Image Analysis Using Mathematical Morphology

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Abstract—**For the purposes of object or defect identification required in industrial vision applications, the operations of mathematical morphology are more useful than the convolution operations employed in signal processing because the morphological operators relate directly to shape. The tutorial provided in this paper reviews both binary morphology and gray scale morphology, covering the operations of dilation, erosion, opening, and closing and their relations. Examples are given for each morphological concept and explanations are given for many of their interrelationships.**

Index Terms—**Closing, dilation, erosion, filtering, image analysis, morphology, opening, shape analysis.**

I. Introduction

MATHEMATICAL morphology provides an approach to the processing of digital images which is based on shape. Appropriately used, mathematical morphological operations tend to simplify image data preserving their essential shape characteristics and eliminating irrelevancies. As the identification of objects, object features, and assembly defects correlate directly with shape, it becomes apparent that the natural processing approach to deal with the machine vision recognition process and the visually guided robot problem is mathematical morphology.

Morphologic operations are among the first kinds of image operators used. Kirsch, Cahn, Ray, and Urban [13] discussed some binary 3×3 morphologic operators. Other early papers include Unger [37] and Moore [21].

Machines which perform morphologic operations are not new. They are the essence of what cellular logic machines such as the Golay logic processor [8], Diff3 [9], PICAP [15], the Leitz Texture Analysis System TAS [14], the CLIP processor arrays [3], and the Delft Image Processor DIP [6] all do. A number of companies now manufacture industrial vision machines which incorporate video rate morphological operations. These companies include Machine Vision International, Maitre, Synthetic Vision Systems, Vicom, Applied Intelligence Systems, Inc., and Leitz.

The 1985 IEEE Computer Society Workshop on Computer Architecture For Pattern Analysis and Image Database Management had an entire session devoted to computer architecture specialized to perform morphological operations. Papers included those by McCubbrey and Lougheed [19], Wilson [39], Kimmel, Jaffe, Manderville, and Lavin [12], Leonard [16], Pratt [27], and Haralick [11]. Gerritsen and Verbeek [7] show how convolution followed by a table look up operation can accomplish binary morphologic operations.

But although the techniques are being used in the industrial world, the basis and theory of mathematical morphology tend to be (with the exception of the highly mathematical books by Matheron [18] and Serra [31]) not covered in the textbooks or journals which discuss image processing or computer vision. It is the intent of this tutorial to help fill this void.

The paper is divided into three parts. Section II discusses the basic operations of dilation and erosion in an N-dimensional Euclidean space. Section III discusses the derived operations of opening and closing. Section IV gives the corresponding definition for the dilation and erosion operations for gray tone images and shows how with these definitions all the properties of dilation and erosion, opening, and closing previously derived and explained in Sections II and III hold.

II. Dilation and Erosion

The language of mathematical morphology is that of set theory. Sets in mathematical morphology represent the shapes which are manifested on binary or gray tone images. The set of all the black pixels in a black and white image, (a binary image) consitutes a complete description of the binary image. Sets in Euclidean 2-space denote foreground regions in binary images. Sets in Euclidean 3-space may denote time varying binary imagery or static gray scale imagery as well as binary solids. Sets in higher dimensional spaces may incorporate additional image information, like color, or multiple perspective imagery. Mathematical morphological transformations apply to sets of any dimensions, those like Euclidean N-space, or those like its discrete or digitized equivalent, the set of N-tuples of integers, Z^N. For the sake of simplicity we will refer to either of these sets as E^N.

Those points in a set being morphologically transformed are considered as the selected set of points and those in the complement set are considered as not selected. Hence, morphology from this point of view is binary morphology. We begin our discussion with the binary morphological operations of dilation and erosion.

A. Dilation

Dilation is the morphological transformation which combines two sets using vector addition of set elements.

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If A and B are sets in N-space (E^N) with elements a and b, respectively, $a = (a_1, \cdots, a_N)$ and $b = (b_1, \cdots, b_N)$ being N-tuples of element coordinates, then the dilation of A by B is the set of all possible vector sums of pairs of elements, one coming from A and one coming from B.

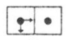
Definition 1: Let A and B be subsets of E^N. The dilation of A by B is denoted by $A \oplus B$ and is defined by

$$A \oplus B = \{c \in E^N \mid c = a + b \text{ for some } a \in A \text{ and } b \in B\}.$$

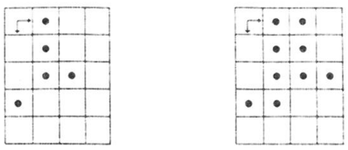
Example: This illustrates an instance of the dilation operation. The coordinate system we use for all the examples in the next few sections is (row, column).

$A = \{(0,1), (1,1), (2,1), (2,2), (3,0)\}$

$B = \{(0,0), (0,1)\}$



B



A $\qquad$ $A \oplus B$

$A \oplus B = \{(0,1),\ (1,1),\ (2,1),\ (2,2),\ (3,0)$

$(0,2),\ (1,2),\ (2,2),\ (2,3),\ (3,1)\}$

Dilation as a set theoretic operation was proposed by H. Minkowski [20] to characterize integral measures of certain open (sparse) sets. Dilation as an image processing operation was employed by several early investigators in image processing as smoothing operations [13], [37], [21], [8], [28], [29]. Dilation as an image operator for shape extraction and estimation of image parameters was explored by Matheron [18] and Serra [30]. All of these early applications dealt with binary images only.

Matheron uses the term "dilatation" for dilation and both Matheron and Serra define dilation slightly differently. In essense, they define the dilation of A by B as the set $\{c \in E^N \mid c = a - b \text{ for some } a \in A \text{ and } b \in B\}$.

In morphological dilation, the roles of the sets A and B are symmetric, that is, the dilation operation is commutative because addition is commutative.

Proposition 2:

$$A \oplus B = B \oplus A.$$

Proof:

$$\begin{aligned} A \oplus B &= \{c \mid c = a + b \text{ for some } a \in A, b \in B\} \\ &= \{c \mid c = b + a \text{ for some } a \in A, \ b \in B\} \\ &= B \oplus A. \end{aligned}$$

In practice, A and B are handled quite differently. The first operand A is considered as the image undergoing analysis, while the second operand B is referred to as the structuring element, to be thought of as constituting a single shape parameter of the dilation transformation. In the remainder of the paper, we will refer to A as the image and B as the structuring element.

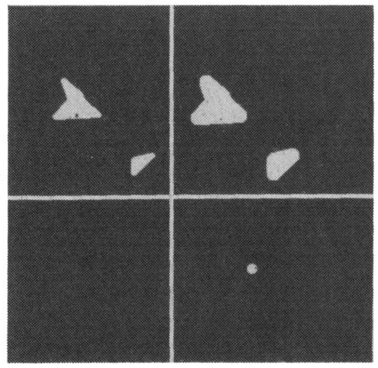


Fig. 1. The upper left shows the input image consisting of two objects. The lower right shows the octagonal structuring element. The upper right shows the input image dilated by the octagonal structuring element.

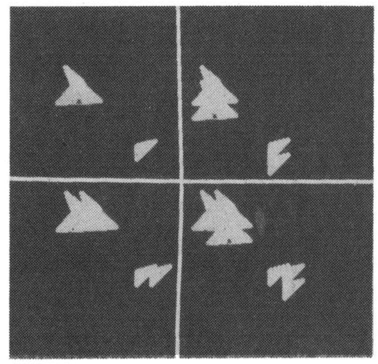


Fig. 2. The upper left shows the input image consisting of two objects. The upper right shows the input image dilated by the structuring element $\{(0, 0), (14, 0)\}$. The lower left shows the input image dilated by the structuring element $\{(0, 0), (0, 14)\}$. The lower right shows the input image dilated by the structuring element $\{(0, 0), (14, 0), (0, 14)\}$.

Dilation by disk structuring elements correspond to isotropic swelling or expansion algorithms common to binary image processing. Dilation by small squares (3×3) is a neighborhood operation easily implemented by adjacency connected array architectures (grids) and is the one many image processing people know by the name "fill," "expand," or "grow." Some example dilation transformations are illustrated in Figs. 1 and 2.

Neighborhood connected image processors such as CLIP [4], Cytocomputer [32], [34], and MPP [1], [26] can implement some dilations (not all) by structuring elements larger than the neighborhood size by iteratively dilating with a sequence of neighborhood structuring elements. In particular, if image A is to be dilated by structuring element D which itself can be expressed as the dilation of B by C, then $A \oplus D$ can be computed as

$$A \oplus D = A \oplus (B \oplus C) = (A \oplus B) \oplus C$$

since addition is associative.

The form $(A \oplus B) \oplus C$ represents a considerable savings in number of operations to be performed when A is the image and $B \oplus C$ is the structuring element. The savings come about because a brute force dilation by $B \oplus C$ might take as many as N^2 operations while first dilating

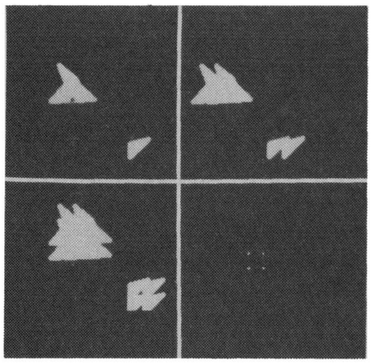


Fig. 3. The upper left shows the input image consisting of two objects. The upper right shows the input image dilated by the structuring element $\{(0, 0)\ (0, 14)\}$. The lower left shows the input image dilated by the structuring element $\{(0, 0), (14, 0), (0, 14)\ (14, 14)\}$, which is shown in the lower right. Notice that the dilated image of the lower left can be obtained by dilating the image shown in the upper right by the structuring element $\{(0, 0), (14, 0)\}$. This is a consequence of the chain rule for dilations and because $\{(0, 0), (14, 0)\} \oplus \{(0, 0), (0, 14)\} = \{(0, 0), (0, 14), (14, 0), (14, 14)\}$.

A by B and then dilating the result by C could take as few as $2N$ operations, where N is the number of elements in B and in C. This computational complexity advantage is not as strong for machines which can implement dilations only as neighborhood operations.

Proposition 3:

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C.$$

Proof: $x \in A \oplus (B \oplus C)$ if and only if there exists $a \in A$, $b \in B$, and $c \in C$ such that $x = a + (b + c)$. $x \in (A \oplus B) \oplus C$ if and only if there exists $a \in A$, $b \in B$, and $c \in C$ such that $x = (a + b) + c$. But $a + (b + c) = (a + b) + c$ since addition is associative. Therefore, $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.

Proposition 3 is commonly referred to as the "chain rule" for dilations. An example of performing a dilation transformation as a chain of dilations is shown in Fig. 3. Notice that this dilation transformation which can be done as a chain of dilations is not able to be done as a chain of neighborhood operations.

Since dilation is commutative, the order of application of the constituent dilations is immaterial.

Dilating an image as an iterative sequence of neighborhood operations is not necessarily the most efficient or universal approach to implementing the dilation transformation. For example, not all structuring elements can be decomposed into iterative neighborhood dilations. An example of a dilation transformation which cannot be implemented as an iterative sequence of neighborhood operations is the dilation by any of the structuring elements $\{(0, 0), (0, 14)\}$, $\{(0, 0), (14, 0)\}$ or $\{(0, 0), (0, 14), (14, 0)\}$ which are shown in Fig. 2.

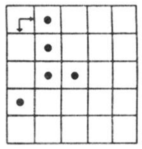
Also, the implementation may not be particularly efficient in terms of processing time or computer hardware requirements. An alternative involves considering dilations in terms of image translations. So first we need the definition for translation.

Definition 4: Let A be a subset of E^N and $x \in E^N$. The translation of A by x is denoted by $(A)_x$ and is defined by

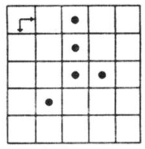
$$(A)_x = \{c \in E^N \mid c = a + x \text{ for some } a \in A\}.$$

Example: This illustrates an instance of translation.

$$A = \{(0, 1), (1, 1), (2, 1), (2, 2), (3, 0)\}$$
$$x = (0, 1)$$
$$(A)_x = \{(0, 2), (1, 2), (2, 2), (2, 3), (3, 1)\}.$$



A



$(A)_{(0,1)}$

The dilation of A by B can be computed as the union of translations of A by the elements of B.

