The Consistent Labeling Problem: Part II

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Abstract—In this second part of a two-part paper, we explore the power and complexity of the ϕ_{KP} and Ψ_{KP} class of look-ahead operators which can be used to speed up the tree search in the consistent labeling problem. For a specified K and P we show that the fixed-point power of ϕ_{KP} and Ψ_{KP} is the same, that ϕ_{KP+1} is at least as powerful as ϕ_{KP}, and that $\Psi_{K+1\,P}$ is at least as powerful at ϕ_{KP}.

Finally, we define a minimal compatibility relation and show how the standard tree search procedure for finding all the consistent labelings is quicker for a minimal relation. This leads to the concept of grading the complexity of compatibility relations according to how much look-ahead work it requires to reduce them to minimal relations and suggests that the reason look-ahead operators, such as Waltz filtering, work so well is that the compatibility relations used in practice are not very complex and are reducible to minimal or near minimal relations by a ϕ_{KP} or Ψ_{KP} look-ahead operator with small value for parameter P.

Index Terms—Backtracking, consistent labeling, constraint satisfaction, graph coloring, homomorphism, look-ahead operators, matching, N-ary relations, relaxation, scene analysis, subgraph isomorphisms, tree search.

I. Introduction

In Part I of this paper [6], we formulated a network constraint problem which we called the consistent labeling problem. We showed how this problem is a generalization of problems such as the subgraph isomorphism problem [10], the graph homomorphism problem [7], the automata homomorphism problem [3], the graph coloring problem [7], the relational homomorphism problem [5], the scene labeling problem [1], and the Boolean satisfiability problem [2]. Then we showed how a two parameter look-ahead operator ϕ_{KP} can be used in some cases to help speed up the tree search required to solve the consistent labeling problem.

In this paper we explore the mathematical properties of the ϕ_{KP} look-ahead operator. We introduce another related look-ahead operator Ψ_{KP}, which is a generalization of the Waltz filtering operator. We show that the fixed-point power of ϕ_{KP} is the same as that of Ψ_{KP}, that operators with greater look-ahead are more powerful (ϕ_{KP+1} is at least as powerful as ϕ_{KP}), and that $\phi_{K+1\,P}$ is at least as powerful as ϕ_{KP}.

The consistent labeling problem is NP-complete and, therefore, worst case problems can be expected to require exponential computational time. The use of the look-ahead operator on these worst case problems does not improve the tree search time. Despite this discouraging situation for worst case problems, the look-ahead operators can improve the tree search time for many specific consistent labeling problems.

In the last part of the paper we explore characterizing the difficulty of any given consistent labeling problem. We define a minimal compatibility relation and show how the standard tree search procedure for finding all the consistent labelings is quicker for a minimal relation. This leads to the concept of grading the complexity of compatibility relations according to how much work it requires to reduce them to minimal relations at each stage of the tree search. This perspective suggests that the reason look-ahead operators tend to work so well on practical problems is that the compatibility relations used in practice are not very complex and are reducible to minimal or near minimal relations by ϕ_{KP} or Ψ_{KP} with small value for parameter P.

A. The Consistent Labeling Problem

To begin our discussion we review some basic definitions. Let $U = \{1, \cdots, M\}$ be a set of M units and let L be a set of labels. If $u_1, \cdots, u_N \in U$ and $l_1, \cdots, l_N \in L$, then we call the N-tuple $(l_1, \cdots, l_N)$ a *labeling* of units $(u_1, \cdots, u_N)$. The problem of labeling is that not all of the labelings are consistent because some of the units are *a priori* known to constrain one another. The compatibility model tells us which units mutually constrain one another N at a time and which labelings are permitted or legal for those units which do constrain one another. One way of representing this compatibility model is by a quadruple (U, L, T, R) where $T \subseteq U^N$ is the set of all N-tuples of units which mutually constrain one another and the constraint relation $R \subseteq (U \times L)^N$ is the set of all N-tuples of unit-label pairs $(u_1, l_1, \cdots, u_N, l_N)$ where $(l_1, \cdots, l_N)$ is a permitted or legal labeling of units $(u_1, \cdots, u_N)$. We call T the *unit constraint relation* and R the *unit-label constraint relation.*

A labeling $(l_1, \cdots, l_P)$ is a *consistent labeling* of units $(u_1, \cdots, u_P)$ with respect to the compatibility model (U, L, T, R) if and only if $\{i_1, \cdots, i_N\} \subseteq \{1, \cdots, P\}$ and $(u_{i_1}, \cdots, u_{i_N}) \in T$ imply the $2N$-tuple

$$(u_{i_1}, l_{i_1}, \cdots, u_{i_N}, l_{i_N}) \in R;$$

that is, the labeling $(l_{i_1}, \cdots, l_{i_N})$ is a permitted or legal labeling of units $(u_{i_1}, \cdots, u_{i_N})$. When U and L are understood, such a labeling $(l_1, \cdots, l_P)$ is called a *(T, R)-consistent labeling* of $(u_1, \cdots, u_P)$. The *consistent labeling* problem is to find all consistent labelings of units $(1, \cdots, M)$ with respect to the compatibility model. We denote this set of consistent labelings by $\mathcal{L}(T, R)$.

