Correspondence

A Metric for Comparing Relational Descriptions

LINDA G. SHAPIRO AND ROBERT M. HARALICK

Abstract—**Relational models are frequently used in high-level computer vision. Finding a correspondence between a relational model and an image description is an important operation in the analysis of scenes. In this paper the process of finding the correspondence is formalized by defining a general relational distance measure that computes a numeric distance between any two relational descriptions—a model and an image description, two models, or two image descriptions. The distance measure is proved to be a metric, and is illustrated with examples of distance between object models. A variant measure used in our past studies is shown not to be a metric.**

Index Terms—**Matching, metric, relational distance, structural description.**

I. Introduction

Relational models of objects are frequently used in computer vision systems that attempt to analyze images of real-world scenes containing these objects. Part of the analysis process deals with 1) constructing a relational description of a portion of the image that is considered a possible two-dimensional view of some object in the knowledge base of the vision system and 2) comparing this relational description to a relational model. The result of the comparison should indicate the likelihood that this portion of the image is a projection of the object represented by the relational model. This type of comparison, also called relational matching, has been utilized in a number of vision systems (see Section II). In this paper, we give a formal description of the general relational distance measure we have been using to compare two relational descriptions (or models), prove it is a metric, and illustrate it with examples of distances between object models. We also show that a variant measure we have used in past studies is not a metric.

Having a measure which is a true distance between two relations is important because it then becomes easy to put the matching problem into a meaningful and applicable Bayesian decision framework. Class conditional probabilities can be defined which are monotonic decreasing functions of the distance between the test relation and a representative relation from the class. This is precisely what happens in the multivariate normal density function where the Mahalanobis distance between a test point and the class mean determines the probability density. In this manner, it becomes possible to have structural relational descriptions of patterns and be able to compute the probability that a pattern arises from a given class typified by a representative prototype pattern. Thereby, structural pattern recognition and decision theory can be married together.

Manuscript received October 11, 1983; revised May 25, 1984. Recommended for acceptance by Y. T. Chieu. This work was supported by the National Science Foundation under Grant MCS-8102874.

The authors are with Machine Vision International, Ann Arbor, MI 48104.

II. Related Literature

One of the early discussions of relational matching was a paper by Barrow, Ambler, and Burstall [1] which compared several strategies for finding a mapping from a relational description of a model to a relational description of an image. The advent of discrete relaxation algorithms (for example, Waltz [24]; Rosenfeld, Hummel, and Zucker [14]; and Haralick and Shapiro [8]) to speed up the exponential process of finding such a mapping helped to popularize relational models and they have been used in a number of systems.

Nevatia and Binford [13] used three-dimensional object models consisting of generalized cylinders with normal cross sections for primitives, plus connectivity relations and global properties. Marr and Nishihara [10] had hierarchical relational models also with generalized cylinder primitives. Schneier [16] represented objects by primitives and relations, but with the special feature that common primitives and relations are shared across models and within models. The matching done in the ACRONYM system (Brooks [2]) was also partly of a relational nature. This system, however, encoded relational and other constraints as symbolic expressions and used symbolic reasoning in the recognition of three-dimensional objects from single perspective views. Relational matching has also been used in two-dimensional shape matching (Shapiro [18], Davis [3]).

The subject of efficiently finding relational matches has been addressed in a number of papers (Ullman [23], Montanari [11], Mackworth [9], Freuder [5], Gaschnig [6], those mentioned above, and others). Haralick and Elliot [7] compared several discrete relaxation operators and found that a very simple operator called forward checking worked best. Shapiro and Haralick [19] extended this work to inexact matching. In [21] we defined a distance measure for graphs and used it to organize a database of models into clusters. An unknown graph could be compared to representatives of each cluster instead of to each graph in the database. The distance measure used in [21] was a special case of the general relational distance defined in this paper. Sanfelieu and Fu [15] developed a different relational distance for graphs and used it to classify muscle tissue patterns. The measure, which they use to compare an input graph to a reference graph, is based on the cost of node recognition plus the cost of the operations necessary to transform the input graph to the reference graph. The measure is defined by

$$\min_{\text{all configurations}} \{w_{nr}c_{nr} + w_{ni}c_{ni} + w_{nd}c_{nd} + w_{bi}c_{bi} + w_{bd}c_{bd}\}$$

where w_{nr}, w_{ni}, w_{nd}, w_{bi}, and w_{bd} are weights summing to 1 and c_{nr}, c_{ni}, c_{nd}, c_{bi}, and c_{bd} are the costs of node recognition, node insertion, node deletion, branch insertion, and branch deletion, respectively. In comparison, the general distance measure is a bidirectional measure which takes into account the last four of these costs, but makes no allowance for node recognition. Sanfelieu's distance measure is not a metric. Our general relational distance is the first measure of the distance between two relational descriptions that is a metric.