Proposition 5:

$$A \oplus B = \bigcup_{b \in B} (A)_b.$$

Proof: Suppose $x \in A \oplus B$. Then for some $a \in A$ and $b \in B$, $x = a + b$. Hence, $x \in (A)_b$ and therefore $x \in \bigcup_{b \in B} (A)_b$.

Suppose $x \in \bigcup_{b \in B} (A)_b$. Then for some $b \in B$, $x \in (A)_b$. But $x \in (A)_b$ implies there exists an $a \in A$ such that $x = a + b$. Now by definition of dilation, $a \in A$, $b \in B$, and $x = a + b$ imply $x \in A \oplus B$.

Historically, the dilation transformation was defined by Minkowski in this manner, hence the name Minkowski addition is applied to Proposition 5 in the literature (for example, see [10]). Unfortunately, Minkowksi failed to define the dual of his set addition operation, and Minkowski subtraction expressed as the intersection of translations of A by the elements of B was not formally proposed until done so by Hadwiger.

Proposition 5 emphasizes the role of image shifting to implement dilation. In pipeline digital image processors employing raster scanning, image shifting is accomplished by delay elements in the transmission path. But delay elements can only cause an image shift in a direction opposite to the row scanning direction of the raster conversion. Thus it is important to know that dilating a shifted image, which arises from previous pipeline delays, shifts the dilated result by an equivalent amount. This fact permits pipeline processors to successively operate morphologically on shifted images and to undo the total resulting shift by performing an opposite shift by the scrolling operation in the output image buffer. We call this property the translation invariance of dilation.

Translation Invariance of Dilation Proposition 6:

$$(A)_x \oplus B = (A \oplus B)_x.$$

Proof: $y \in (A)_x \oplus B$ if and only if for some $z \in (A)_x$ and $b \in B$, $y = z + b$. But $z \in (A)_x$ if and only if $z = a + x$ for some $a \in A$. Hence, $y = (a + x) + b = (a +$

$b) + x$. Now by definition of dilation and translation $y \in (A \oplus B)_x$.

A corollary to Proposition 6 applies to dilations implemented through the chain rule (Proposition 3). The corollary states that shifting any one of the structuring elements in a dilation decomposition shifts the dilated image by an equivalent amount.

Corollary 7:

$$A \oplus B_1 \oplus \cdots \oplus (B_n)_x \oplus \cdots \oplus B_N$$
$$= (A \oplus B_1 \oplus \cdots \oplus B_n \oplus \cdots \oplus B_N)_x.$$

Image shift can be compensated for in the definition of the structuring element. In particular, let the structuring element B be compensating for a shift in the image A by taking B to be shifted in the opposite direction. Then the shift in B compensates for the shift in A.

Proposition 8:

$$(A)_x \oplus (B)_{-x} = A \oplus B.$$

Proof:

$$(A)_x \oplus (B)_{-x} = \left(A \oplus (B)_{-x}\right)_x$$
$$= (A \oplus B)_{x-x}$$
$$= A \oplus B.$$

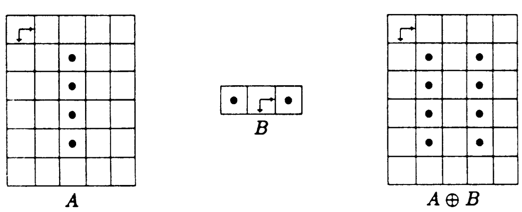
Similarly, compensating shifts within the sequence of decomposed structuring element dilations can balance image shifts and cause an unshifted result.

Corollary 9:

$$(A)_x \oplus B_1 \oplus \cdots \oplus (B_n)_{-x} \oplus \cdots \oplus B_N$$
$$= A \oplus B_1 \oplus \cdots \oplus B_n \oplus \cdots \oplus B_N$$

In addition to being commutative, the dilation transformation is necessarily extensive when the origin belongs to the structuring element, extensivity meaning that the dilated result contains the original.

Example: This example shows that when the origin is not in the structuring element B, it may happen that the dilation of A by B has nothing in common with A.



A corollary to Proposition 10 states that if the origin belongs to each of the structuring elements, in a dilation composition, each structuring element in the decomposition is necessarily contained in the original composed structuring element.

Corollary 11: If $0 \in B_1, \cdots, B_N$ then $B_m \in B_1 \oplus \cdots \oplus B_N$, $m = 1, \cdots, N$.

The dilation transformation is increasing, that is, containment relationships are maintained through dilation.

Dilation Is Increasing Proposition 12: $A \subseteq B$ implies $A \oplus D \subseteq B \oplus D$.

Proof: Suppose $A \subseteq B$. Let $x \in A \oplus D$. Then for some $a \in A$ and $d \in D$, $x = a + d$. Since $a \in A$ and $A \subseteq B$, $a \in B$. But $a \in B$ and $d \in D$ implies $x \in B \oplus D$.

Corollary 13: $A \subseteq B$ implies $D \oplus A \subseteq D \oplus B$.

The order of an image intersection operation and a dilation operation cannot be interchanged. Rather, the result of intersecting two images followed by a dilation of the intersection result is contained in the intersection of the dilation of the two images.

Proposition 14:

$$(A \cap B) \oplus C \subseteq (A \oplus C) \cap (B \oplus C)$$
$$(A \oplus (B \cap C) \subseteq (A \oplus B) \cap (A \oplus C)$$

Proof: Suppose $x \in (A \cap B) \oplus C$. Then for some $y \in A \cap B$ and $c \in C$, $x = y + c$. Now $y \in A \cap B$ implies $y \in A$ and $y \in B$. But $y \in A$, $c \in C$, and $x = y + c$ implies $x \in A \oplus C$; $y \in B$, $c \in C$, and $x = y + c$ implies $x \in B \oplus C$. Hence $x \in (A \oplus C) \cap (B \oplus C)$.

$(A \oplus (B \cap C) \subseteq (A \oplus B) \cap (A \oplus C)$ comes about immediately from the previous result since dilation is commutative.

On the other hand, the order of image union and dilation can be interchanged. The dilation of the union of two images is equal to the union of the dilations of these images.

Proposition 15:

$$(A \cup B) \oplus C = (A \oplus C) \cup (B \oplus C)$$

Proof:

$$(A \cup B) \oplus C = \bigcup_{x \in A \cup B} (C)_x$$
$$= \left[\bigcup_{x \in A} (C)_x\right] \cup \left[\bigcup_{x \in B} (C)_x\right]$$
$$= (A \oplus C) \cup (B \oplus C).$$

By the commutativity of dilation, we immediately have the following.

Corollary 16:

$$A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C).$$

This equality is significant. It permits for a further decomposition of a structuring element into a union of structuring elements. Previously we saw that the decomposition of a structuring element into the dilation of elemental structuring elements led to a chain rule for dilation. Here we see that decomposing a structuring element into the union of elemental structuring elements leads to another method of evaluating the dilation.

The distinction between structuring element decomposition by dilation and by union deserves further mention. The issue bears upon the efficiency of computing the dilations. Consider the structuring element of Fig. 4.

Structuring element B of Fig. 4 top consists of 16 points, hence it can be decomposed into the union of 16 structuring elements, each structuring element consisting

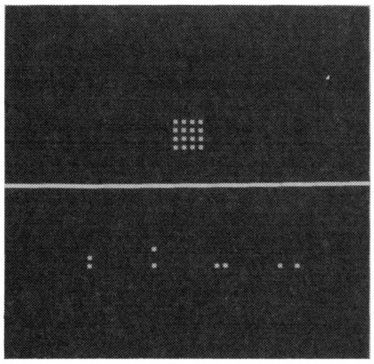


Fig. 4. This figure shows how the chain rule dilation decomposition can save operations. To dilate an image by the structuring element shown in the top half requires 15 operations. To dilate using the chain decomposition shown in the bottom half requires only 4 operations.

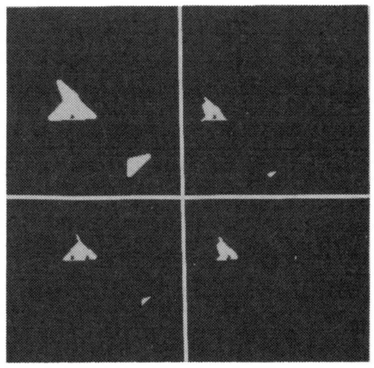


Fig. 5. The upper left shows the input image consisting of two blobs. The upper right shows the input image eroded by the structuring element $\{(0, 0), (-14, 0)\}$. The lower left shows the input image eroded by the structuring element $\{(0, 0)(0, -14)\}$. The lower right shows the input image eroded by the structuring element $\{(0, 0), (0, -14), (-14, 0)\}$.

of a single point which is suitably displaced from the origin. Dilation by a structuring element consisting of a single point is simply a shift of the original image, hence Proposition 15 becomes equivalent to the expression of Proposition 5 for the dilation, involving 15 shifts and 15 unions. By contrast, the decomposition of structuring element B into the four elemental structuring elements of Fig. 4 bottom permits dilation by B through the chain rule of Proposition 3. Here we see that only four shifts and four unions are required. Computationally, the difference involves a shift and union of the previously computed result in the case of Proposition 3's chain rule, while decomposition by union as in Proposition 15 independently accumulates the individual shifts of the original image.

B. Erosion

Erosion is the morphological dual to dilation. It is the morphological transformation which combines two sets using the vector subtraction of set elements. If A and B are sets in Euclidean N-space, then the erosion of A by B is the set of all elements x for which $x + b \in A$ for every $b \in B$. Some image processing people use the name shrink or reduce for erosion.

Definition 17: The erosion of A by B is denoted by $A \ominus B$ and is defined by

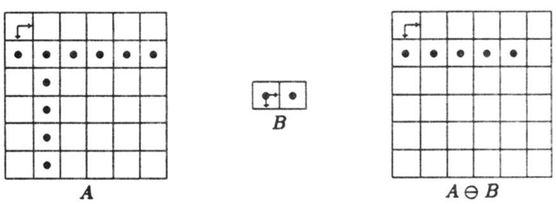
$$A \ominus B = \{x \in E^N \mid x + b \in A \text{ for every } b \in B\}$$

Example: This illustrates an instance of erosion.

$$A = \{(1, 0), (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (3, 1), (4, 1), (5, 1),\}$$

$$B = \{(0, 0), (0, 1)\}$$

$$A \ominus B = \{(1, 0), (1, 1), (1, 2), (1, 3), (1, 4)\}$$



Expressed as a difference of elements a and b, Definition 17 becomes

$$A \ominus B = \{x \in E^N \mid \text{for every } b \in B, \text{ there exists an } a \in A \text{ such that } x = a - b\}$$

This is the definition used for erosion by [10].

The utility of the erosion transformation is better appreciated when the erosion is expressed in a different form. The erosion of an image A by a structuring element B is the set of all elements x of E^N for which B translated to x is contained in A. In fact, this was the definition used for erosion by [18]. The proof is immediate from the definition of erosion and the definition of translation.

Proposition 18:

$$A \ominus B = \{x \in E^N \mid (B)_x \subseteq A\}.$$

Thus the structuring element B may be visualized as a probe which slides across the image A, testing the spatial nature of A at every point. Where B translated to x can be contained in A (by placing the origin of B at x), then x belongs to the erosion $A \ominus B$. The erosion transformation is illustrated in Fig. 5.

The careful reader should beware that the symbol $\ominus$ used by [31] does not designate erosion. Rather it designates the Minkowski subtraction which is the intersection of all translations of A by the elements $b \in B$. Whereas the dilation transformation and the Minkowski addition of sets are identical, the erosion transformation and the Minkowski subtraction differ in a significant way. Erosion of an image A by a structuring element B is the intersection of all translations of A by the points $-b$, where $b \in B$.

