \bigcap \mathbf{Z}^2 = H(\mathbf{A}) \bigcap \mathbf{Z}^2 = A \bigcup (\mathbf{A} \ominus \mathbf{K}).$$

Hence, any two points of $A \bigcup (\mathbf{A} \ominus \mathbf{K})$ are connected by a digital line segment in $\mathbf{A} \bigcup (\mathbf{A} \ominus K)$. O.E.D.

Corollary — If A is convex then any two points in A are connected by a DLS in $A \bigcup (A \ominus K)$.

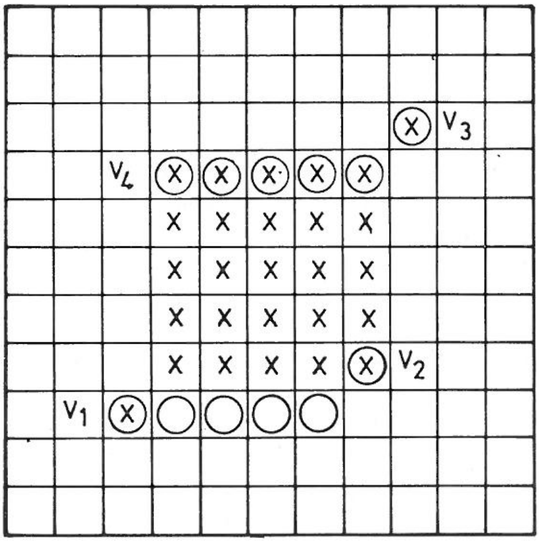


FIG. 12. A simply connected convex object.

In Proposition 2, we have seen that when a DLS is dilated by another straight line segment, the resultant object can have at most point holes. Now we will see that resultant object must also be convex. Before that let us restate another result from Kim and Rosenfeld (Theorem 7 of [7]).

Proposition 5 — A DLS is convex.

PROOF : Proof can be found in the above mentioned literature. Q.E.D.

Proposition 6 — If L_1 and L_2 are two DLSs, then $L_1 \oplus L_2$ is convex.

PROOF : Suppose L_1 is a sequence of lattice points $p_0, p_1, p_2, ..., p_m$ and L_2 is an other sequence of lattice points $p_0, p_1, p_2, ..., p_n$. Let $L = L_1 \oplus L_2$. As explained in the proof of Proposition 2, L will be either

1. a DLS with end points $(p_0 + q_0)$ and $(p_m + q_n)$, or
2. a parallelogram with vertices $(p_0 + q_0)$, $(p_m + q_0)$, $(p_m + q_n)$ and $(p_0 + q_n)$, and is bounded by L_1, L_2, a DLS parallel to L_1, and another DLS parallel to L_2.

Now in the first case, when L is a DLS, according to Proposition 5, L is convex.

In the second case, L can have at most point holes, so $L \bigcup (L \ominus K)$ is simply connected. Again, since, boundary pixels of L form a parallelogram which by definition is convex, then bounday of L and that of $H(L) \bigcap \mathbf{Z}^2$ are same. That means $L \bigcup (L \ominus K) = H(L) \bigcap \mathbf{Z}^2$.

Hence, $L_1 \oplus L_2$ [or L] is convex. Q.E.D.

Proposition 7 — If A and B are convex, then $A \oplus B$ is also convex.

PROOF : Since **A** and **B** are convex, then **A** and **B** can have at most point holes. So according to Proposition 3, $A \oplus B$ can have at most point holes. Let $\mathbf{C} = \mathbf{A} \oplus \mathbf{B}$. Therefore, $\mathbf{C} \bigcup (\mathbf{C} \ominus \mathbf{K})$ is simply connected. Now if we can show that any two points in **C** are connected by a DLS in $\mathbf{C} \bigcup (\mathbf{C} \ominus \mathbf{K})$, then convexity is proved.

Let us take arbitrary points c_1 and c_2 of **C**. For every pair of c_1, c_2, there exist two points b_1, b_2 in B and a_1, a_2 in A such that $a_1 + b_1 = c_1$ and $a_2 + b_2 = c_2$. Since **A** and **B** are convex, then DLSs connecting a_1 and a_2, and b_1 and b_2 lie in $\mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K})$ and $\mathbf{B} \bigcup (\mathbf{B} \ominus \mathbf{K})$, respectively. So we can imagine that DLS connecting a_1 and a_2 is dilated by the DLS connected b_1 and b_2, and results in a parallelogram with vertices $(a_1 + b_1), (a_1 + b_2), (a_2 + b_2)$ and $(a_2 + b_1)$. Thus c_1 and c_2 is in the parallelogram. Hence, DLS connecting c_1 and c_2 is in $\mathbf{C} \bigcup (\mathbf{C} \ominus \mathbf{K})$.