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B. The ϕ_{KP} Look-Ahead Operator

Haralick *et al.* [4] define a look-ahead operator ϕ_P which when applied to a constraint relation R, removes $2N$-tuples which do not contribute to a globally consistent labeling. We can generalize this look-ahead operator ϕ_P not only to work within the framework of the (U, L, T, R) model, but also to have an additional parameter K to give us more control over the use of the operator.

Let $U = \{1, \cdots, M\}$ be a set of units, L be a set of labels, $T \subseteq U^N$, and $R \subseteq (U \times L)^N$. Let $K \leqslant N \leqslant P$ with $K < P$. The *look-ahead operator* ϕ_{KP} is defined by $\phi_{KP}R = \{(u_1, l_1, \cdots, u_N, l_N) \in R \mid$ for every combination $j_1, \cdots, j_K$ of $1, \cdots, N$ and for every $u'_{K+1}, \cdots, u'_P \in U$, there exists $l'_{K+1}, \cdots, l'_P \in L$ such that $(l_{j_1}, \cdots, l_{j_K}, l'_{K+1}, \cdots, l'_P)$ is a (T, R)-consistent labeling of $(u_{j_1}, \cdots, u_{j_K}, u'_{K+1}, \cdots, u'_P)\}$.

Thus to apply the ϕ_{KP} operator to R, we individually check each N-tuple of R. We fix any K of the N units $(u_{j_1}, \cdots, u_{j_K})$ to their labels in the N-tuple and check every set of $P - K$ units to determine if there are $P - K$ labels which make the K fixed labels $(l_{j_1}, \cdots, l_{j_K})$ plus the $P - K$ free labels $(l'_{K+1}, \cdots, l'_P)$ a consistent labeling of all P units

$$(u_{j_1}, \cdots, u_{j_K}, u'_{K+1}, \cdots, u'_P).$$

We repeat this process for every combination of K out of the N units in the N-tuple. If for any combination of K fixed units and labels, there is a set of $P - K$ units for which $P - K$ labels to make a consistent labeling cannot be found, the $2N$-tuple is thrown out. Mackworth [8] uses the ϕ_{23} operator, the ϕ_P operator of Haralick *et al.* [4] is the ϕ_{NP} operator, Haralick and Kartus [5] use the ϕ_{1N} operator, and Ullman [10] uses the ϕ_{12} operator.

II. Properties of the ϕ_{KP} Operator

The ϕ_{KP} operator is a generalization of the ϕ_P operator in the Haralick *et al.* paper [4]. Thus, we would anticipate that the characterizations of the ϕ_P operator also apply to the ϕ_{KP} operator. In this section, we will show that this is so. In addition, we will show that the power of the ϕ_{KP} operator depends on both P and K, the power to remove N-tuples of unit-label pairs from R increasing as K or P increase. We continue the numbering of the theorems and propositions from Part I of this paper.

Let $U = \{1, \cdots, M\}$ be a set of units, L be a set of labels, $T \subseteq U^N$, and $R \subseteq (U \times L)^N$. First we define the smallest order-N relation $S_{TR} \subseteq (U \times L)^N$ that defines the same set of (T, R)-consistent labelings as R does. We do so by noting that the intersection of unit-label constraints, all of which have the identical set of consistent labelings, is a unit-label constraint having the same set of consistent labelings. The relation S_{TR} is defined as the intersection over all relations R' which have the same set of consistent labelings that R does.

Proposition 4: Let $R_1, R_2 \subseteq (U \times L)^N$ be such that $\mathcal{L}(T, R_1) = \mathcal{L}(T, R_2)$. Then $\mathcal{L}(T, R_1 \cap R_2) = \mathcal{L}(T, R_1)$.

Proof: First, $R_1 \cap R_2 \subseteq R_1$, so by Lemma 1, $\mathcal{L}(T, R_1 \cap R_2) \subseteq \mathcal{L}(T, R_1)$. Now suppose $(l_1, \cdots, l_M) \in \mathcal{L}(T, R_1)$. By the hypothesis, $(l_1, \cdots, l_M) \in \mathcal{L}(T, R_2)$. Then for every N-tuple $(u_1, \cdots, u_N) \in T$, the $2N$-tuple $(u_1, l_{u_1}, \cdots, u_N, l_{u_N})$ is in R_1 and R_2 and therefore in $R_1 \cap R_2$. Thus $(l_1, \cdots, l_M) \in \mathcal{L}(T, R_1 \cap R_2)$.