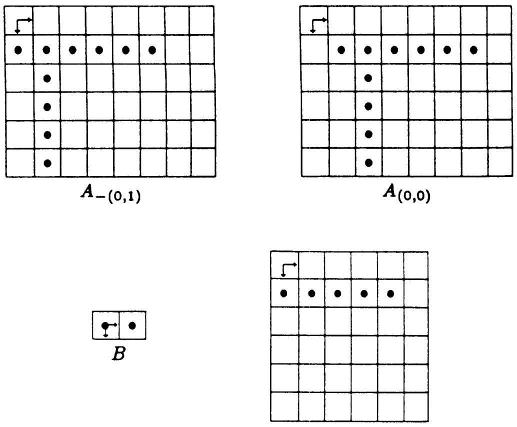
Proposition 19:

$$A \ominus B = \bigcap_{b \in B} (A)_{-b}.$$

Proof: Let $x \in A \ominus B$. Then for every $b \in B$, $x + b \in A$. But $x + b \in A$ implies $x \in (A)_{-b}$. Hence for every $b \in B$, $x \in (A)_{-b}$. This implies $x \in \bigcap_{b \in B} (A)_{-b}$.

Let $x \in \bigcap_{b \in B} (A)_{-b}$. Then for every $b \in B$, $x \in (A)_{-b}$. Hence, for every $b \in B$, $x + b \in A$. Now by definition of erosion $x \in A \ominus$ B.

Example: This illustrates how erosion can by computed as an intersection of translates of A.



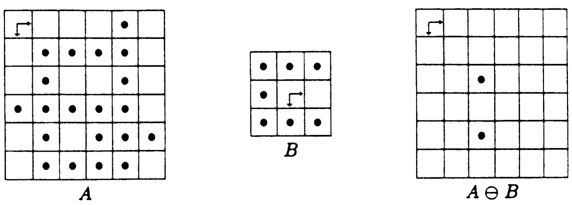
$A \ominus B = A_{(0,0)} \cap A_{-(0,1)}$

The erosion transformation is popularly conceived of as a shrinking of the original image. In set terms, the eroded set is often thought of as being contained in the original set. A transformation having this property is called anti-extensive. However, the erosion transformation is necessarily anti-extensive only if the origin belongs to the structuring element.

Proposition 20: If $0 \in B$, then $A \ominus B \subseteq A$.

Proof: Let $x \in A \ominus B$. Then $x + b \in A$ for every $b \in B$. Since $0 \in B$, $x + 0 \in A$. Hence $x \in A$.

Example: This illustrates how eroding with a structuring element which does not contain the origin can lead to a result which has nothing in common with the set being eroded.



Like dilation, erosion is a translation invariant and increasing transformation.

Translation Invariance of Erosion Proposition 21:

$$A_x \ominus B = (A \ominus B)_x$$

$$A \ominus B_x = (A \ominus B)_{-x}.$$

Proof: $y \in A_x \ominus$ B if and only if for every $b \in B$, $y + b \in A_x$. But $y + b \in A_x$ if and only if $y + b - x \in A$. Now, $y + b - x = (y - x) + b$. Hence for every $b \in B$, $(y - x) + b \in A$. By definition of erosion, $y - x \in A \ominus B$ and, therefore, $y \in (A \ominus B)_x$.

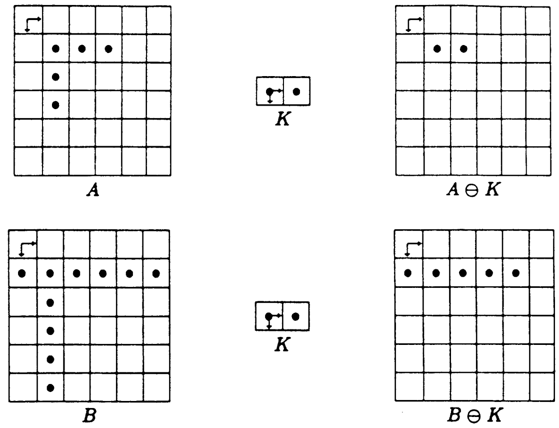
$y \in A \ominus B_x$ if and only if $y + b \in A$ for every $b \in B_x$. But $y + b \in A$ for every $b \in B_x$ if and only if $y - x \in A \ominus B$. Finally $y - x \in A \ominus B$ if and only if $y \in (A \ominus B)_{-x}$.

If image A is contained in image B, then the erosion of A is contained in the erosion of B.

Erosion Is Increasing Proposition 22: $A \subseteq B$ implies $A \ominus K \subseteq B \ominus K$.

Proof: Let $x \in A \ominus K$. Then $x + k \in A$ for every $k \in K$. But $A \subseteq B$. Hence, $x + k \in B$ for every $k \in K$. By definition of erosion, $x \in B \ominus K$.

Example: This illustrates an instance showing the increasing property of erosion.

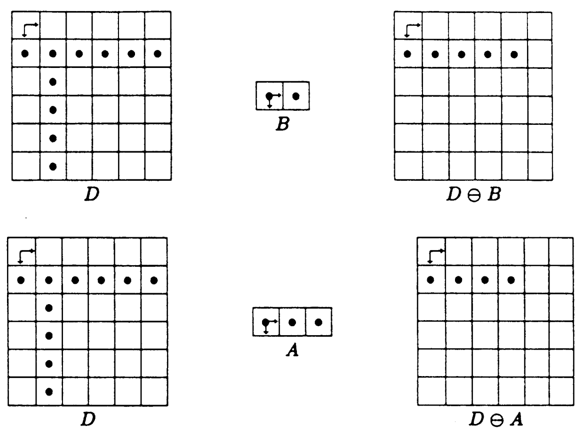


On the other hand, if A and B are structuring elements and B is contained in A, then the erosion of an image D by A is necessarily more severe than erosion by B, that is, D eroded by A will necessarily be contained in D eroded by B.

Proposition 23: $A \supseteq B$ implies $D \ominus A \subseteq D \ominus B$.

Proof: Let $x \in D \ominus A$. Then $x + a \in D$ for every $a \in A$. But $B \subseteq A$. Hence, $x + a \in D$ for every $a \in B$. Now by definition of erosion, $x \in D \ominus B$.

Example: This illustrates an instance showing that larger structuring elements have a more severe effect than smaller ones on the erosion process.



This proposition leads to a natural ordering of the erosions by structuring elements having the same shape but different sizes. It is the basis of the morphological distance transformations. Fig. 6 illustrates these distance relationships.

The dilation and erosion transformations bear a marked similarity, in that what one does to the image foreground the other does to the image background. Indeed, their similarity can be formalized as a duality relationship. Recall that two operators are dual when the negation of a

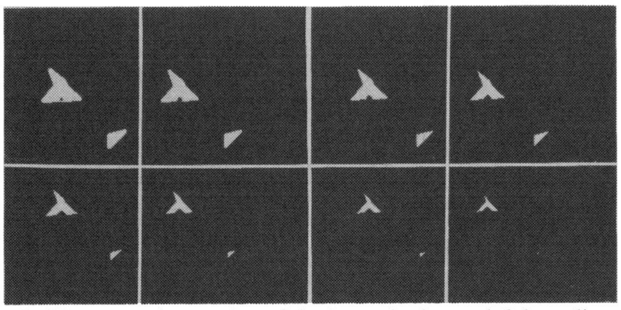


Fig. 6. The successive erosion of the image in the top left by a diamond shaped structuring element. Pixels which are white in the ith erosion are white pixels in the input image which have a 4-distance of greater than i pixels to the black background.

formulation employing the first operator is equal to that formulation employing operator on the negated variables. An example is DeMorgan's law, illustrating the duality of union and intersection,

$$(A \cup B)^c = A^c \cap B^c.$$

Here the negation of a set A is its complement,

$$A^c = \{x \in E^n \mid x \notin A\}.$$

In morphology, negation of a set is considered in a geometrical sense: that of reversing the orientation of the set with respect to its coordinate axes. Such reversing is called reflection.

Definition 24: Let $B \subseteq E^N$. The reflection of B is denoted by $\check{B}$ and is defined by

$$\check{B} = \{x \mid \text{for some } b \in B,\ x = -b\}.$$

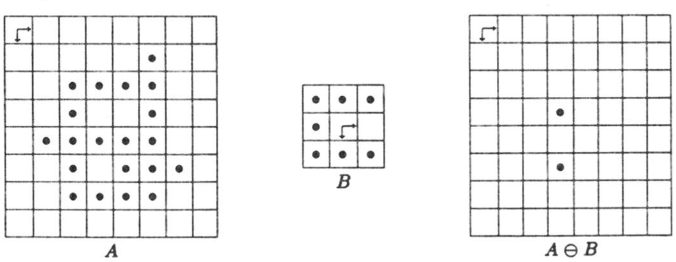
The reflection occurs about the origin. Matheron [18] refers to $\check{B}$ as "the symmetrical set of B with respect to the origin." Serra [31] refers to $\check{B}$ as "B transpose."

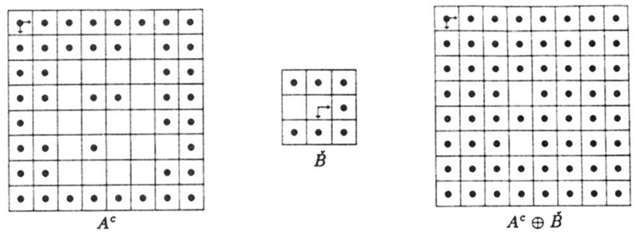
As given in Theorem 25, the duality of dilation and erosion employs both logical and geometric negation because of the different roles of the image and the structuring element in an expression employing these morphological operators.

Erosion Dilation Duality Theorem 25: $(A \ominus B)^c = A^c \oplus \check{B}$.

Proof: $x \in (A \ominus B)^c$ if and only if $x \notin A \ominus B$. $x \notin A \ominus B$ if and only if there exists $b \in B$ such that $x + b \notin A$. There exists $b \in B$ such that $x + b \in A^c$ if and only if there exists $b \in B$ such that $x \in (A^c)_{-b}$. There exists $b \in B$ such that $x \in (A^c)_{-b}$ if and only if $x \in \bigcup_{b \in B} (A^c)_{-b}$. Now, $x \in \bigcup_{b\in B} (A^c)_{-b}$ if and only if $x \in \bigcup_{b \in \check{B}} (A^c)_b$; and $x \in \bigcup_{b \in \check{B}} (A^c)_b$ if and only if $x \in A^c \oplus \check{B}$.

Example: This illustrates an instance of the relationship $(A \ominus B)^c = A^c \oplus \check{B}$.





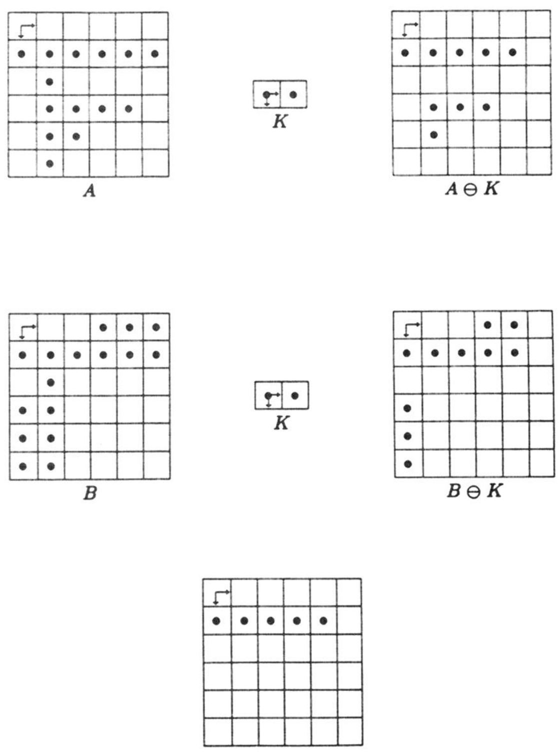
The difference between the dilation and erosion transformations is illuminated in the algebraic properties of the erosion as contrasted with the dilation. First, the erosion of the intersection of two images is equal to the intersection of their erosions. This contrasts with Proposition 14 where the relationship is one of containment.