III. A General Relational Distance

A *relational description* D_X is a sequence of relations $D_X = \{R_1, \cdots, R_I\}$ where for each $i = 1, \cdots, I$, there exists a posi-

0162-8828/85/0100-0090$01.00

tive integer n_i with $R_i \subseteq X^{n_i}$ for some set X. Intuitively, X is a set of the parts of the entity being described and the relations R_i indicate various relationships among the parts. A relational description may be used to describe an object model, a group of regions on an image, a two-dimensional shape, a Chinese character, or anything else having structure to it. We wish to define a distance measure for pairs of relational descriptions.

Let $D_A = \{R_1, \cdots, R_I\}$ be a relational description with part set A. Let $D_B = \{S_1, \cdots, S_I\}$ be a second relational description with part set B. We will assume that $|A| = |B|$; if this is not the case, we will add enough dummy parts to the smaller set to make it the case. We will see later, that the relational distance measure is only a metric if the mapping from A to B that defines the distance is one-one and onto. Thus, the mapping will be a permutation or relabeling function.

Let f be any one-one, onto mapping from A to B. For any $R \subseteq A^N$ and $S \subseteq B^N$, N a positive integer, we define the *composition* $R \circ f$ by $R \circ f = \{(b_1, \cdots, b_N) \in B^N |$ there exists $(a_1, \cdots, a_N) \in R$ with $f(a_n) = b_n$, $n = 1, \cdots, N\}$. Thus, the composition operator takes N-tuples of R and maps them, component by component, into N-tuples of B^N.

The *structural error* of f for the ith pair of corresponding relations (R_i and S_i) in D_A and D_B is given by

$$E_s^i(f) = |R_i \circ f - S_i| + |S_i \circ f^{-1} - R_i|.$$

The structural error indicates how many tuples in R_i are not mapped by f to tuples in S_i and how many tuples in S_i are not mapped by f^{-1} to tuples in R_i.

The *total error* of f with respect to D_A and D_B is the sum of the structural errors for each pair of corresponding relations. That is,

$$E(f) = \sum_{i=1}^{I} E_s^i(f).$$

The total error gives a quantitative idea of the difference between the two relational descriptions D_A and D_B with respect to the mapping f.

The *relational distance* between D_A and D_B is then given by

$$GD(D_A, D_B) = \min_{\substack{1\text{-}1 \\ f: A \to B \\ \text{onto}}} E(f).$$

That is, the relational distance is the minimal total error obtained for any one-one, onto mapping f from A to B. We call a mapping f that minimizes total error a *best mapping* from D_A to D_B. If there is more than one best mapping, we arbitrarily select one as the designated best mapping. More than one best mapping will occur when the relational descriptions involve certain kinds of symmetries.

We have chosen, in this paper, to represent relations as sets of N-tuples and to use the composition operator ($\circ$) to represent the application of a function to each component of one or more N-tuples. We made this choice because this notation is consistent with all of our past work on matching and because we feel this is the most general and most suitable notation for the problem. This notation is also extendable to allow the addition of labels or attributes to N-tuples of parts or units.

The reader should note that it is also possible to think of an N-ary, relation $R \subseteq A^N$ as a bit vector. The bit vector has a position for each possible N-tuple $(a_1, \cdots, a_N) \in A^N$. Those positions representing N-tuples of R have value 1 and the rest have value 0. The composition operation $R \circ f$ is achieved by a permutation of the bit vector of R, resulting in the new bit vector $\boldsymbol{R} \circ \boldsymbol{f}$. The structural error of a permutation $f: A \to B$ with respect to relations $R \subseteq A^N$ and $S \subseteq B^N$ is merely the number of one bits in the bit vector $\boldsymbol{R} \circ \boldsymbol{f}$ XOR $\boldsymbol{S}$, where XOR stands for the Exclusive-OR operation. The total error and relational distance definitions remain the same.