Theorem 3: Let $R_i \subseteq (U \times L)^N$, $i = 1, \cdots, n$, be such that $\mathcal{L}(T, R_i) = \mathcal{L}(T, R)$, $i = 1, \cdots, n$. Then

$$\mathcal{L}\left(T, \bigcap_{i=1}^{n} R_i\right) = \mathcal{L}(T, R).$$

Proof: The proof is by induction on n.

Theorem 3 motivates the following definition for S_{TR}.

S_{TR} is a *minimal relation* with respect to T and R if and only if

$$S_{TR} = \cap \{R' \mid \mathcal{L}(T, R') = \mathcal{L}(T, R)\}.$$

Clearly, S_{TR} is contained in R since R is trivially one of the R' such that $\mathcal{L}(T, R') = \mathcal{L}(T, R)$, and S_{TR} is the smallest relation $R' \subseteq R$ such that $\mathcal{L}(T, R') = \mathcal{L}(T, R)$. The following theorem shows that every N-tuple of unit-label pairs in S_{TR} contributes to some (T, R)-consistent labeling.

Theorem 4: Let U, L, T, R, and S_{TR} be defined as above. Let $S = \{(u_1, c_1, \cdots, u_N, c_N) \in (U \times L)^N \mid (u_1, \cdots, u_N) \in T$ and for some $(l_1, \cdots, l_M) \in \mathcal{L}(T, R)$, $c_n = l_{u_n}$, $n = 1, \cdots, N\}$. Then $S_{TR} = S$.

Proof: Let $(u_1, c_1, \cdots, u_N, c_N) \in S$. By the definition of a consistent labeling, $(u_1, c_1, \cdots, u_N, c_N)$ must be an element of every $R' \subseteq (U \times L)^N$ such that $\mathcal{L}(T, R') = \mathcal{L}(T, R)$. Thus $(u_1, c_1, \cdots, u_N, c_N) \in S_{TR'}$ and $S \subseteq S_{TR}$.

Conversely, the definition of S implies that $\mathcal{L}(T, S) = \mathcal{L}(T, R)$. Since S_{TR} is the smallest relation with this property, we have $S_{TR} \subseteq S$. Thus $S_{TR} = S$.

An immediate consequence of Theorem 4 is that $S_{TR} \neq \emptyset$ if and only if there exists a (T, R)-consistent labeling of $(1, \cdots, M)$.

Since S_{TR} contains only those $2N$-tuples of R that contribute to a consistent labeling, and since the job of the ϕ_{KP} operator is to remove $2N$-tuples from R that do not contribute to a consistent labeling, we could expect relationships to exist between the reduced relation S_{TR} and the partly reduced relation $\phi_{KP}R$. The following theorems state these relationships: repeated application of ϕ_{KP} cannot reduce R to something smaller than S_{TR} and S_{TR} is a fixed point of ϕ_{KP}.

Theorem 5: Let $T \subseteq U^N$ and $R \subseteq (U \times L)^N$. Let S_{TR} be the minimal relation corresponding to (T, R). Suppose $K \leqslant N \leqslant P$ and $K < P$. Then $S_{TR} \subseteq \phi_{KP}^m R$ for every positive integer m.

Proof: Since $\mathcal{L}(T, R) = \mathcal{L}(T, \phi_{KP}R)$ by Proposition 3, a simple induction yields $\mathcal{L}(T, R) = \mathcal{L}(T, \phi_{KP}^m R)$ for every positive integer m. Since S_{TR} is minimal and $\mathcal{L}(T, S_{TR}) = \mathcal{L}(T, \phi_{KP}^m R)$, we must have $S_{TR} \subseteq \phi_{KP}^m R$.

Theorem 6: $\phi_{KP}(S_{TR}) = S_{TR}$.

Proof: By definition of ϕ_{KP}, $\phi_{KP}S_{TR} \subseteq S_{TR}$. By Proposition 3, $\mathcal{L}(T, S_{TR}) = \mathcal{L}(T, \phi_{KP}S_{TR})$. Since S_{TR} is minimal and $\mathcal{L}(T, S_{TR}) = \mathcal{L}(T, \phi_{KP}S_{TR})$, $S_{TR} \subseteq \phi_{KP}S_{TR}$. Hence, $S_{TR} = \phi_{KP}S_{TR}$.