Q.E.D.

Like connectively, convexity property is also translation invariant. Since convexity of a digital object and containment of DLS connecting any two points of it are closely related to each other, let us examine some problems associated with DLS at this point. In the continuous domain a straight line connecting two points is unique. But in the discrete domain, we have two different DLSs L_a and L_b connecting two points [Fig. 2]. So the DLS connecting two points is not unique. However, L_a and L_b are individually unique for any two points. Secondly in the continuous domain, if the intersection of any two convex object is nonempty then it must be convex. According to Proposition 5, a DLS is convex. But if we take the intersection of two DLSs connecting the same pair of points, we may get a nonempty set which is not, in general, connected; convexity is out of question. These two problems may be solved if we restrict our definition of DLS either to L_a or to L_b. Suppose the sequence of pixels or lattice points corresponding to L_b [of previous definition] define the DLS.

Now let us see what happens to convex digital objects is terms of this constrained definition of digital straight line. Let us consider a simple digital convex object A as shown in Fig. 12. It has vertices v_1 v_2, v_3 and v_4. The sequence of pixels marked with O characterize DLS connecting two endpoints. In the figure we see that DLS connecting v_3 and v_4 is in A, but that connecting v_1 and v_2 is not in A. As a result $A \bigcap A_{(-5,-1)}$ will give two well separated points corresponding to $v_1(v_4)$ and $v_2(v_3)$. We may call the edge segment $\overline{v_3 v_4}$ bound and $\overline{v_1 v_2}$ open. Similarly, $\overline{v_2 v_3}$ is open and $\overline{v_4 v_1}$ is bound. In fact, Kim and Rosenfeld[7] considered both L_a and L_b to prove line containment property of convex objects. The convexity problem of intersection of two convex objects may be solved by binding the open edge segments of objects. 'Binding open edge segment' means taking the union of the object with the DLSs connecting the vertices associated with the open edge segments. However, augmenting pixels to an object means an increase in the area of the object which is undesirable too.

Hence, we drop the idea of augmenting pixels to the object and accept both L_a and L_b as DLS connecting same pair of points. This leads us to following two propositions which are most close to the property of intersection of two convex objects in the continuous domain but not exactly same because of the connectivity problem.

Proposition 8 — If **A** and **B** are simply connected objects and $A \bigcap B$ is connected, then $A \bigcap B$ must be simply connected.

PROOF : Suppose $\mathbf{A} \bigcap \mathbf{B}$ is connected but not simply connected. Then there exists at least a hole in $\mathbf{A} \bigcap \mathbf{B}$. This implies that either **A** or **B** or both contain a hole. Accordingly, either **A** or **B** or both are not simply connected. Q.E.D.

Hence, $A \bigcap B$ is simply connected.

Proposition 9 — If A and B are two convex objects, and

If $\mathbf{C} = (\mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K})) \bigcap (\mathbf{B} \bigcup (\mathbf{B} \ominus \mathbf{K}))$, then **C** is a convex object if it is connected.

PROOF : If A and B are convex, then each can have at most point holes. Thus $\mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K})$ and $\mathbf{B} \bigcup (\mathbf{B} \ominus \mathbf{K})$ are simply connected. Now

$$\mathbf{C} = (\mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K})) \bigcap (\mathbf{B} \bigcup (\mathbf{B} \ominus \mathbf{K}))$$

and C is not empty. Then it can have

1. one pixel only, or
2. more than one pixel but not connected, or
3. more than one pixel and connected.

In the first case, **C** is always convex. We discard the second since **C** is not connected. In the third case, since **C** is connected, according to Proposition 8, C must be simply connected. Let us consider any two points p and q in **C**, and possible DLS connecting them are denoted by L_a^C and L_b^C. Since $\mathbf{C} \subseteq \mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K})$ and **A** is convex, then $p, q \in \mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K})$ and either L_a^A or L_b^A or both connecting p and q must be in $\mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K})$. Similarly, $\mathbf{B} \bigcup (\mathbf{B} \ominus \mathbf{K})$ also contains p and q as well as either L_a^B or L_b^B or both connecting p and q. Now since L_a and L_b connecting p and q are unique, then $L_a^A = L_a^B\ L_b^B = L_a^C$ and $a\, L_b^A = L_b^B = L_b^C$. Since C is simply connected then either L_a^C or L_b^C or both must be in C. Hence, C or $(A \bigcup (A \ominus K)) \bigcap (\mathbf{B} \bigcup (\mathbf{B} \ominus \mathbf{K}))$ is convex.