Proposition 26:

$$(A \cap B) \ominus K = (A \ominus K) \cap (B \ominus K).$$

Proof: $x \in (A \cap B) \ominus K$ if and only if for every $k \in K$, $x + k \in A \cap B$. $x + k \in A \cap B$ if and only if $x + k \in A$ and $x + k \in B$. $x + k \in A$ for every $k \in K$ if and only if $x \in A \ominus K$. $x + k \in A$ for every $k \in K$ if and only if $x \in B \ominus K$. $x + k \in A$ for every $k \in K$ and $x + k \in B$ for every $k \in K$ if and only if $x \in (A \ominus K) \cap (B \ominus K)$.

Example: This illustrates an instance of the relationship $(A \cap B) \ominus K = (A \ominus K) \cap (B \ominus K)$.



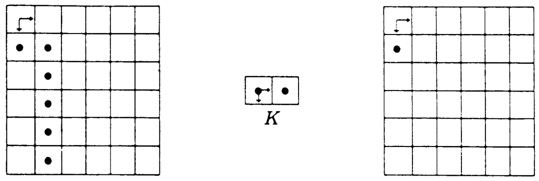
$(A \cap B) \ominus K = (A \ominus K) \cap (B \ominus K)$

On the other hand, whereas the dilation of the unions of two images is equal to the union of their dilations (Proposition 15), for the erosion transformation the relationship is one of containment.

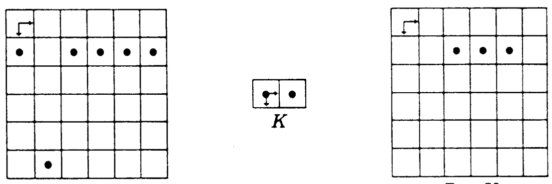
Proposition 27: $(A \cup B) \ominus K \supseteq (A \ominus K) \cup (B \ominus K)$.

Proof: Let $x \in (A \ominus K) \cup (B \ominus K)$. Then $x \in A \ominus K$ or $x \in B \ominus K$. If $x \in A \ominus K$ then since $A \cup B \supseteq A$, $x \in (A \cup B) \ominus K$. If $x \in B \ominus K$ then since $A \cup B \supseteq B$, $x \in (A \cup B) \ominus K$.

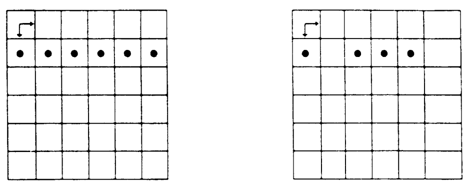
Example: This illustrates an instance in which $(A \cup B) \ominus K$ strictly contains $(A \ominus K) \cup (B \ominus K)$.



A K $A \ominus K$



B K $B \ominus K$



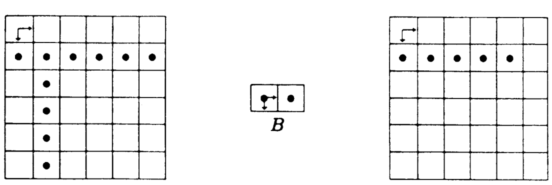
$(A \cup B) \ominus K$ $(A \ominus K) \cup (B \ominus K)$

But erosion is not commutative, $A \ominus B \neq B \ominus A$. Hence the behavior of $A \ominus (B \cup C)$ indicated in the equality of Proposition 28 is different than the behavior of $(A \cup B) \ominus C$ as indicated in Proposition 27.

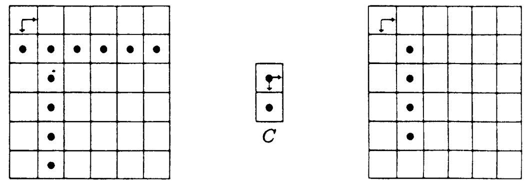
Proposition 28: $A \ominus (B \cup C) = (A \ominus B) \cap (A \ominus C)$

Proof: $x \in A \ominus (B \cup C)$ if and only if $x + y \in A$ for every $y \in B \cup C$. $x + y \in A$ for every $y \in B \cup C$ if and only if $x + y \in A$ for every $y \in B$ and $x + y \in A$ for every $y \in C$. $x + y \in A$ for every $y \in B$ if and only if $x \in A \ominus B$. $x + y \in A$ for every $y \in C$ if and only if $x \in A \ominus C$. $x + y \in A$ for every $y \in B$ and $x + y \in A$ for every $y \in C$ if and only if $x \in (A \ominus B) \cap (A \ominus C)$.

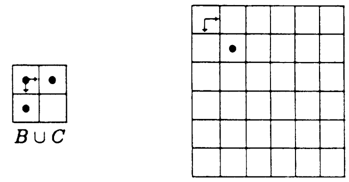
Example: This illustrates an instance of the relationship $A \ominus (B \cup C) = (A \ominus B) \cap (A \ominus C)$.



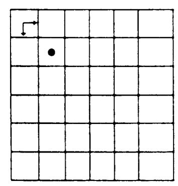
A B $A \ominus B$



A C $A \ominus C$



$B \cup C$ $A \ominus (B \cup C)$



$(A \ominus B) \cap (A \ominus C)$

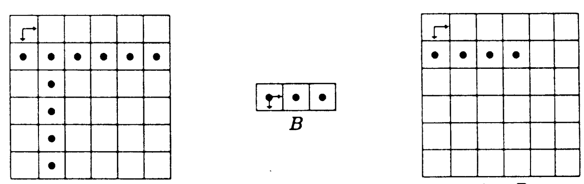
The practical utility of Proposition 28 is that it indicates how to compute erosions with structuring elements which can only be decomposed as the union of individual structuring elements.

Although structuring elements can be decomposed through union into simpler structuring elements to simplify the erosion transformation, structuring elements cannot be decomposed through intersection and maintain an equality. Rather, the intersection decomposition leads to a containment relationship.

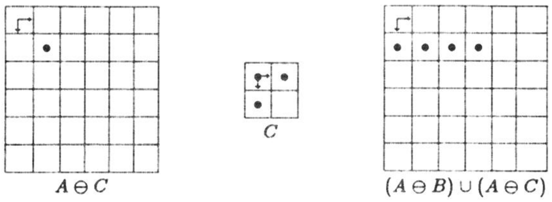
Proposition 29: $A \ominus (B \cap C) \supseteq (A \ominus B) \cup (A \ominus C)$.

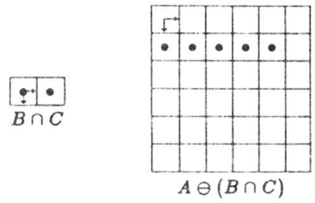
Proof: Let $x \in (A \ominus B) \cup (A \ominus C)$. Then $x \in A \ominus B$ or $x \in A \ominus C$. If $x \in A \ominus B$, then $x + b \in A$ for every $b \in B$. If $x \in A \ominus C$, then $x + b \in A$ for every $b \in C$. Hence, $x + b \in A$ for every $b \in B \cap C$. Now by definition of erosion, $x \in A \ominus (B \cap C)$.

Example: This illustrates an instance in which $A \ominus (B \cap C)$ strictly contains $(A \ominus B) \cup (A \ominus C)$.



A B $A \ominus B$





Finally, with respect to structuring element decomposition, a chain rule for erosion holds when the structuring element is decomposable through dilation,

$$A \ominus (B \oplus C) = (A \ominus B) \ominus C.$$

This relation is as important as the chain rule relation for dilation because it permits a large erosion to be computed by two successive smaller erosions.

Proposition 30: $(A \ominus B) \ominus C = A \ominus (B \oplus C)$.

Proof: Let $x \in (A \ominus B) \ominus C$. Then for every $c \in C$, $x + c \in A \ominus B$. But $x + c \in A \ominus B$ implies $x + c + b \in A$ for every $b \in B$. But $x + b + c \in A$ for every $b \in B$ and $c \in C$ implies $x + d \in A$ for every $d \in B \oplus C$.

Let $x \in A \ominus (B \oplus C)$. Then $x + d \in A$ for every $d \in B \oplus C$. Hence $x + b + c \in A$ for every $b \in B$ and $c \in C$. Now $(x + c) + b \in A$ for every $b \in B$ implies $x + c \in A \ominus B$. But $x + c \in A \ominus B$ for every $c \in C$ implies $x \in (A \ominus B) \ominus C$.

Corollary 31 extends this result to structuring elements decomposed as the dilation of K structuring elements.

Corollary 31: $A \ominus (B_1 \oplus \cdots \oplus B_K) = (\cdots (A \ominus B_1) \ominus \cdots \ominus B_K)$.

It is immediately apparent from the corollary that because dilation is commutative, the order in which successive erosions are applied is immaterial.

Fig. 7 illustrates the utility of the chain rule for erosions.

Reversing the position of dilation and erosion in Proposition 30 does not lead to an equality as in Proposition 30 but leads to a containment relation as given in Proposition 32. In some sense this indicates that when performing erosion and dilation, performing erosion first is more severe than performing dilation first.

Proposition 32: $A \oplus (B \ominus C) \subseteq (A \oplus B) \ominus C$.

Proof: Let $x \in A \oplus (B \ominus C)$. Then for some $a \in A$ and $B \ominus C$, $x = a + y$. But $y \in B \ominus C$ implies $y + c \in B$ for every $c \in C$. Now $y + c \in B$ and $a \in A$ implies $y + c + a \in A \oplus B$. Finally $y + c + a \in A \oplus B$ for every $c \in C$ implies $x = y + a \in (A \oplus B) \ominus C$.

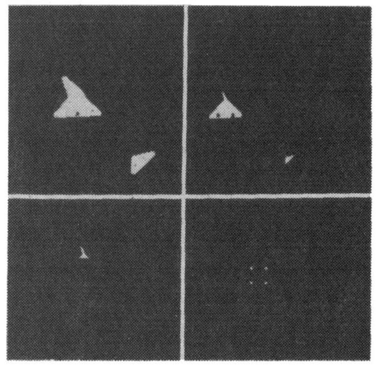
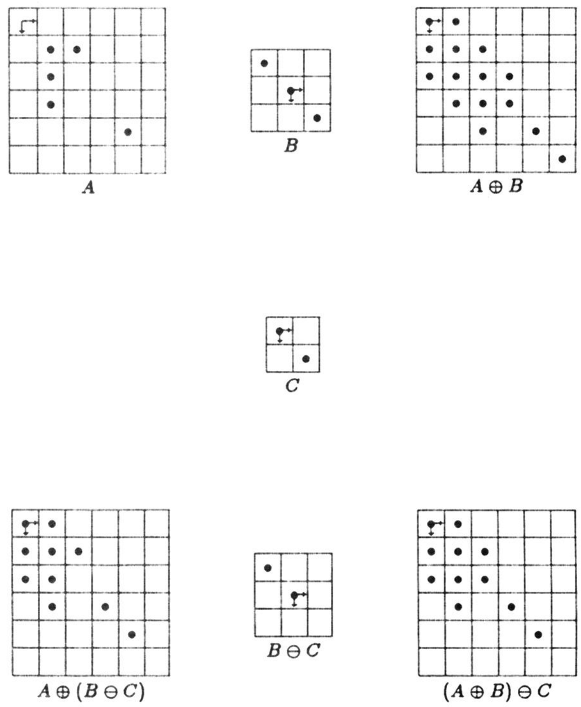


Fig. 7. The upper left shows the input image. The upper right shows the input image eroded by the structuring element $\{(0, 0), (0, -14)\}$. The lower left shows the eroded image of the upper right eroded by the structuring element $\{(0, 0), (-14, 0)\}$. This result is equivalent to eroding the input image by the structuring element $\{(0, 0), (0, -14), (-14, 0), (-14, -14)\}$ which is shown in the lower right.

Example: This illustrates an instance in which $A \oplus (B \ominus C)$ is strictly contained in $(A \oplus B) \ominus C$.