If we fix K, then larger values of P represent more work for the ϕ_{KP} operator. Thus we would expect ϕ_{KP} to be more powerful for larger values of P. The following proposition shows that ϕ_{KP+1} takes at least as many elements out of R as does ϕ_{KP}.

Proposition 5: Let $R \subseteq (U \times L)^N$, $K \leqslant N$, and $P > K$. Then $\phi_{KP+1} R \subseteq \phi_{KP} R$.

Now if we fix P, we might expect to obtain a related result for K. However, it is not true that $\phi_{K+1P} R \subseteq \phi_{KP} R$ or $\phi_{KP} R \subseteq \phi_{K+1P} R$. To see this, consider binary relation R of Fig. 1(a) and let $T = \{1, 2, 3\}^2$. Then $\phi_{13} R = R$, but the $\phi_{23} R = R - \{(1, b, 2, a), (2, a, 1, b)\}$. Thus $\phi_{23} R \subseteq \phi_{13} R$. Now consider the binary relation R' of Fig. 1(b) and let $T = \{1, 2, 3, 4\}^2$. Then $\phi_{13} R' = \emptyset$ and $\phi_{23} R' = \{(1, a, 2, a), (2, a, 1, a)\}$. Thus $\phi_{13} R' \subseteq \phi_{23} R'$. The relationship that does exist is that if ϕ_{K+1P} cannot reduce R anymore, then ϕ_{KP} will certainly not be able to reduce R anymore. In order to prove this kind of fixed-point result, we need one condition on R and one condition on T. The condition on R is that R contains no N-tuples of unit-label pairs corresponding to N-tuples of units which do not constrain one another; that is, N-tuples of units not in T. This just says that R has no "excess baggage." R contains the constraints and if an N-tuple of units is not in T (do not constrain one another), then it is illogical to put its constraints in R since it does not have any. The condition on T is one of constraint connectivity. T must be sufficiently connected so that if $(u_1, \cdots, u_N) \in T$, then any K units from $u_1, \cdots, u_N$ plus any $N - K$ additional units from U must form an N-tuple of units in T.

The proof that $\phi_{K+1P} R = R$ implies $\phi_{KP} R = R$ also relies on the fact that if $(u_1, l_1, \cdots, u_N, l_N) \in R$ and $\phi_{K+1P} R = R$, then by finding any Q-tuple of units from $u_1, \cdots, u_N$, $Q \leqslant K$, and by adding to these Q units any $N - Q$ units which make the resulting N-tuple of units in T, we can always find labels for the N-Q units so that the N-tuple of unit-label pairs formed by the initial Q units and their labels plus $N - Q$ units and the labels we find for them are in R.

Proposition 6: Let $R \subseteq (U \times L)^N$, $K \leqslant N \leqslant P$, $K < P$, $Q \leqslant K$, and $P - K \geqslant N - Q$. Suppose $R = \phi_{KP} R$, $j_1, \cdots, j_Q$ is a combination of $1, \cdots, N$ and $v_1, \cdots, v_{N-Q} \in U$. Then $(u_1, l_1, \cdots, u_N, l_N) \in R$ implies there exists $l'_1, \cdots, l'_{N-Q} \in L$ such that $(u_{j_1}, l_{j_1}, \cdots, u_{j_Q}, l_{j_Q}, v_1, l'_1, \cdots, v_{N-Q}, l'_{N-Q}) \in R$ when $(u_{j_1}, \cdots, u_{j_Q}, v_1, \cdots, v_{N-Q}) \in T$.

Proof: Let $(u_1, l_1, \cdots, u_N, l_N) \in R$. Since $R = \phi_{KP} R$, $(u_1, l_1, \cdots, u_N, l_N) \in \phi_{KP} R$. Let $j_1, \cdots, j_Q, \cdots, j_K$ be an arbitrary combination of $1, \cdots, N$ and $v_1, \cdots, v_{N-Q}, \cdots, v_{P-K}$ be arbitrary units of U. Since $(u_1, l_1, \cdots, u_N, l_N) \in \phi_{KP} R$, there exist labels $l'_1, \cdots, l'_{N-Q}, \cdots, l'_{P-K} \in L$ such that $(l_{j_1}, \cdots, l_{j_K}, l'_1, \cdots, l'_{N-Q}, \cdots, l'_{P-K})$ is a (T, R)-consistent labeling of units $(u_{j_1}, \cdots, u_{j_K}, v_1, \cdots, v_{N-Q}, \cdots, v_{P-K})$. Hence if $(u_{j_1}, \cdots, u_{j_Q}, v_1, \cdots, v_{N-Q}) \in T$, $(u_{j_1}, l_{j_1}, \cdots, u_{j_Q}, l_{j_Q}, v_1, l'_1, \cdots, v_{N-Q}, l'_{N-Q}) \in R$.