Corollary 2 — If **A** and **B** are simply connected convex objects, then $\mathbf{A} \bigcap \mathbf{B}$ is also a simply connected convex object if it is connected.

PROOF : Suppose $\mathbf{A} \bigcap \mathbf{B}$ is connected. Since A and B are simply connected and $\mathbf{A} \bigcap \mathbf{B}$ is connected then, according to Proposition 8, $\mathbf{A} \bigcap \mathbf{B}$ is simply connected. Again since **A** and **B** are simply connected, then $\mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K}) = \mathbf{A}$ and $\mathbf{B} \bigcup (\mathbf{B} \ominus \mathbf{K}) = \boldsymbol{B}$. According to Proposition 9, since **A** and **B** are convex, then $(\mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K})) \bigcap (\mathbf{B} \bigcup (\mathbf{B} \ominus \mathbf{K}))$ is convex if $(\mathbf{A} \bigcup (\mathbf{A} \ominus \mathbf{K})) \bigcap (\mathbf{B} \bigcup (\mathbf{B} \ominus \mathbf{K}))$ is connected, i.e., $\mathbf{A} \bigcap \mathbf{B}$ is convex if $\mathbf{A} \bigcap \mathbf{B}$ is connected.

Hence, $A \bigcap B$ is simply connected convex. Q.E.D.

Corollary 3 — If $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, ..., \mathbf{A}_n$ are simply connected convex objects, then $A_1 \bigcap A_2 \bigcap A_3 \bigcap .... A_n$ is also convex if it is connected.

Corollary 4 — If **A** is a simply connected convex object, then for any structuring element **B**, $\mathbf{A} \ominus \mathbf{B}$ is a simply connected convex object if it is connected.

PROOF : From definitio of erosion we can write

$$\mathbf{A} \ominus \mathbf{B} = \bigcap_{b \in \mathbf{B}} \mathbf{A}_{-b} = A_{b_1} \bigcap A_{b_2} \bigcap A_{b_3} \bigcap .... \bigcap A_{b_n}$$

where $-b_1, -b_2, -b_3, ..., -b_n$ are elements of B. Now since convexity is translation invariant property of objects, A translated by b_i is simply connected convex for all i. Suppose $\mathbf{A} \ominus \mathbf{B}$ is connected, i.e.,

$$\mathbf{A}_{b_1} \bigcap \mathbf{A}_{b_2} \bigcap \mathbf{A}_{b_3} \bigcap \dots \bigcap A_{b_n}$$

is connected. Therefore, according to Corollary-3,

$$\mathbf{A}_{b_1} \bigcap \mathbf{A}_{b_2} \bigcap \mathbf{A}_{b_3} \bigcap \dots \bigcap \mathbf{A}_{b_n}$$

is simply connected convex. Q.E.D.

Hence, $\mathbf{A} \ominus \mathbf{B}$ is simply connected convex.

Proposition 10 — If $\mathbf{A}$ is a simply connected convex object, then for any structuring element $\mathbf{B}$, $\mathbf{A} \ominus \mathbf{B}$ is connected if it is non-empty.

PROOF : To prove this proposition let us consider the definition of erosion in another form that states that

$$\mathbf{A} \ominus \mathbf{B} = \{p \in \mathbf{Z}^2 \mid \mathbf{B}_p \subseteq \mathbf{A}\}$$

Suppose $\mathbf{A}$ is a simply connected convex object, and also suppose that $C = \mathbf{A} \ominus \mathbf{B}$ and $\mathbf{C}$ is not connected. Then there exists at least a pair of pixels p and q such that $p, q \in C$ and p and q are not connected. So we can say that there exists a pixel r that lies on the DLS connecting p and q, and $r \notin C$. Now consider a pixel of the structuring element, i.e., $b \in \mathbf{B}$, then $(b + p) \in B_p$, $(b+q) \in B_q$ and $(b+r) \in \mathbf{B}_r$. Thus $(b + r)$ also lies on the digital straight line segment connecting $(b + p)$ and $(b + q)$. Since p, $q \in \mathbf{C}$ and $r \notin \mathbf{C}$, then $\mathbf{B}_p \subset A$, $\mathbf{B}_q \subset A$, and $\mathbf{B}_r \not\subset A$, respectively. That means for some value of b, $(b + p)$, $(b + q) \in \mathbf{A}$ and $(b+r) \notin \mathbf{A}$. Therefore, there exists at least a pair of pixels (i.e. $(b + p)$ and $(b + q)$) such that the DLS connecting them is not contained in $\mathbf{A}$. If this is true, according to Proposition 3, $\mathbf{A}$ is not simply connected convex.