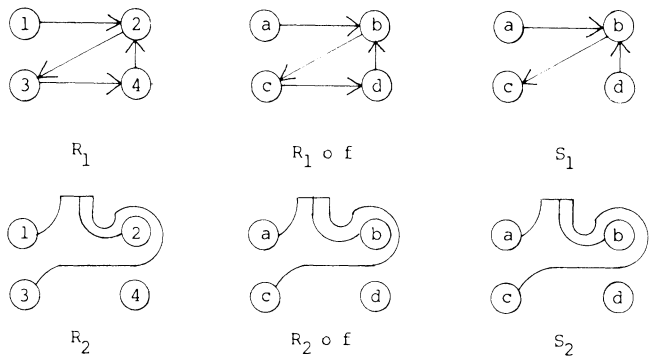

Fig. 1. Illustrates the relations $R_1, R_1 \circ f, S_1, R_2, R_2 \circ f$, and S_2. The notation ⌜⌝ indicates a hyperarc representing a triple.

The second notational approach was used in our discussion of graph matching in [21] since it is the standard adjacency matrix representation of graphs. It is not as naturally extendable to the addition of labels or attributes to the relations as is the notation we use here.

Examples

Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$. Let $D_A = \{R_1 \subseteq A^2, R_2 \subseteq A^3\}$, and $D_B = \{S_1 \subseteq B^2, S_2 \subseteq B^3\}$. Let $R_1 = \{(1, 2), (2, 3), (3, 4), (4, 2)\}$ and $S_1 = \{(a, b), (b, c), (d, b)\}$. Let $R_2 = \{(1, 2, 3)\}$ and $S_2\{(a, b, c)\}$. Let f be defined by $f(1) = a, f(2) = b, f(3) = c, f(4) = d$. These relations are illustrated in Fig. 1. Then we have

$$|R_1 \circ f - S_1| = |\{(a, b), (b, c), (c, d), (d, b)\} - \{(a, b), (b, c), (d, b)\}| = 1$$

$$|S_1 \circ f^{-1} - R_1| = |\{(1, 2), (2, 3), (4, 2)\} - \{(1, 2), (2, 3), (3, 4), (4, 2)\}| = 0,$$

$$E_s^1(f) = 1 + 0 = 1,$$

$$|R_2 \circ f - S_2| = |\{(a, b, c)\} - \{(a, b, c)\}| = 0,$$

$$|S_2 \circ f^{-1} - R_2| = |\{(1, 2, 3)\} - \{(1, 2, 3)\}| = 0,$$

$$E_s^2(f) = 0 + 0 = 0,$$

$$E(f) = E_s^1(f) + E_s^2(f) = 1.$$

We note that f is the best mapping and therefore $GD(D_A, D_B) = 1$.

For a simple but practical example, consider a set of object models constructed from simple parts with two binary relations: the connection relation and the parallel relation. Fig. 2 illustrates a model ($M1$) and two other models ($M2$ and $M3$) that are each a relational distance of 1 from the first model. The model $M4$ shown in Fig. 3 is a variation of $M3$, but its relational distance from $M3$ is 6, due to several missing relationships induced by the additional two parts. Finally, the two models ($M5$ and $M6$) of Fig. 4 have more dissimilarity and a relational distance of 12.

IV. The Relational Distance is a Metric

In this section we prove that GD is a metric over the space of relational descriptions. Let f be a one-one and onto function from A to B. We say that f is a *relational isomorphism* if $E(f) = 0$. In this case D_A and D_B are said to be isomorphic. Intuitively, this means that the parts of D_A perfectly match the parts of D_B with respect to all required attributes and relationships.

Lemma: Let GD be the relational distance measure, and let D_A, D_B, and D_C be arbitrary relational descriptions.

1) $GD(D_A, D_B) = 0$ if and only if D_A and D_B are isomorphic.

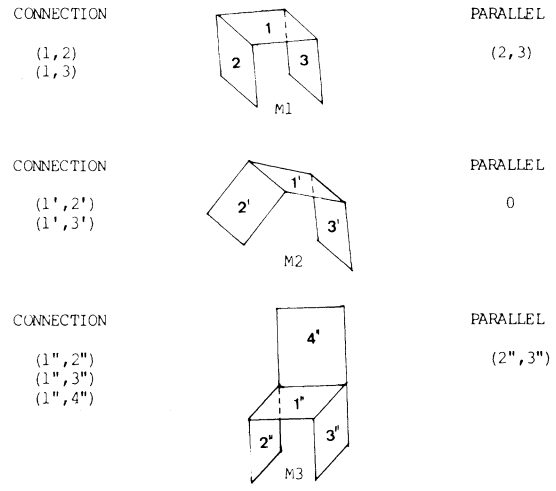

Fig. 2. Illustrates an object model $M1$ and two other models, $M2$ and $M3$, that are each a relational distance of 1 from $M1$.