Taking $Q = K$ in Proposition 6, we have the following corollary.

Corollary 1: Let $R \subseteq (U \times L)^N$, $K + 1 \leqslant N < P$. Suppose $R = \phi_{K+1P} R$, $j_1, \cdots, j_K$ is a combination of $1, \cdots, N$ and $v_1, \cdots, v_{N-K} \in U$. Then $(u_1, l_1, \cdots, u_N, l_N) \in R$ implies there exists $l'_1, \cdots, l'_{N-K} \in L$ such that $(u_{j_1}, l_{j_1}, \cdots, u_{j_K}, l_{j_K}, v_1, l'_1, \cdots, v_{N-K}, l'_{N-K}) \in R$ when $(u_{j_1}, \cdots, u_{j_K}, v_1, \cdots, v_{N-K}) \in T$.

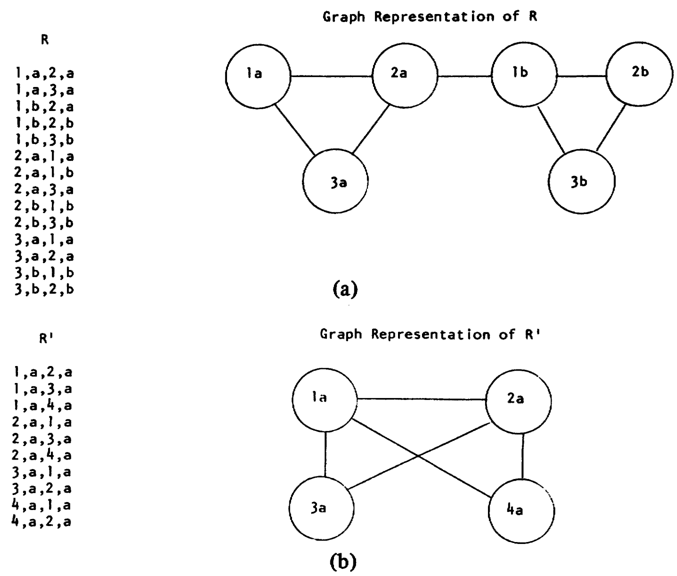

Fig. 1. Illustrates that $\phi_{K+1P} R \not\subseteq \phi_{KP} R$ and $\phi_{KP} R \not\subseteq \phi_{K+1P} R$. (a) Relation R where $\phi_{23} R \subseteq \phi_{13} R$. (b) Relation R' where $\phi_{13} R' \subseteq \phi_{23} R'$.

We can now give sufficient conditions on R and T which guarantee that $\phi_{K+1P} R = R$ implies $\phi_{KP} R = R$.

Theorem 7: Let $R \subseteq (U \times L)^N$, $T \subseteq U^N$, and $K + 1 \leqslant N < P$. Suppose $(u_1, l_1, \cdots, u_N, l_N) \in R$ implies $(u_1, \cdots, u_N) \in T$ and suppose $(u_1, \cdots, u_N) \in T$, $j_1, \cdots, j_K$ a combination of $1, \cdots, N$, and $u'_{K+1}, \cdots, u'_N \in U$ imply $(u_{j_1}, \cdots, u_{j_K}, u'_{K+1}, \cdots, u'_N) \in T$. Then $\phi_{K+1P} R = R$ implies $\phi_{KP} R = R$.

Proof: Let $\phi_{K+1P} R = R$. Since $\phi_{KP} R \subseteq R$, by definition of ϕ_{KP}, we need only prove $R \subseteq \phi_{KP} R$.

Let $(u_1, l_1, \cdots, u_N, l_N) \in R$. Since $R = \phi_{K+1P} R$, $(u_1, l_1, \cdots, u_N, l_N) \in \phi_{K+1P} R$.

Let $j_1, \cdots, j_K$ be an arbitrary combination of $1, \cdots, N$. Let $u'_{K+1}, \cdots, u'_P$ be arbitrary units of U. By supposition on R, $(u_1, l_1, \cdots, u_N, l_N) \in R$ implies $(u_1, \cdots, u_N) \in T$. By supposition on T, $(u_1, \cdots, u_N) \in T$, $j_1, \cdots, j_K$ a combination of $1, \cdots, N$ and $u'_{K+1}, \cdots, u'_N \in U$ imply $(u_{j_1}, \cdots, u_{j_K}, u'_{K+1}, \cdots, u'_N) \in T$. Since $R = \phi_{K+1P} R$, $j_1, \cdots, j_K$ a combination of $1, \cdots, N$, $u'_{K+1}, \cdots, u'_N \in U$, and $(u_1, l_1, \cdots, u_N, l_N) \in R$ imply by the corollary to Proposition 6 that there exists labels $l'_{K+1}, \cdots, l'_N \in L$ such that $(u_{j_1}, l_{j_1}, \cdots, u_{j_K}, l_{j_K}, u'_{K+1}, \cdots, u'_N, l'_N) \in R$ when $(u_{j_1}, \cdots, u_{j_K}, u'_{K+1}, \cdots, u'_N) \in T$.