Hence, $\mathbf{A} \ominus \mathbf{B}$ (or C) is connected. Q.E.D.

Proposition 11 — If A is simply connected convex object, then for any structuring element B, $A \ominus B$ is a simply connected convex object.

PROOF : Proof is straightforward and can be done just by combining the proofs of Corollary-4 and Proposition 10. Q.E.D.

Proposition 12 — If A and B are simply connected convex objects then $A \circ B$ is also a simply connected convex object.

PROOF : Proof is straighforward and can be done just by combining the Proposition 11 followed by Proposition 7. Q.E.D.

Corollary 5 — If $\mathbf{A}$ is simply connected and B is convex, then $A \circ B$ is convex.

Proposition 13 — If $\mathbf{A}$ is a simply connected object and L is DLS then $A \circ L$ is a convex object.

PROOF : Since L is a DLS, L is convex. By definition, $\mathbf{A} \circ \mathbf{L} = (\mathbf{A} \ominus L) \oplus L$. Now $(\mathbf{A} \ominus L)$ can be : (i) a point; (ii) a DLS; or (iii) an extended region. let us consider each case separately.

Case 1 : If $(\mathbf{A} \ominus L)$ is a point, then $(\mathbf{A} \ominus L) \oplus L$ is a DLS and, hence, is digitally convex.

Case 2 : If $(\mathbf{A} \ominus L)$ is a DLS, then $(\mathbf{A} \ominus L) \oplus L$, according to Proposition 6, is convex. However, according to Proposition 2, $(\mathbf{A} \ominus L) \oplus L$ can have point holes, i.e., is not cimply connected.

Case 3 : If $(\mathbf{A} \ominus L)$ is an extended region, $(\mathbf{A} \ominus L) \oplus L$ (as shown in Case 2 of the proof of Proposition 2) is simply connected. According to Proposition 7, $(\mathbf{A} \ominus L) \oplus L$ is convex.

Therefore, in general, $\mathbf{A} \circ L$ convex. Q.E.D.

Proposition 14 — If **A** and **B** are simply connected convex, then $\mathbf{A} \bullet \mathbf{B}$ too is simply connected convex.

PROOF : Proof is straightforward and can be done just by combining the Proposition 7 followed by Proposition 11. Q.E.D.

Proposition 15 — If **A** and **B** are convex, then $\mathbf{A} \bullet \mathbf{B}$ is also convex.

PROOF : Since both **A** and **B** are convex, they can have, at most, point holes. Consequently, according to Proposition 7, $(\mathbf{A} \oplus \mathbf{B})$ is convex and can have at most point holes. Now we know that $\mathbf{A} \subseteq \mathbf{A} \bullet \mathbf{B}$. That means $A \bullet B$ can have at most point holes. So the erosion of $(\mathbf{A} \oplus \mathbf{B})$ by **B** does not create any hole. Secondly, the closing operation smoothes the boundary of **A**. So the erosion of $(\mathbf{A} \oplus \mathbf{B})$ by B does not create any concavity on the boundary.

Hence $\mathbf{A} \bullet \mathbf{B}$ is convex.

5. CONCLUSION

In this paper, we have presented morphological algorithms for computing some topological properties like connectivity number and genus of digital objects. For connectivity number both the definitions suggested by Yokoi *et al.*[17] and Rutovitz[14] are considered. For computing the genus of an binary image we propose a new algorithm. This algorithm auggests that the simple erosion operation with appropriate structuring elements is capable of computing genus. At this point we recall that Gray[4] and Serra[15] computed genus using hit-and-miss transform. However, the present discussion reveals that both connectivity number and genus of an binary image can be computed using erosion only.

We have also presented a new definition for digital convexity. This definition has the property that when a digital convex object is dilated by another digital convex object the resultant object will always be digitally convex. The same is true for erosion, opening and closing also, if the resultant object is nonempty. So this definition has a direct resemblance with that in the continuous domain except for the fact that it takes into account also the objects that have point holes. Secondly, the definition is equivalent to existing definitions of digital convexity in the case of simply connected objects. In fact, the proposed definition is more general and holds the line containment property. However, this definition of digital convexity, like other definitions except that suggested by Freeman and Shapira[3], lacks the intersection property. The intersection of two digital objects, which are declared convex by the proposed definition, may not be connected when it is nonempty. On the other hand, the definition of digital convexity suggested by Freeman and Shapira covers only a subset of digital objects which are digital versions of convex objects in the continuous domain. We have seen that the above mentioned intersection problem is actually caused by the ambiguity in digitization of straight line segment.

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