Although dilation and erosion are dual, this does not imply that we can freely perform cancellation on morphological equalities. For example, if $A = B \ominus C$, then dilating both sides of the expression by C results in $A \oplus C = B \ominus C \oplus C \neq B$. However, a containment relationship is maintained, as indicated in Proposition 33.

Proposition 33: $A \subseteq B \ominus C$ if and only if $B \supseteq A \oplus C$.

Proof: Suppose $A \subseteq B \ominus C$. Let $x \in A \oplus C$. Then there exists $a \in A$ and $c \in C$ such that $x = a + c$. But $a \in A$ and $A \subseteq B \ominus C$ implies $a \in B \ominus C$. Hence, for every $c' \in B$, $a + c' \in B$. In particular, $c \in C$. Thus $a + c \in B$. But $x = a + c$. Therefore, $x \in B$.

Suppose $A \oplus C \subseteq B$. Let $x \in A$. Let $c \in C$. Then $x +$

$c \in A \oplus C$. But $A \oplus C \subseteq B$ so that $x + c \subseteq B$. Finally $x + c \subseteq B$ for any $c \in C$ implies $x \in B \ominus C$.

The containment is maintained for chained erosions, as follows.

Corollary 34:

$$A \subseteq \left(\cdots (B \ominus C_1) \ominus \cdots \right) \ominus C_N \text{ if and only if } \left(\cdots (A \oplus C_1) \oplus \cdots \right) \oplus C_N \subseteq B.$$

III. Openings and Closings

In practice, dilations and erosions are usually employed in pairs, either dilation of an image followed by the erosion of the dilated result, or image erosion followed by dilation. In either case, the result of iteratively applied dilations and erosions is an elimination of specific image detail smaller than the structuring element without the global geometric distortion of unsuppressed features. For example, opening an image with a disk structuring element smooths the contour, breaks narrow isthmuses, and eliminates small islands and sharp peaks or capes. Closing an image with a disk structuring element smooths the contours, fuses narrow breaks and long thin gulfs, eliminates small holes, and fills gaps on the contours.

Of particular significance is the fact that image transformations employing iteratively applied dilations and erosions are idempotent, that is, their reapplication effects no further changes to the previously transformed result. The practical importance of idempotent transformations is that they comprise complete and closed stages of image analysis algorithms because shapes can be naturally described in terms of under what structuring elements they can be opened or can be closed and yet remain the same. Their functionality corresponds closely to the specification of a signal by its bandwidth. Morphologically filtering an image by an opening or closing operation corresponds to the ideal nonrealizable bandpass filters of conventional linear filtering. Once an image is ideal bandpassed filtered, further ideal bandpass filtering does not alter the result.

This property motivates the importance for having definitions of opening and closing, concepts first studied by Matheron [17], [18] who was interested in axiomatizing the concept of size. Both Matheron's [18] definitions and Serra's [31] definitions for opening and closing are identical to the ones given here, but their formulas appear different because they use the symbol $\ominus$ to mean Minkowski subtraction rather than erosion.

Definition 35: The opening of image B by structuring element K is denoted by $B \circ K$ and is defined as $B \circ K = (B \ominus K) \oplus K$.

Definition 36: The closing of image B by structuring element K is denoted by $B \bullet K$ and is defined by $B \bullet K = (B \oplus K) \ominus K$.

If B is unchanged by opening it with K, we say that B is open with respect to K, while if B is unchanged by closing it with K, then B is closed with respect to K.

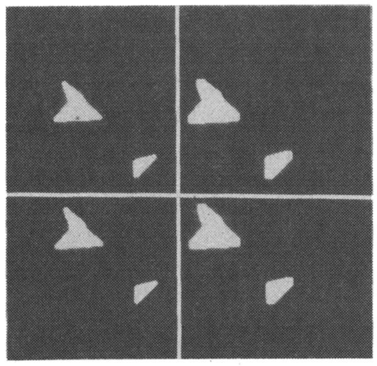


Fig. 8. The upper left shows the input image. In the upper right, the input image is dilated by a structuring element consisting of a 5 × 5 square. In the lower left, the dilated image is eroded by a 5 × 5 square structuring element. It is the closing of the input image. In the lower right, the closed image is dilated by a 5 × 5 square structuring element. It is the same as the initially dilated image shown in the upper right.

We approach the issue of idempotency of opening and closing by first discussing a class of sets which are unaltered by erosion followed by dilation with a given structuring element K. This class consists of all sets which can be expressed as some set dilated by K.

Proposition 37: $A \oplus K = (A \oplus K) \circ K = (A \bullet K) \oplus K$.

Proof: Let $G = A \oplus K$, $H = G \ominus K$, and $I = H \oplus K$. Now, by Proposition 33, $G = A \oplus K$ implies $A \subseteq G \ominus K = H$; $H = G \ominus K$ implies $G \supseteq H \oplus K = I$. But $A \subseteq H$ implies $A \oplus K \subseteq H \oplus K$. Since $G = A \oplus K$ and $I = H \oplus K$, $G \subseteq I$. Finally, $G \supseteq I$ and $G \subseteq I$ imply $G = I$. Hence,

$$\begin{aligned} A \oplus K &= H \oplus K = (G \ominus K) \oplus K \\ &= \left((A \oplus K) \ominus K\right) \oplus K = (A \bullet K) \oplus K. \end{aligned}$$

Proposition 37 is illustrated in Fig. 8. The idempotency of closing follows immediately as given by Theorem 38.

Theorem 38: $(A \bullet K) \bullet K = A \bullet K$.

Proof:

$$\begin{aligned} A \oplus K &= (A \bullet K) \oplus K \\ (A \oplus K) \ominus K &= \left((A \bullet K) \oplus K\right) \ominus K \\ A \bullet K &= \left((A \bullet K) \bullet K\right). \end{aligned}$$

Similarly, images eroded by K are unaltered by further dilation and erosion by K.

Proposition 39: $A \ominus K = (A \circ K) \ominus K = (A \ominus K) \bullet K$.

Proof: Let $G = A \ominus K$, $H = G \oplus K$, and $I = H \ominus K$. Now, $G = A \ominus K$ implies $A \supseteq G \oplus K = H$; $H = G \oplus K$ implies $G \subseteq H \ominus K = I$. But $A \supseteq H$ implies $A \ominus K \supseteq H \ominus K$ so that $G \supseteq I$. Finally $G \subseteq I$ and $G \supseteq I$ imply $G = I = H \ominus K = (G \oplus K) \ominus K$. Since $G = A \ominus K$, $A \ominus K = ((A \ominus K) \oplus K) \ominus K = (A \circ K) \ominus K$. The idempotency of opening follows immediately as given by Theorem 40.

Theorem 40: $A \circ K = (A \circ K) \circ K$.

Proof:

$$A \ominus K = (A \circ K) \ominus K$$
$$(A \ominus K) \oplus K = \big((A \circ K) \ominus K\big) \oplus K$$
$$A \circ K = (A \circ K) \circ K.$$

As with chained dilations and erosions, we can extend these results for chained openings and closings.

Openings and closings have other properties. For example, it follows immediately from the increasing property of dilation (Proposition 12) and the increasing property of erosion (Proposition 22) that both opening and closing are increasing.

It follows immediately from the translation invariance of dilation (Proposition 6) and the translation invariance of erosion (Proposition 21) that both opening and closing are translation invariant. Unlike dilation and erosion, opening and closing are invariant to translations of the structuring elements. That is, $A \circ (B)_x = A \circ B$ and $A \bullet (B)_x = A \bullet B$. This also follows directly from Propositions 6 and 21. As stated in Proposition 41, the opening transformation is antiextensive, i.e., the opening of A by structuring element B is necessarily contained in A, regardless of whether or not the origin belongs to B.

Antiextensivity of Opening Proposition 41: $A \circ B \subseteq A$.

Proof: Let $x \in A \circ B$. Then $x \in (A \ominus B) \oplus B$. Hence there exist $u \in A \ominus B$ and $v \in B$ such that $x = u + v$. Now $u \in A \ominus B$ implies $u + b \in A$ for every $b \in B$. In particular, $v \in B$. Thus $u + v \in A$. But $x = u + v$ so that $x \in A$.

Example: Illustrates how opening can produce a result which is strictly contained in the original.

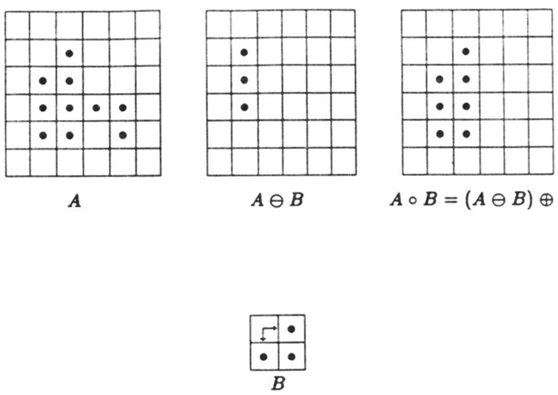


Fig. 9 illustrates an opening by a structuring element which does not include its origin.

The closing transformation is extensive, i.e., the closing of A by structuring element B contains A regardless of whether or not B contains its origin.

Extensivity of Closing Proposition 42: $A \subseteq A \bullet B$.

Proof: Let $a \in A$. Let $b \in B$. Then $a + b \in A \oplus B$. But $a + b \in A \oplus B$ for every $b \in B$ implies $a \in (A \oplus B) \ominus B$.

Example: This illustrates an instance of the relationship $A \subseteq A \bullet B$.

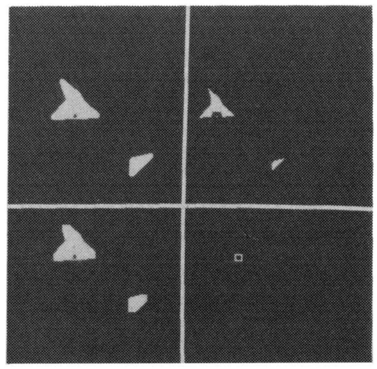


Fig. 9. The upper left shows the input image. In the upper right, the input image is eroded by the box boundary structuring element shown in the lower right. Notice that because the box is big enough to surround the hole and still be inside the blob with the hole, the eroded image has one white point whose position is in the hole. The lower left shows the image of the upper right dilated by the box boundary structuring element. This is the opening of the input image by the box boundary structuring element.

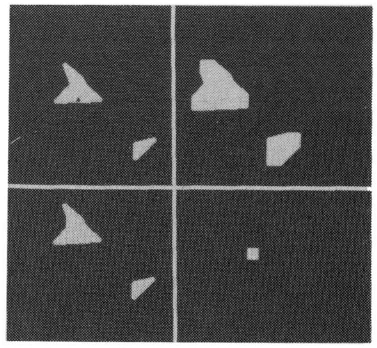


Fig. 10. The upper left shows the input image. In the upper right, the input image is dilated by a structuring element consisting of an 11 × 11 square shown in lower right. In the lower left, the dilated image is eroded by a 11 × 11 square structuring element. This results in the closing of the input image.

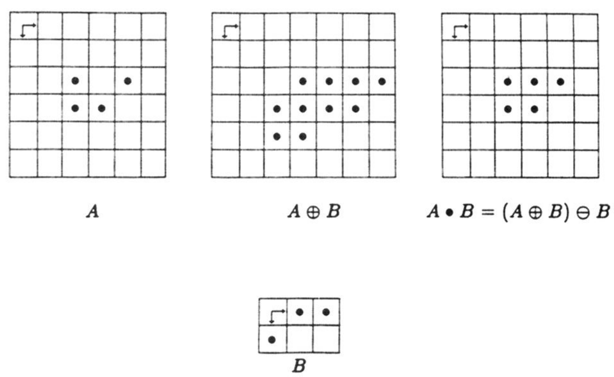


Fig. 10 illustrates a closing by a square structuring element.

Openings and closings, like erosion and dilations, are dual transformations. The complement of the closing of A by B is the opening of A^c by $\check{B}$. This is illustrated in Fig. 11.

Duality of Opening and Closing Theorem 43: $(A \bullet B)^c = A^c \circ \check{B}$.