Because $R = \phi_{K+1P} R$, $(u_{j_1}, l_{j_1}, \cdots, u_{j_K}, l_{j_K}, u'_{K+1}, l'_{K+1}, \cdots, u'_N, l'_N) \in R$, $1, \cdots, K + 1$ is a combination of $1, \cdots, N$, and $u'_{K+2}, \cdots, u'_P \in U$, by the definition of ϕ_{K+1P}, there exists labels $l''_{K+2}, \cdots, l''_P$ such that $(l_{j_1}, \cdots, l_{j_K}, l'_{K+1}, l''_{K+2}, \cdots, l''_P)$ is a (T, R)-consistent labeling of $(u_{j_1}, \cdots, u_{j_K}, u'_{K+1}, \cdots, u'_P)$. Now since $(u_1, l_1, \cdots, u_N, l_N) \in R$ and $j_1, \cdots, j_K$ is an arbitrary combination of $1, \cdots, N$ and $u'_{K+1}, \cdots, u'_P \in U$, and we have shown the existence of labels $l'_{K+1}, l''_{K+2}, \cdots, l''_P \in L$ which make $(l_{j_1}, \cdots, l_{j_K}, l'_{K+1}, l''_{K+2}, \cdots, l''_P)$ a (T, R)-consistent labeling of $(u_{j_1}, \cdots, u_{j_K}, u'_{K+1}, \cdots, u'_P)$, we must have by the definition of ϕ_{KP}, $(u_1, l_1, \cdots, u_N, l_N) \in \phi_{KP} R$. Hence $R \subseteq \phi_{KP} R$.

Corollary 2: If $T = U^N$, then $\phi_{K+1P} R = R$ implies $\phi_{KP} R = R$.

Theorem 7 implies that successive application of ϕ_{K+1P} to any unit-label constraint relation will produce a relation certainly no larger than the successive application of ϕ_{KP} to the same relation. Hence, the fixed-point behavior of ϕ_{K+1P} is better than the fixed-point behavior of ϕ_{KP}. To show this fact, we will need a precise way of talking about the successive application of ϕ_{KP} to the point where $\phi_{KP}^{m+1}R = \phi_{KP}^{m}R$.

To help us do this, we adopt the following notational convention:

$$\phi_{KP}^{\infty}R = \bigcap_{m=1}^{\infty} \phi_{KP}^{m}R.$$

It is directly verifiable from the definition of ϕ_{KP} that

$$\phi_{KP}(R \cap S) = (\phi_{KP}R) \cap (\phi_{KP}S).$$

A simple induction on this relationship yields $\phi_{KP}(\cap_{i=1}^{\infty} R_i) = \cap_{i=1}^{\infty} \phi_{KP}R_i$. From this it is easy to see that $\phi_{KP}^{\infty}R$ is a fixed-point of ϕ_{KP}:

$$\phi_{KP}(\phi_{KP}^{\infty}R) = \phi_{KP}\left(\bigcap_{m=1}^{\infty} \phi_{KP}^{m}R\right) = \bigcap_{m=1}^{\infty} \phi_{KP}(\phi_{KP}^{m}R)$$

$$= \bigcap_{m=1}^{\infty} \phi_{KP}^{m+1}R = \bigcap_{m=2}^{\infty} \phi_{KP}^{m}R$$

$$= \bigcap_{m=1}^{\infty} \phi_{KP}^{m}R = \phi_{KP}^{\infty}R.$$

Because all sets we consider are finite and $\phi_{KP}R \subseteq R$, $\phi_{KP}^{\infty}R = \phi_{KP}^{m}R$ for any positive integer m satisfying $\phi_{KP}^{m}R = \phi_{KP}^{m+1}R$. Therefore, we are justified in thinking that $\phi_{KP}^{\infty}R$ is the result of successively applying ϕ_{KP} to R until a fixed point is reached.

Corollary 3 to Theorem 8 states that $\phi_{K+1P}^{\infty}R \subseteq \phi_{KP}^{\infty}R$; the fixed-point behavior of ϕ_{K+1P} is at least as good as the fixed-point behavior of ϕ_{KP}. Fig. 2 shows an example where the fixed-point behavior of ϕ_{23} is strictly better than the fixed-point behavior of ϕ_{13}, thereby demonstrating that the containment relation in $\phi_{K+1P}^{\infty}R \subseteq \phi_{KP}^{\infty}R$ cannot be made into the stronger equality relation. One practical consequence of Corollary 3 is that for $N > 2$, use should be made of the ϕ_{N-1N} operator rather than the ϕ_{1N} operator used by Ullman [10] and Haralick and Kartus [5], since ϕ_{1N} and ϕ_{N-1N} require the same amount of work.