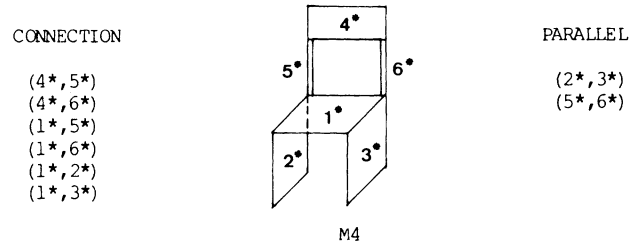

Fig. 3. Illustrates a model, $M4$, that differs from $M3$ by a relational distance of 6.

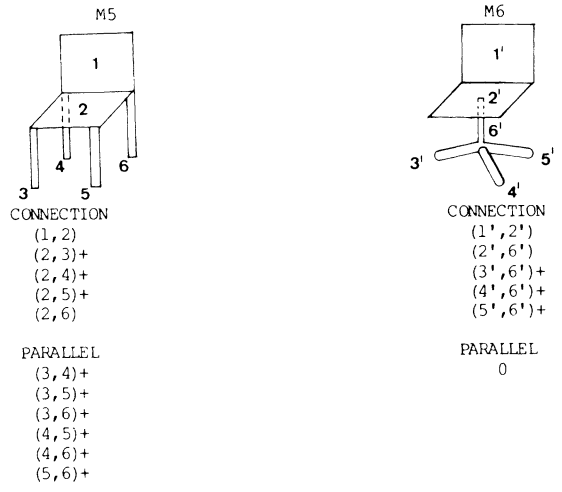

Fig. 4. Illustrates two models, $M5$ and $M6$, which differ from each other by a relational distance of 12. The tuples marked with a plus sign (+) are those that caused errors.

2) $GD(D_A, D_B) = GD(D_B, D_A)$.

3) $GD(D_A, D_B) \leqslant GD(D_A, D_C) + GD(D_C, D_B)$.

Proof:

1) If f is an isomorphism between D_A and D_B then $E(f) = 0$. Thus, $GD(D_A, D_B) = \min_f E(f) = 0$. If $GD(D_A, D_B) = 0$, then there exists one-one onto f with $E(f) = 0$. Thus, f is an isomorphism between D_A and D_B.

2) $GD(D_A, D_B)$

$$= \min_f \sum_{i=1}^{I} |R_i \circ f - S_i| + |S_i \circ f^{-1} - R_i|$$

$$f: A \overset{1\text{-}1}{\underset{\text{onto}}{\rightarrow}} B$$

$$= \min_{f^{-1}} \sum_{i=1}^{I} |R_i \circ f^{-1} - S_i| + |S_i \circ (f^{-1})^{-1} - R_i|$$

$$f^{-1}: A \overset{1\text{-}1}{\underset{\text{onto}}{\rightarrow}} B$$

$$= \min_{f^{-1}} \sum_{i=1}^{I} |S_i \circ f - R_i| + |R_i \circ f^{-1} - S_i|$$

$$f^{-1}: A \overset{1\text{-}1}{\underset{\text{onto}}{\rightarrow}} B$$

$$= \min_{f} \sum_{i=1}^{I} |S_i \circ f - R_i| + |R_i \circ f^{-1} - S_i|$$

$$f: B \overset{1\text{-}1}{\underset{\text{onto}}{\rightarrow}} A$$

$$= GD(D_B, D_A).$$

3) Let $D_A = \{R_1, \cdots, R_I\}$, $D_B = \{S_1, \cdots, S_I\}$, and $D_C = \{T_1, \cdots, T_I\}$ where for each $i = 1, \cdots, I$, $R_i \subseteq A^{n_i}$, $S_i \subseteq B^{n_i}$, and $T_i \subseteq C^{n_i}$. Let $f_1 \subseteq A \times C$ be one-one, onto, and that f_1 that minimizes $GD(D_A, D_C)$. Let $f_2 \subseteq C \times B$ be one-one, single-valued, and that f_2 that minimizes $GD(D_B, D_C)$.