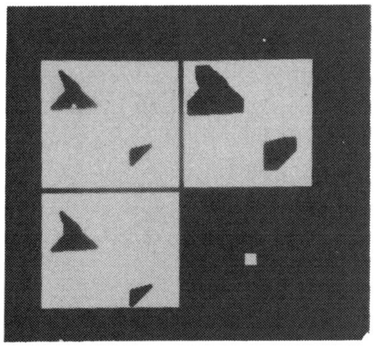


Fig. 11. The upper left shows the input image which is the complement of the input image of Fig. 10. In the upper right, the input image is eroded by a structuring consisting of an 11 × 11 square (shown in lower right). In the lower left, the eroded image is dilated by the 11 × 11 square structuring element. This results in the opening of the input image. Comparing Fig. 11 to Fig. 10, the opening of the complement is the complement of the closing for the symmetric square structuring element.

Proof:

$$\begin{aligned}(A \bullet B)^c &= \left[(A \oplus B) \ominus B\right]^c \\ &= (A \oplus B)^c \oplus \check{B} \\ &= (A^c \ominus \check{B}) \oplus \check{B} \\ &= A^c \circ \check{B}.\end{aligned}$$

Proposition 44 gives a geometric characterization to the opening operation. The opening of A by B is the union of all translations of B that are contained in A.

Proposition 44:

$$\begin{aligned}A \circ B &= \left\{x \in A \,\middle|\, \text{for some } y,\ x \in B_y \subseteq A\right\} \\ &= \bigcup_{\{y \mid B_y \subseteq A\}} B_y.\end{aligned}$$

Proof: Suppose $x \in A \circ B$, then $x \in (A \ominus B) \oplus B$. Hence there exists a $y \in A \ominus B$ and $b \in B$ such that $x = y + b$. Since $y \in A \ominus B$, $y + b \in A$ with $b \in B$ implies $x \in B_y$.

Suppose $x \in A$ and for some y, $x \in B_y \subseteq A$, then for every $z \in B$, $z + y \in A$ and there must exist some $b \in B$ such that $x = b + y$. But $y + z \in A$ for every $z \in B$ implies by definition of erosion that $y \in A \ominus B$. And $x = b + y$ implies by definition of dilation that $x \in (A \ominus B) \oplus B = A \circ B$.

By the duality of opening and closing, it is immediate that the closing of A by B is the complement of the union of all translations of $\check{B}$ that are contained in A^c. That is, $A \bullet B = (A^c \circ \check{B}) = [\bigcup_{\{y \mid \check{B}_y \subseteq A^c\}} \check{B}_y]^c$. Proposition 45 gives another geometric characterization to the closing operation.

Proposition 45:

$$\begin{aligned}A \bullet B &= \left\{x \in E^N \,\middle|\, x \in \check{B}_y \text{ implies } \check{B}_y \cap A \neq \emptyset\right\} \\ &= \bigcap_{\{y \mid \check{B}_y \cap A = \emptyset\}} \check{B}_y^c\end{aligned}$$

Proof: By Theorem 43, $A \bullet B = (A^c \circ \check{B})^c$.

By Proposition 44, $A^c \circ \check{B} = \{ x \in E^N \mid \text{for some } y,\ x \in \check{B}_y \subseteq A^c \}$. Hence,

$$\begin{aligned}A \bullet B = (A^c \circ \check{B})^c &= \left\{x \in E^N \,\middle|\, \text{for some } y,\ x \in \check{B}_y \text{ and } \check{B}_y \subseteq A^c\right\}^c \\ &= \left\{x \in E^N \,\middle|\, \text{for some } y,\ x \in \check{B}_y \text{ and } \check{B}_y \cap A = \emptyset\right\}^c \\ &= \left\{x \in E^N \,\middle|\, x \in \check{B}_y \text{ implies } \check{B}_y \cap A \neq \emptyset\right\}\end{aligned}$$

Propositions 44 and 45 immediately imply that $A \circ B_x = A \circ B_y$ and $A \bullet B_x = A \bullet B_y$ for any x and y in E^N. Hence the origin of the structuring element makes no difference in the results of an opening or closing.

Also,

$$\left[\bigcup_{\{y \mid \check{B}_y \subseteq A^c\}} \check{B}_y\right]^c = \left[\bigcup_{\{y \mid \check{B}_y \cap A = \emptyset\}} \check{B}_y\right]^c.$$

By DeMorgan's Law,

$$\left[\bigcup_{\{y \mid \check{B}_y \cap A^c = \emptyset\}} \check{B}_y\right]^c = \bigcup_{\{y \mid \check{B}_y \cap A^c = \emptyset\}} \check{B}_y^c.$$

IV. Gray Scale Morphology

The binary morphological operations of dilation, erosion, opening, and closing are all naturally extended to gray scale imagery by the use of a min or max operation. Nakagawa and Rosenfeld [22] first discussed the use of neighborhood min and max operators. The general extensions, due to Sternberg [33], [35], keep all the relationships discussed in Sections II and III. Peleg and Rosenfeld [24] use gray scale morphology to generalize the medial axis transform to gray scale imaging. Peleg, Naor, Hartley, and Avnir [23] use gray scale morphology to measure changes in texture properties as a function of resolution. Werman and Peleg [38] use gray scale morphology for texture feature extraction. Favre, Muggli, Stucki, and Bonderet [5] use gray scale morphology for the detection of platelet thrombosis detection in cross sections of blood vessels. Coleman and Sampson [2] use gray scale morphology on range data imagery to help mate a robot gripper to an object.

We will develop the extension in the following way. First we introduce the concept of the top surface of a set and the related concept of the umbra of a surface. Then gray scale dilation will be defined as the surface of the dilation of the umbras. From this definition we will proceed to the representation which indicates that gray scale dilation can be computed in terms of a maximum operation and a set of addition operations. A similar plan is followed for erosion which can be evaluated in terms of a minimum operation and a set of subtraction operations.

Of course, having a definition and a means of evaluating the defined operations does not imply that the properties of gray scale dilation and erosion are the same as binary dilation and erosion. To establish that the relationships are identical, we explore some of the relationships between the umbra and surface operation. Our explana-

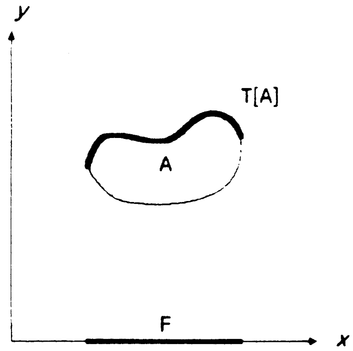


Fig. 12. The concept of top or top surface of a set.

tion shows that umbra and surface operations are essentially inverses of each other. Then we illustrate how the umbra operation is a homomorphism from the gray scale morphology to the binary morphology. Having the homomorphism in hand, all the interesting relationships follow by appropriately unwrapping and wrapping the involved sets or functions.

A. Gray Scale Dilation and Erosion

We begin with the concepts of surface of a set and the umbra of a surface. Suppose a set A in Euclidean N-space is given. We adopt the convention that the first $(N - 1)$ coordinates of the N-tuples of A constitute the spatial domain of A and the Nth coordinate is for the surface. For gray scale imagery, $N = 3$. The top or top surface of A is a function defined on the projection of A onto its first $(N - 1)$ coordinates. For each $(N - 1)$-tuple x, the top surface of A at x is the highest value y such that the N-tuple $(x, y) \in A$. This is illustrated in Fig. 12. If the space we work in is Euclidean, we can express this using the concept of supremum. If the space is discrete, we use the more familiar concept of maximum. Since we have suppressed the underlying space in what follows, we use maximum throughout. The careful reader will want to translate maximum to supremum under the appropriate circumstances.

Definition 46: Let $A \subseteq E^N$ and $F = \{x \in E^{N-1} | \text{for some } y \in E, (x, y) \in A\}$. The top or top surface of A, denoted by $T[A]: F \to E$, is defined by

$$T[A](x) = \max \{y | (x, y) \in A\}.$$

Definition 47: A set $A \subseteq E^{N-1} \times E$ is an umbra if and only if $(x, y) \in A$ implies that $(x, y) \in A$ for every $z \leq y$.

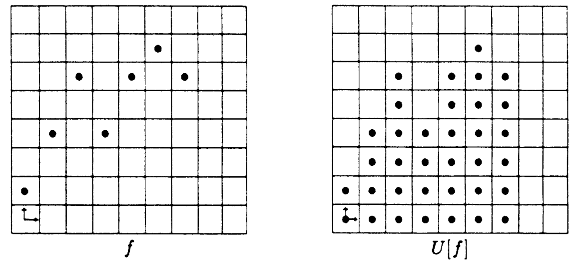
For any function f defined on some subset F of Euclidean $(N - 1)$-space the umbra of f is a set consisting of the surface f and everything below the surface.

Definition 48: Let $F \subseteq E^{N-1}$ and $f: F \to E$. The umbra of f, denoted by $U[f]$, $U[f] \subseteq F \times E$, is defined by

$$U[f] = \{(x, y) \in F \times E | y \leq f(x)\}.$$

Obviously, the umbra of f is an umbra.

Example: This illustrates a discretized one-dimensional function f defined as a domain consisting of seven successive column positions and a finite portion of its umbra which lies on or below the function f. The actual umbra has infinite extent below f. The reader should note that because the gray scale morphology so closely involves functions defined on the real line or plane, our example illustrations use the ordinary (x, y) coordinate frame instead of the row column coordinate frame employed in the examples of the binary morphology.

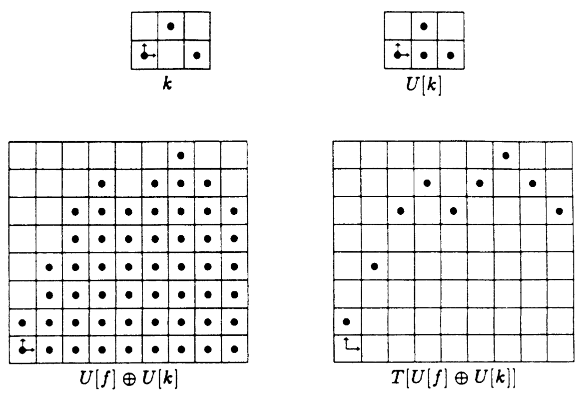


Having defined the operations of taking a top surface of a set and the umbra of a surface, we can define gray scale dilation. The gray scale dilation of two functions is defined as the surface of the dilation of their umbras.

Definition 49: Let $F, K \subseteq E^{N-1}$ and $f: F \to E$ and $k: K \to E$. The dilation of f by k is denoted by $f \oplus k$, $f \oplus k: F \oplus K \to E$, and is defined by

$$f \oplus k = T[U[f] \oplus U[k]].$$

Example: This illustrates a second discretized one-dimensional function k defined on a domain consisting of three successive column positions and a finite portion of its umbra which lies on or below the function k. The dilation of the umbras of f (from the previous example) and k are shown and the surface of the dilation of the umbras of f and k are shown.



The definition of gray scale dilation tells us conceptually how to compute the gray scale dilation, but this conceptual way is not a reasonable way to compute it in hardware. The following theorem establishes that gray scale dilation can be accomplished by taking the maximum of a set of sums. Hence, gray scale dilation has the same complexity as convolution. However, instead of doing the summation of products as in convolution, a maximum of sums is performed.