Theorem 8: Suppose ϕ_A and ϕ_B are operators which satisfy the following:

1) $\phi_A S = S$ implies $\phi_B S = S$;

2) $\phi_A S \subseteq S$;

3) $S_1 \subseteq S_2$ implies $\phi_B S_1 \subseteq \phi_B S_2$.

Let R be given. Then $\phi_A^{\infty}R \subseteq \phi_B^{\infty}R$.

Proof: Let R be given. Certainly $\phi_A(\phi_A^{\infty}R) = (\phi_A^{\infty}R)$. Hence, by property 1), $\phi_B(\phi_A^{\infty}R) = (\phi_A^{\infty}R)$. By a simple induction, for every $n \geq 1$, $\phi_B^{n}(\phi_A^{\infty}R) = (\phi_A^{\infty}R)$. Hence $\phi_B^{\infty}(\phi_A^{\infty}R) = (\phi_A^{\infty}R)$. By successive application of property 2), $\phi_A^{\infty}R \subseteq R$. Now by property 3), $\phi_B(\phi_A^{\infty}R) \subseteq \phi_B R$. By successive application of property 3), $\phi_B^{\infty}(\phi_A^{\infty}R) \subseteq \phi_B^{\infty}R$. But $\phi_A^{\infty}R = \phi_B^{\infty}(\phi_A^{\infty}R)$. Therefore, $\phi_A^{\infty}R \subseteq \phi_B^{\infty}R$.

Corollary 3: For any R, $\phi_{K+1P}^{\infty}R \subseteq \phi_{KP}^{\infty}R$.

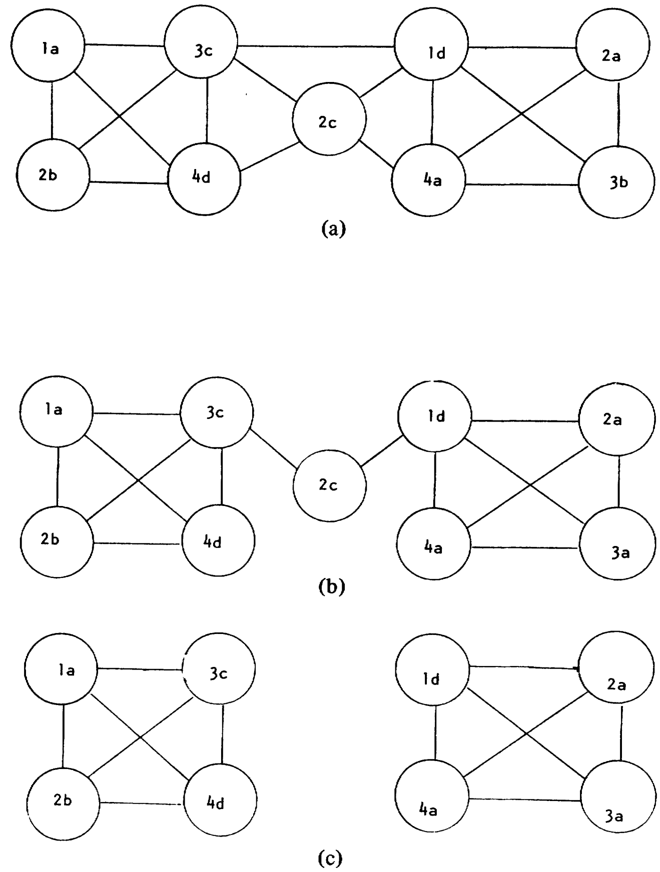

Fig. 2. Shows that ϕ_{23} is more powerful than ϕ_{13}. Let $U = \{1, 2, 3, 4\}$ and $L = \{a, b, c, d\}$. Take $T = U \times U$. (a) The relation R is at a fixed point under ϕ_{13}. (b) The relation is the result after one iteration by ϕ_{23}. (c) The relation is the fixed point reached after two iterations by ϕ_{23}.

III. The Ψ_{KP} Look-Ahead Operator and Its Properties

The task of the ϕ_{KP} operator is to reduce the relation R by removing N-tuples of unit-label pairs that do not contribute to any consistent labeling. We can also define a look-ahead operator which removes K-tuples of unit-label pairs that do not contribute to any consistent labeling from projections of R. Such an operator is the generalization to the Ψ_P operator of Haralick *et al.* [4] which removes from a set of unit-label pairs those pairs which do not contribute to a consistent labeling. In this section we will show the fixed-point equivalence of ϕ_{KP} and Ψ_{KP}.