Let $f: A \rightarrow B = f_1 \circ f_2$. Then f is one-one and onto and produces some error $E(f)$ with respect to D_A and D_B. $E(f)$ is greater than or equal to the minimum error $GD(D_A, D_B)$. We wish to show that

$$GD(D_A, D_B) \leqslant GD(D_A, D_C) + GD(D_C, D_B).$$

We will do this by showing that

$$E(f) \leqslant GD(D_A, D_C) + GD(D_C, D_B).$$

Let $x \in R_i \circ f_1 \circ f_2 - S_i$. Then $x \in R_i \circ f_1 \circ f_2$ and $x \notin S_i$. Since $x \in R_i \circ f_1 \circ f_2$ and f_2 is one-one and onto, there exists a unique $y \in R_i \circ f_1$ such that $\{y\} = \{x\} \circ f_2^{-1}$ and $\{x\} = \{y\} \circ f_2$. If $y \notin T_i$, then $y \in R_i \circ f_1 - T_i$. If $y \in T_i$, then $\{x\} = \{y\} \circ f_2$ is an element of $T_i \circ f_2$. Hence $x \in T_i \circ f_2 - S_i$.

Since for each $x \in R_i \circ f_1 \circ f_2 - S_i$ either $x \in T_i \circ f_2 - S_i$ or $y = x \circ f_2^{-1} \in R_i \circ f_1 - T_i$, we have

$$|R_i \circ f_1 \circ f_2 - S_i| \leqslant |R_i \circ f_1 - T_i| + |T_i \circ f_2 - S_i|.$$

Thus

$$\sum_{i=1}^{I} |R_i \circ f_1 \circ f_2 - S_i| \leqslant \sum_{i=1}^{I} |R_i \circ f_1 - T_i| + \sum_{i=1}^{I} |T_i \circ f_2 - S_i|.$$

Similarly we can show

$$|S_i \circ f_2^{-1} \circ f_1^{-1} - R_i| \leqslant |T_i \circ f_1^{-1} - R_i| + |S_i \circ f_2^{-1} - T_i|.$$

Thus

$$\sum_{i=1}^{I} |S_i \circ f_2^{-1} \circ f_1^{-1} - R_i| \leqslant \sum_{i=1}^{I} |T_i \circ f_1^{-1} - R_i| + \sum_{i=1}^{I} |S_i \circ f_2^{-1} - T_i|.$$

Adding, we get

$$\sum_{i=1}^{I} (|R_i \circ f_1 \circ f_2 - S_i| + |S_i \circ f_2^{-1} \circ f_1^{-1} - R_i|)$$

$$\leqslant \sum_{i=1}^{I} (|R_i \circ f_1 - T_i| + |T_i \circ f_1^{-1} - R_i|)$$

$$+ \sum_{i=1}^{I} (|T_i \circ f_2 - S_i| + |S_i \circ f_2^{-1} - T_i|)$$

which says

$$E(f) \text{ wrt } D_A \text{ and } D_B \leqslant GD(D_A, D_C) + GD(D_C, D_B).$$

But

$$GD(D_A, D_B) \leqslant E(f).$$

So

$$GD(D_A, D_B) \leqslant GD(D_A, D_C) + GD(D_C, D_B).$$

Thus, the relational distance of two relational descriptions (whether they consist of one relation or several relations) is a metric up to isomorphism.

V. A Variant Relational Distance That Is Not a Metric

In one of our past studies [20], we defined a similar relational distance measure, but did not require $|A| = |B|$. In this case, the mapping f was a binary relation and, while required to be single-valued and one-one, was not necessarily onto. Thus, some parts from set A mapped to nothing in B and vice versa. In this case, we defined the composition operation, the structural error, a completeness error, the total error, and the relational distance as follows.

$R_i \circ f = \{(b_1, \cdots, b_{n_i}) \in B^{n_i} |$ there exists $(a_1, \cdots, a_{n_i}) \in R_i$ with $(a_j, b_j) \in f$ for $j = 1, \cdots, n_i\}$.

$E_s^{i\prime}(f) = |R_i \circ f - S_i| + |S_i \circ f^{-1} - R_i|$ is the *structural error* of f for the ith pair of corresponding relations in D_A and D_B where $|x|$ denotes the cardinality of set x.