Proposition 50: Let $f: F \to E$ and $k: K \to E$. Then $f \oplus$

$k: F \oplus K \rightarrow E$ can be computed by

$$(f \oplus k)(x) = \max_{\substack{z \in K \\ x-z \in F}} \{f(x-z) + k(z)\}.$$

Proof: Suppose $z = (f \oplus k)(x)$. Then $z = T[U[f] \oplus U[k]](x)$. By definition of surface,

$$z = \max \{y \mid (x, y) \in [U[f] \oplus U[k]]\}.$$

By definition of dilation,

$$z = \max \{a + b \mid \text{for some } u \in K \text{ satisfying } x - u \in F, (x-u, a) \in U[f] \text{ and } (u, b) \in U[k]\}.$$

By definition of umbra, the largest a such that $(x - u, a) \in U[f]$ is $a = f(x - u)$. Likewise, the largest b such that $(u, b) \in U[k]$ is $b = k(u)$. Hence

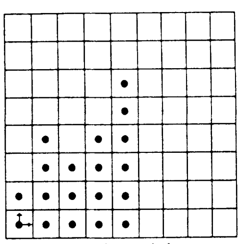
$$z = \max \{f(x-u) + k(u) \mid u \in K, (x-u) \in F\} = \max_{\substack{u \in K \\ (x-u) \in F}} \{f(x-u) + k(u)\}.$$

The definition for gray scale erosion proceeds in a similar way to the definition of gray scale dilation. The gray scale erosion of one function by another is the surface of the binary erosions of the umbra of one with the umbra of the other.

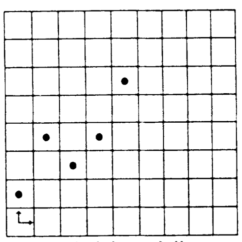
Definition 51: Let $F \subseteq E^{N-1}$ and $K \subseteq E^{N-1}$. Let $f: F \rightarrow E$ and $k: K \rightarrow E$. The erosion of f by k is denoted by $f \ominus k$, $f \ominus k: F \ominus K \rightarrow E$, and is defined by

$$f \ominus k = T[U[f] \ominus U[k]].$$

Example: Using the same function f and k of the previous example, illustrated here is the erosion of f by k by taking the surface of the erosion of the umbra of f by the umbra of k.



$U[f] \ominus U[k]$



$T[U[f] \ominus U[k]]$

Evaluating a gray scale erosion is accomplished by taking the minimum of a set of differences. Hence its complexity is the same as dilation. Its form is like correlation with the summation of correlation replaced by the minimum operation and the product of correlation replaced by a subtraction operation. If the underlying space is Euclidean, substitute infimum for minimum.

Proposition 52: Let $f: F \rightarrow E$ and $k: K \rightarrow E$. Then $f \ominus k: F \ominus K \rightarrow E$ can be computed by $(f \ominus k)(x) = \min_{z \in K} \{f(x+z) - k(z)\}$.

Proof: Suppose $z = (f \ominus k)(x)$. Then, $z = T[U[f] \ominus U[k]](x)$. By definition of surface, $z = \max \{y \mid (x, y) \in U[f] \ominus U[k]\}$. By definition of erosion

$$z = \max \{y \mid \text{for every } (u, v) \in U[k], (x, y) + (u, v) \in U[f]\}.$$

By definition of umbra,

$$z = \max \{y \mid \text{for every } u \in K, v \leq k(u), y + v \leq f(x+u)\}$$
$$= \max \{y \mid \text{for every } u \in K, v \leq k(u), y \leq f(x+u) - v\}.$$

But $y \leq f(x + u) - v$ for every $v \leq k(u)$ implies $y \leq f(x + u) - k(u)$. Hence,

$$z = \max \{y \mid \text{for every } u \in K, y \leq f(x+u) - k(u)\}.$$

But $y \leq f(x + u) - k(u)$ for every $u \in K$ implies

$$y \leq \min_{u \in K} [f(x+u) - k(u)].$$

Now,

$$z = \max \left\{ y \mid y \leq \min_{u \in K} [f(x+u) - k(u)] \right\} = \min_{u \in K} [f(x+u) - k(u)].$$

Fig. 13 illustrates an example of gray scale dilation and erosion.

The basic relationship between the surface and umbra operations is that they are, in a certain sense, inverses of each other. More precisely, the surface operation will always undo the umbra operation. That is, the surface operation is an inverse to the umbra operation as given in the next proposition.

Proposition 53: Let $F \subseteq E^{N-1}$ and $f: F \rightarrow E$. Then $T[U[f]] = f$.

Proof: Let $y \in T[U[f]](x)$. Then $y = \max \{z \mid (x, z) \in U[f]\}$. Now $(x, z) \in U[f]$ implies $z \leq f(x)$. Also, $(x, f(x)) \in U[f]$. Thus y cannot get larger than $f(x)$ and since $(x, f(x)) \in U[f]$, y can get as large as $f(x)$. Thus $y = f(x)$.

Corollary 54: $U[T[U[f]]] = U[f]$.

However the umbra operation is not an inverse to the surface operation. Without any constraints on the set A, the strongest statement which can be made is that the umbra of the surface of A contains A. This is illustrated in Fig. 14.

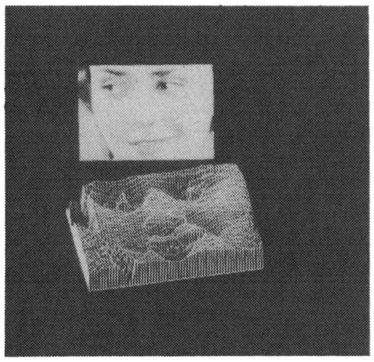
Proposition 55: Let $A \subseteq E^N$. Then $A \subseteq U[T[A]]$.

Proof: Let $x \in E^{N-1}$ and $y \in E$. Suppose, $(x, y) \in A$. Let $z = T[A](x) = \max \{v \mid (x, v) \in A\}$. Hence, $z \geq y$. But by definition of the umbra operation $z = T[A](x)$ implies $(x, w) \in U[T[A]]$ for all $w \leq z$. In particular, $y \leq z$. Hence, $(x, y) \in U[T[A]]$.

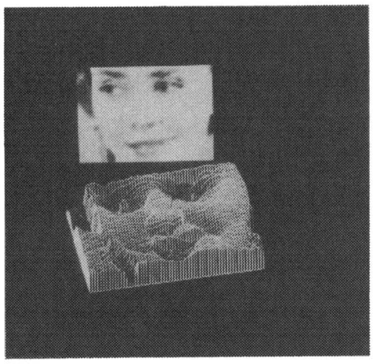
When the set A is an umbra, then the umbra of the surface of A is itself A. In this case the umbra operation is an inverse to the surface operation.

Proposition 56: If A is an umbra, then $A = U[T[A]]$.

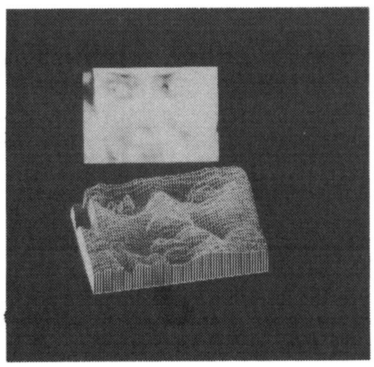
Proof: By Proposition 54, $A \subseteq U[T[A]]$. So we just need to show that $A \supseteq U[T[A]]$. Suppose $(x, y) \in$



(a)



(b)



(c)

Fig. 13. A woman's face in the image form and in a perspective projection surface plot form. This image is morphologically processed with a paraboloid structuring element given by $6(8 - r^2 - c^2)$, $-2 \le r \le 2$, $-2 \le c \le 2$. (b) The erosion of the girl's face in image form and perspective projection surface plot form. (c) The dilation of the girl's face in image form and perspective projection surface plot form.

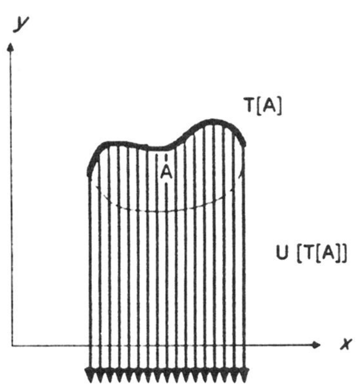


Fig. 14. The umbra of the top surface of a set.

$U[T[A]]$ and A is an umbra. By definition of the umbra operation $(x, y) \in U[T[A]]$ implies $y \le T[A](x)$ and there exists some z such that $(x, z) \in A$. Now if there exists some z such that $(x, z) \in A$, $(x, T[A](x)) \in A$. Since A is an umbra, $(x, T[A](x)) \in A$ implies that $(x, w) \in A$ for every $w \le T[A](x)$. In particular $y \le T[A](x)$. Hence $(x, y) \in A$.

Having established that the surface operation is always an inverse to the umbra operation and that the umbra operation is the inverse to the surface operation when the set being operated on itself is an umbra, we are almost ready to develop the umbra homomorphism theorem. First we need to establish that the dilation of one umbra by another is an umbra and that the erosion of one umbra by another is also an umbra.

Proposition 57: Suppose A and B are umbras. Then $A \oplus B$ and $A \ominus B$ are umbras.

Proof: Suppose $(x, y) \in A \oplus B$. Let $w \le y$. We need to demonstrate that $(x, w) \in A \oplus B$. By definition of dilation, $(x, y) \in A \oplus B$ implies that there exists $(u, v) \in B$ such that $(x - u, y - v) \in A$. Now $w \le y$ implies $w - v \le y - v$. And A is an umbra so that $(x - u, y - v) \in A$ and $w - v \le y - v$ implies $(x - u, w - v) \in A$. But by definition of dilation, $(x - u, w - v) \in A$ and $(u, v) \in B$ implies $(x, w) \in A \oplus B$ which means that $A \oplus B$ is an umbra.

Suppose $(x, y) \in A \ominus B$. Let $w \le y$. We need to demonstrate that $(x, w) \in A \ominus B$. By definition of erosion, $(x, y) \in A \ominus B$ implies that for every $(u, v) \in B$, $(x, y) + (u, v) = (x + u, y + v) \in A$. Now $w \le y$ implies $w + v \le y + v$. Since A is an umbra and $(x + u, y + v) \in A$ and $w + v \le y + v$, then $(x + u, w + v) \in A$. But by definition of erosion if $(x, w) + (u, v) \in A$ for every $(u, v) \in B$ then $(x, w) \in A \ominus B$. Hence $A \ominus B$ is an umbra.

Now we are ready for the umbra homomorphism theorem which states that the operation of taking an umbra is a homomorphism from the gray scale morphology to the binary morphology.

Umbra Homomorphism Theorem 58: Let $F, K \subseteq E^{N-1}$ and $f: F \to E$ and $k: K \to E$. Then

$$1)\ U[f \oplus k] = U[f] \oplus U[k]$$

and

$$2)\ U[f \ominus k] = U[f] \ominus U[k]$$

Proof: 1) $f \oplus k = T[U[f] \oplus U[k]]$ so that $U[f \oplus k] = U[T[U[f] \oplus U[k]]]$. But $U[f] \oplus U[k]$ is an umbra and for sets which are umbras the umbra operation undoes the surface operation. Hence $U[f \oplus k] = U[T[U[f] \oplus U[k]]] = U[f] \oplus U[k]$.

2) $f \ominus k = T[U[f] \ominus U[k]]$ so that $U[f \ominus k] = U[T[U[f] \ominus U[k]]]$. But $U[f] \ominus U[k]$ is an umbra and for sets which are umbras, the umbra operation undoes the surface operation. Hence,

$$U[f \ominus k] = U\big[T[U[f] \ominus U[k]]\big] = U[f] \ominus U[k].$$

To illustrate how the umbra homomorphism property is used to prove relationships by first wrapping the relationship by re-expressing it in terms of umbra and surface operations and then transforming it through the umbra homomorphism property and finally by unwrapping it using the definitions of gray scale dilation and erosion, we state and prove the commutivity and associativity of gray scale dilation and the chain rule for gray scale erosion.

Proposition 59: $f \oplus k = k \oplus f$.

Proof:

$$\begin{aligned} f \oplus k &= T[U[f] \oplus U[k]] \\ &= T[U[k] \oplus U[f]] \\ &= k \oplus f. \end{aligned}$$

Proposition 60: $k_1 \oplus (k_2 \oplus k_3) = (k_1 \oplus k_2) \oplus k_3$.