$E_c^{i\prime}(f) = |S_i - R_i \circ f| + |R_i - S_i \circ f^{-1}|$ is the *completeness error* of f for the ith pair of corresponding relations in D_A and D_B.

$E'(f) = \Sigma_{i=1}^{I} c_1 E_s^{i\prime}(f) + c_2 E_c^{i\prime}(f)$ is the *total error* of f with respect to D_A and D_B where c_1 and c_2 are nonnegative real numbers.

$GD'(D_A, D_B) = \min_f E'(f)$ is the *relational distance* of D_A and D_B.

A Problem

Suppose we want to insist that the relational distance GD' satisfy the property that the best mapping used in the definition of GD' must have some minimum number of pairs. This is important since the empty mapping always has $\emptyset$ structural error and is likely to be the minimum error mapping in many cases. However, restricting the mapping can cause a problem.

Suppose for ease of illustration that GD' is based only on structural error, not completeness error.

Consider the following example:

$$A = \{1, 2, 3\},\ C = \{1', 2', 3', 4', 5', 6'\},\ B = \{4, 5, 6\},$$

$$D_A = \{R_1, R_2\}, D_C = \{T_1, T_2\}, D_B = \{S_1, S_2\}$$

$$R_1 = \{(1, 2), (2, 1), (1, 3), (2, 3), (1, 1), (2, 2), (3, 3)\}$$

$$R_2 = \emptyset$$

$$T_1 = \{(1', 2'), (2', 1'), (1', 3'), (2', 3'), (1', 1'), (2', 2'), (3', 3')\}$$

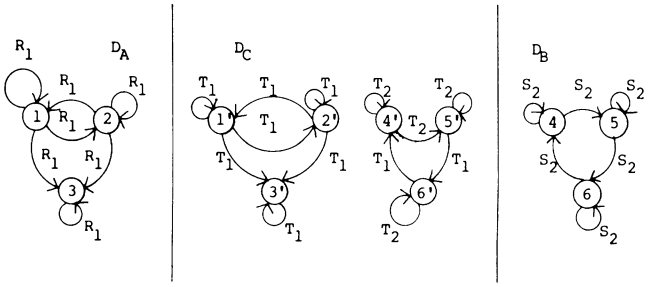

Fig. 5. Illustrates three relational descriptions, each containing two binary relations.

$$T_2 = \{(4', 5'), (5', 6'), (6', 4'), (4', 4'), (5', 5'), (6', 6')\}$$

$$S_1 = \emptyset$$

$$S_2 = \{(4, 5), (5, 6), (6, 4), (4, 4), (5, 5), (6, 6)\}.$$

This is illustrated in Fig. 5.

The best mapping with respect to D_A and D_C is $f_1 \subseteq A \times C$ defined by $f_1 = \{(1, 1'), (2, 2'), (3, 3')\}$ and the structural error is 0. The best mapping with respect to D_C and D_B is $f_2 \subseteq C \times B$ defined by $f_2 = \{(4', 4), (5', 5), (6', 6)\}$, and its structural error is also 0. The composition $f = f_1 \circ f_2$ of these mappings is $\emptyset$. If we had set no constraints on the size of the best mapping, this would cause no problems.

But we have some constraint. If this constraint requires even one pair to be in the best mapping $f' \subseteq A \times B$, then f' will have positive structural error since any pair (x_A, x_B) with $x_A \in A$, and $x_B \in B$ will have one arc (x_A, x_A) in R_1 that is not in S_1 and one arc (x_B, x_B) in S_2 that is not in R_2. Thus, for any mapping f' constrained to have even one pair, we have $E'(f') \geqslant 2$. Since the general distance would be the minimum error of all such constrained f's, the general distance would be $\geqslant 2$ and the triangle inequality would fail. ($2 \not\leqslant \emptyset + \emptyset$). In this example, the only way the triangle inequality can hold is for f' to be $\emptyset$ with error $E'(f') = 0$.

Now, what if we consider completeness error in this example?