Proof:

$$\begin{aligned} k_1 \oplus (k_2 \oplus k_3) &= T[U[k_1] \oplus U[k_2 \oplus k_3]] \\ &= T[U[k_1] \oplus (U[k_2] \oplus U[k_3])] \\ &= T[(U[k_1] \oplus U[k_2]) \oplus U[k_3]] \\ &= T[U[k_1 \oplus k_2] \oplus U[k_3]] \\ &= (k_1 \oplus k_2) \oplus k_3. \end{aligned}$$

Proposition 61: $(f \ominus k_1) \ominus k_2 = f \ominus (k_1 \oplus k_2)$.

Proof:

$$\begin{aligned} (f \ominus k_1) \ominus k_2 &= T[U[f \ominus k_1] \ominus U[k_2]] \\ &= T[(U[f] \ominus U[k_1]) \ominus U[k_2]] \\ &= T[U[f] \ominus (U[k_1] \oplus U[k_2])] \\ &= T[U[f] \ominus U[k_1 \oplus k_2]] \\ &= f \ominus (k_1 \oplus k_2). \end{aligned}$$

Gray scale opening and closing are defined in an analogous way to opening and closing in the binary mophology and they have similar properties.

Definition 62: Let $f: F \to E$ and $k: K \to E$. The gray scale opening of f by structuring element k is denoted by $f \circ k$ and is defined by $f \circ k = (f \ominus k) \oplus k$.

Definition 63: Let $f: F \to E$ and $k: K \to E$. The gray scale closing of f by structuring element k is denoted by $f \bullet k$ and is defined by $f \bullet k = (f \oplus k) \ominus k$.

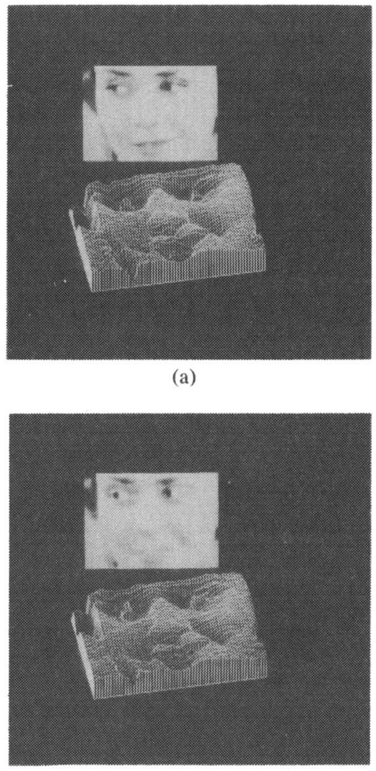
Fig. 15 shows an example of gray scale opening and closing.

To prove the idempotency of gray scale opening and closing, we need the following property relating functions to their umbras.

Proposition 64: Let $f: F \to E$ and $g: G \to E$. Suppose $F \subseteq G$. Then $f \le g$ if and only if $U[f] \subseteq U[g]$.

Proof: Suppose $f \le g$. Let $(x, y) \in U[f]$. Then by definition of umbra, $y \le f(x)$. But $x \in F$ and $F \subseteq G$ so that $x \in G$. By supposition, $f(x) \le g(x)$. Hence, $y \le g(x)$. Now by definition of umbra, $(x, y) \in U[g]$.

Suppose $U[f] \subseteq U[g]$. Let $y = f(x)$. Certainly, $(x, y) \in U[f]$. But $U[f] \subseteq U[g]$ so that $(x, y) \in U[g]$. Now by definition of umbra, $y \le g(x)$.



(a)

(b)

Fig. 15. The gray scale opening and closing operation. (a) The gray scale opening of the girl's face in image form and in perspective projection surface plot form. The structuring element is the paraboloid described in Fig. 13. (b) The gray scale closing of the girl's face in image form and in perspective projection surface plot form.

Having this property, the analog to Proposition 33 follows.

Proposition 65: $g \le f \ominus k$ if and only if $f \ge g \oplus k$.

Proof: $g \le f \ominus k$ if and only if $U[g] \subseteq U[f \ominus k]$. But $U[f \ominus k] = U[f] \ominus U[k]$. Now $U[g] \subseteq U[f] \ominus U[k]$ if and only if $U[f] \supseteq U[g] \oplus U[k]$. But $U[g] \oplus U[k] = U[g \oplus k]$. Finally, $U[f] \supseteq U[g \oplus k]$ if and only if $f \ge g \oplus k$.

Another property which is immediately obvious is that if one set is contained in a second, then the surface of the first will be no higher at each point than the surface of the second.

Proposition 66: Let $A \subseteq E^{N-1} \times E$ and $D \subseteq E^{N-1} \times E$. Then $A \subseteq D$ implies $T[A](x) \le T[D](x)$.

Proof: Let $x \in E^{N-1}$ be given. Then, since $A \subseteq D$,

$$T[A](x) = \max_{(x,z) \in A} z \le \max_{(x,z) \in D} z = T[D](x).$$

From this fact, it quickly follows that the gray scale opening of a function must be no larger than the function at each point in their common domain. This is the gray scale analog to the antiextensivity property of the binary morphology opening.

Proposition 67: $(f \circ k)(x) \le f(x)$ for every $x \in F \circ K$.

Proof:

$$f \circ k = (f \ominus k) \oplus k = T[U(f \ominus k) \oplus U[k]] = T[(U[f] \ominus U[k]) \oplus U[k]].$$

But $(U[f] \ominus U[k]) \oplus U[k] \subseteq U[f]$, hence by Proposition 63,

$$T[(U[f] \ominus U[k]) \oplus U[k]](x) \leq T[U[f]](x)$$

for every $x \in (F \ominus K) \oplus K$. Since $T[U[f]] = f$, $T[(U[f] \ominus U[k]) \oplus U[k]](x) \leq f(x)$.

Likewise, the gray scale closing of a function must be no smaller than the function at each point in their common domain. This is the gray scale analog to the extensivity property of the binary morphology closing.

Proposition 68: $f(x) \leq (f \bullet k)(x)$ for every $x \in F$.

Proof:

$$f \bullet k = (f \oplus k) \ominus k = T[U[f \oplus k] \ominus U[k]] = T[(U[f] \oplus U[k]) \ominus U[k]].$$

But $(U[f] \oplus U[k]) \ominus U[k] \supseteq U[f]$; hence $T[(U[f] \oplus U[k]) \ominus U[k]](x) \geq T[U[f]](x)$ for every $x \in F$. Since $T[U[f]] = f$, $f(x) \leq T[(U[f] \oplus U[k]) \ominus U[k]](x)$.

Now the idempotency property of opening and closing can be proved by the umbra homomorphism theorem.

Proposition 69: $(f \circ k) \circ k = f \circ k$.

Proof:

$$\begin{aligned}(f \circ k) \circ k &= T[(U[f \circ k] \ominus U[k]) \oplus U[k]] \\ &= T[(((U[f] \ominus U[k]) \oplus U[k]) \ominus U[k]) \oplus U[k]] \\ &= T[(U[f] \circ U[k]) \circ U[k]] \\ &= T[U[f] \circ U[k]] \\ &= T[(U[f] \ominus U[k]) \oplus U[k]] \\ &= T[U[f \ominus k] \oplus U[k]] \\ &= T[U[(f \ominus k) \oplus k]] \\ &= T[U[f \circ k]] \\ &= f \circ k.\end{aligned}$$

Proposition 70: $(f \bullet k) \bullet k = f \bullet k$.

Proof:

$$\begin{aligned}(f \bullet k) \bullet k &= T[(U[f \bullet k] \oplus U[k]) \ominus U[k]] \\ &= T[(((U[f] \oplus U[k]) \ominus U[k]) \oplus U[k]) \ominus U[k]] \\ &= T[(U[f] \bullet U[k]) \bullet U[k]] \\ &= T[U[f] \bullet U[k]] \\ &= T[(U[f] \oplus U[k]) \ominus U[k]] \\ &= T[U[(f \oplus k) \ominus k]] \\ &= T[U[f \bullet k]] \\ &= f \bullet k.\end{aligned}$$

There is a geometric interpretation to the gray scale opening and to the gray scale closing in the same manner that there is a geometric meaning to the binary morphological opening and closing (Propositions 44 and 45). To obtain the opening of f by a paraboloid structuring element, for example, take the paraboloid, apex up, and slide it under all the surface of f pushing it hard up against the surface. The apex of the paraboloid may not be able to touch all points of f. For example, if f has a spike narrower than the paraboloid, the top of the apex may only reach as far as the mouth of the spike. The opening is the surface of the highest points reached by any part of the paraboloid as it slides under all the surface of f. The formal statement of this is given in Proposition 71.

Proposition 71:

$$f \circ k = T\left[\bigcup_{\{y \mid U[k]_y \subseteq U[f]\}} U[k]_y\right].$$

Proof:

$$\begin{aligned}f \circ k &= T[U[f \ominus k] \oplus U[k]] \\ &= T[(U[f] \ominus U[k]) \oplus U[k]] \\ &= T[U[f] \circ U[k]] \\ &= T\left[\bigcup_{\{z \mid (U[k])_z \subseteq U[f]\}} (U[k])_z\right].\end{aligned}$$

We have not mentioned the duality relationship between gray scale dilation and erosion. We need this in order to give the geometric interpretation to closing. The duality relationship is analogous to the relationship given in Theorem 25. Before stating and proving it, we need the definition of gray scale reflection.

Definition 72: Let $f: F \rightarrow E$. The reflection of f is denoted by $\check{f}$, $\check{f}: \check{F} \rightarrow E$, and is defined by $\check{f}(x) = f(-x)$.

Gray Scale Dilation Erosion Duality Theorem 73: Let $f: F \rightarrow E$ and $k: K \rightarrow E$. Let $x \in (F \oplus K) \cap (F \ominus \check{K})$ be given. Then $-(f \oplus k)(x) = ((-f) \ominus \check{k})(x)$.

Proof:

$$\begin{aligned}-(f \oplus k)(x) &= -\max_{\substack{z \in K \\ x - z \in F}} [f(x - z) + k(z)] \\ &= \min_{\substack{z \in K \\ x - z \in F}} [-f(x - z) - k(z)] \\ &= \min_{\substack{z \in \check{K} \\ x + z \in F}} [-f(x + z) - \check{k}(z)] \\ &= ((-f) \ominus \check{k})(x).\end{aligned}$$

It follows immediately from the gray scale dilation and erosion duality that there is a gray scale opening and closing duality.

Gray Scale Opening and Closing Duality Theorem 74: $-(f \circ k) = (-f) \bullet \check{k}$.

Proof:

$$\begin{aligned} -(f \circ k) &= -((f \ominus k) \oplus k) \\ &= (-(f \ominus k)) \ominus \check{k} \\ &= ((-f) \oplus \check{k}) \ominus \check{k} \\ &= (-f) \bullet \check{k}. \end{aligned}$$

Having the gray scale opening and closing duality, we immediately have $f \bullet k = -((-f) \circ \check{k})$. In essence, this means that we can think of closing like opening. To close f with a paraboloid structuring element, we take the reflection of the paraboloid in the sense of Definition 72, turn it upside down (apex down), and slide it all over the top of the surface of f. The closing is the surface of all the lowest points reached by the sliding paraboloid.

V. Summary

We have developed the basic relationships in binary morphology and have then developed the extensions of these relationships in gray scale morphology. We have shown that morphological openings are increasing, antiextensive, translation invariant, and idempotent. We have shown that morphological closings are increasing, extensive, translation invariant, and idempotent. For further algebraic depth on opening and closings, see [18] or [31].

We intend to publish two follow-on tutorials to the present one. The first will discuss a variety of topics including sieves, sampling, morphologic topography, thickenings, thinnings, boundaries, skeletons, connectivity, convexity, morphologic derivative estimation, and bounding derivatives by gray scale morphologic openings and closings. The second will be on application where we will discuss morphologic solutions to a variety of industrial vision problems.

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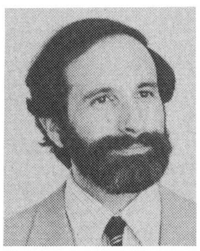

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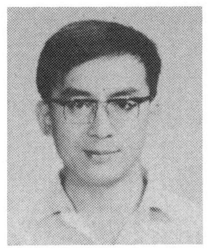

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