Although $f_1 = \{(1, 1'), (2, 2'), (3, 3')\}$ is still the best mapping with respect to D_A and D_C, its completeness error is $|R_1 - T_1 \circ f_1^{-1}| + |R_2 - T_2 \circ f_2^{-1}| + |T_1 - R_1 \circ f_1| + |T_2 - R_2 \circ f_2| = \emptyset + \emptyset + \emptyset + |T_2| = 6$. Similarly, the completeness error of $f_2 = \{(4', 4), (5', 5), (6', 6)\}$ is $|T_1| = 7$. So $E'(f_1) = 6$ and $E'(f_2) = 7$ and $GD'(D_A, D_C) = 6$ and $GD'(D_C, D_B) = 7$.

Again, suppose we require that the best mapping have at least one pair. We want to find the best mapping with respect to D_A and D_B. Again any mapping f' with exactly one pair will have structural error 2. And no matter how many pairs in f', its completeness error will be $|R_1 - S_1 \circ f'^{-1}| + |R_2 - S_2 \circ f'^{-1}| + |S_1 - R_1 \circ f'| + |S_2 - R_2 \circ f'| = |R_1| + |S_2| = 7 + 6 = 13$. Thus, an f' with exactly one pair has total error $= 2 + 13 = 15$! And $15 \not\leqslant 6 + 7$. Similarly, as we add more pairs to f', the structural error can only get worse while the completeness error will remain 13. So, the sum of structural and completeness error is also not a metric when size constrained.

Since it is extremely important that the null mapping not be always chosen as the best mapping in comparing relational descriptions, we have the choice of 1) adding dummy parts so that all parts sets are all the same size and forcing f to be a one-one, onto function from A to B or 2) allowing f to be a size-constrained, one-one, single-valued binary relation and realizing that GD' is not a metric, but may still be useful in comparing relational descriptions. In choosing alternative 1), we force the variant relational distance to become the relational distance metric defined in this paper.

VI. Conclusions

In this paper we have reviewed relational modeling and relational matching. We have given a general form for measuring

the degree to which two sequences of relations are isomorphic. We have proved the measure to be a distance function or metric in the formal sense.

The existence of a relational distance opens new decision theoretic approaches to structural pattern recognition. It makes it possible to work with class conditional distributions of the relational models of objects. Given that we are observing an object of class A, a class which has an ideal or representative relational model R, the probability that the observed object has relational model S can be computed as a monotonically decreasing function of the relational distance between R and S. With a mechanism to compute such probabilities it becomes easy to perform identification of the observed object in a Bayesian manner by maximizing an expected utility or by maximizing the probability of being correct. In future work, we expect to derive estimation procedures for the monotonic functions and give results for the identification of a variety of three-dimensional objects.

REFERENCES

[1] H. G. Barrow, A. P. Ambler, and R. M. Burstall, "Some techniques for recognizing structures in pictures," in *Frontiers of Pattern Recognition*, S. Watanabe, Ed. New York: Academic, 1972, pp. 1-29.

[2] R. A. Brooks, "Symbolic reasoning among three-dimensional models and two-dimensional images," *Artif. Intell. (Special Volume on Computer Vision)*, AI 17, 1981.

[3] L. S. Davis, "Shape matching using relaxation techniques," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. PAMI-1, pp. 60-72, Jan. 1979.

[4] C. D. Feustel and L. G. Shapiro, "The nearest neighbor problem in an abstract metric space," *Pattern Recog. Lett.*, vol. 1, no. 2, pp. 125-128, Dec. 1982.

[5] E. C. Freuder, "Synthesizing constraint expression," *Commun. Ass. Comput. Mach.*, vol. 21, no. 11, 1978.

[6] J. Gaschnig, "A general backtrack algorithm that eliminates most redundant tests," in *Proc. 5th Int. Joint Conf. Artif. Intell.*, 1972.

[7] R. M. Haralick and G. Elliot, "Increasing tree search efficiency for constraint satisfaction problems," in *Proc. 6th Int. Joint Conf. Artif. Intell.*, 1979.

[8] R. M. Haralick and L. G. Shapiro, "The consistent labeling problem: Part I," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. PAMI-1, pp. 173-184, Apr. 1979.

[9] A. Mackworth, "Consistency in network of relations," *Artif. Intell.*, vol. 8, pp. 99-118, 1977.

[10] D. Marr and H. K. Nishihara, "Spatial disposition of axes in a generalized cylinder representation of objects that do not encompass the viewer," AI Lab., Mass. Inst. Technol., Cambridge, MA, Memo. 341, Dec. 1975.

[11] U. Montanari, "Networks of constraints: Fundamental properties and applications to picture processing," *Inform. Sci.*, vol. 7, pp. 95-132, 1974.

[12] P. G. Mulgaonkar, L. G. Shapiro, and R. M. Haralick, "Identification of man-made objects using geometric and relational constraints," Dep. Comput. Sci., Virginia Polytech. Inst., State Univ., Blacksburg, Sept. 1982.

[13] R. Nevatia and T. O. Binford, "Description and recognition of curved objects," *Artif. Intell.*, vol. 8, pp. 77-90, 1977.

[14] A. Rosenfeld, R. A. Hummel, and S. W. Zucker, "Scene labeling by relaxation operations," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-6, pp. 420-433, June 1976.

[15] A. Sanfelieu and K. S. Fu, "A distance measure between attributed relational graphs for pattern recognition," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-13, May-June 1983.

[16] M. Schneier, "A compact relational structure representation," in *Workshop Represent. Three-Dimensional Obj.*, R. Bajcy, Director, Univ. Penn., Philadelphia, PA, May 1-2, 1979.

[17] L. G. Shapiro, J. D. Moriarty, P. G. Mulgaonkar, and R. M. Haralick, "A generalized blob model for three-dimensional object representation," in *IEEE Workshop Pict. Data Descript. Management*, Asilomar, CA, Aug. 1980.

[18] L. G. Shapiro, "A structural model of shape," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. PAMI-2, pp. 111-126, Mar. 1980.

[19] L. G. Shapiro and R. M. Haralick, "Structural descriptions and inexact mathing," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. PAMI-3, pp. 504-519, Sept. 1981.

[20] L. G. Shapiro, J. D. Moriarty, R. M. Haralick, and P. G. Mulgaonkar, "Matching three-dimensional objects using a relational paradigm," *Pattern Recognition*, vol. 17, no. 4, pp. 385-405, 1984.

[21] L. G. Shapiro and R. M. Haralick, "Organization of relational models for scene analysis," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. PAMI-4, Nov. 1982.

[22] —, "Decomposition of two-dimensional shapes by graph-theoretic clustering," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. PAMI-1, pp. 10-20, Jan. 1979.

[23] J. R. Ullman, "An algorithm for subgraph homomorphisms," *J. Ass. Comput. Mach.*, vol. 23, pp. 31-42, Jan. 1976.

[24] D. Waltz, "Understanding line drawings of scenes with shadows," in *The Psychology of Computer Vision*, P. Winston, Ed. New York: McGraw-Hill, 1975, pp. 19-91.

A Top-Down Quadtree Traversal Algorithm

HANAN SAMET

***Abstract*—Many standard image processing operations can be implemented using quadtrees as a simple tree traversal where, at each terminal node, a computation is performed involving some of that node's neighbors. Most of this work has involved the use of bottom-up neighbor-finding techniques which search for a nearest common ancestor. Recently, top-down techniques have been proposed which make use of a neighbor vector as the tree is traversed. A simplified version of the top-down method for a quadtree in the context of a general-purpose tree traversal algorithm is presented. It differs, in part, from' prior work in its ability to compute diagonally adjacent neighbors rather than just horizontally and vertically adjacent neighbors. It builds a neighbor vector for each node using a minimal amount of information. Analysis of the algorithm shows that its execution time is directly proportional to the number of nodes in the tree. However, it does require some extra storage. Use of the algorithm leads to lower execution time bounds for some common quadtree image processing operations such as connected component labeling.**

***Index Terms*—Connected component labeling, image processing, image representation, perimeter, quadtrees.**

I. INTRODUCTION

The quadtree [7] (e.g., Fig. 1) is a hierarchical representation which has been the subject of much research in recent years [15]. It has been found to be useful in such applications as image processing, computer graphics, pattern recognition, and cartography. Many algorithms for standard operations in these domains can be expressed as simple tree traversals where at each node a computation is performed involving the use of "bottom-up" neighbor-finding techniques [11]. Recently, "top-down" methods which build a neighbor vector as the tree is traversed have been independently proposed [5], [8], [13].

Manuscript received December 7, 1983; revised July 23, 1984. Recommended for acceptance by S. L. Tanimoto. This work was supported by the U.S. Army Engineer Topographic Laboratories under Contract DAAG-70-81-C-0059.

The author is with the Department of Computer Science, University of Maryland, College Park, MD 20742.