In order to define the operator Ψ_{KP}, we first define a generalized projection operator π_K. Let $Q \subseteq A^N$ be an N-ary relation and $K \leq N$. We define the generalized *projection operator* π_K by

$$\pi_K Q = \{(a_1, \cdots, a_K) \in A^K \mid \text{for some } (q_1, \cdots, q_N) \in Q \text{ and some combination } j_1, \cdots, j_K \text{ of } 1, \cdots, N, \; a_i = q_{j_i}, \; i = 1, \cdots, K\}.$$

The operator π_K will be used to transform N-tuples of units from T to K-tuples of units and N-tuples of unit-label pairs from R to K-tuples of unit-label pairs. Notice that it follows directly from the definition of the π_K operator that $S \subseteq R$ implies $\pi_K S \subseteq \pi_K R$.

We now define the generalized look-ahead operator Ψ_{KP}.

Let $U = \{1, \cdots, M\}$ be a set of units, L be a set of labels, $T \subseteq U^N$, $R \subseteq (U \times L)^N$, $K \leqslant N \leqslant P$, $K < P$, and $D \subseteq (U \times L)^K$. The *look-ahead operator* Ψ_{KP} is defined by

$$\begin{aligned}\Psi_{KP}D = \{(u_1, l_1, \cdots, u_K, l_K) \in D \mid & \text{ for every } u_{K+1}, \cdots, u_P \in U \text{ there exist labels } l_{K+1}, \cdots, l_P \in L \text{ such that}\\ & 1)\ (l_1, \cdots, l_K, l_{K+1}, \cdots, l_P) \text{ is a } (T,R)\text{-consistent labeling of } (u_1, \cdots, u_K, u_{K+1}, \cdots, u_P),\\ & 2)\ \text{for every combination } j_1, \cdots, j_N \text{ of } 1, \cdots, P \text{ satisfying } (u_{j_1}, \cdots, u_{j_N}) \in T,\\ & \text{then } \pi_K(u_{j_1}, l_{j_1}, \cdots, u_{j_N}, l_{j_N}) \in D\}.\end{aligned}$$

As with the ϕ_{KP} operator, the Ψ_{KP} operator is stronger for larger values of P.

Proposition 7: Let $R \subseteq (U \times L)^N$, $T \subseteq U^N$, $K \leqslant N \leqslant P$ and $P > K$, and $D \subseteq (U \times L)^K$. Then $\Psi_{KP+1}(D) \subseteq \Psi_{KP}(D)$.

The example in Fig. 2 helps compare the power between the ϕ and Ψ operators. It shows a case where ϕ_{23} is strictly stronger than ϕ_{13} or Ψ_{13} indicating that ϕ operators with larger K's can be more powerful than Ψ operators with the same fixed P. It is also possible to construct examples where operators like Ψ_{23} are more powerful than ϕ_{13}. In the latter part of this section, we will show the equivalence between ϕ_{KP} and Ψ_{KP}. But before we do this, we characterize some more properties of Ψ_{KP}. As we defined a minimal relation S_{TR} which was related to the fixed point of ϕ_{KP}, we can define a minimal set of K-tuples A_{TR} and relate it to the fixed point of Ψ_{KP}. Let $R \subseteq (U \times L)^N$, $T \subseteq U^N$, $K \leqslant N \leqslant P$, and $P > K$. We define the set $A_{TR} \subseteq (U \times L)^K$ by

$$\begin{aligned}A_{TR} = \{(u_1, c_1, \cdots, u_K, c_K) & \in (U \times L)^K \mid (u_1, \cdots, u_K) \in \pi_K T\\ & \text{and for some labeling}\\ & (l_1, \cdots, l_M) \in \mathcal{L}(T,R), c_i = l_{u_i}, i = 1, \cdots, K\}.\end{aligned}$$

The following relations exist between Ψ_{KP} and A_{TR}: repeated application of Ψ_{KP} cannot reduce $\pi_K R$ to something smaller than A_{TR} and A_{TR} is a fixed point of Ψ_{KP}.

Theorem 9: Let $R \subseteq (U \times L)^N$ and $T \subseteq U^N$. Let $K \leqslant N \leqslant P$ and $K < P$. Let A_{TR} be as above. Then $A_{TR} \subseteq \Psi_{KP}^m(\pi_K R)$ for every nonnegative integer m.

Proof: The proof is by induction on m. For $m = 0$, we must show that $A_{TR} \subseteq \pi_K(R)$. Let $(u_1, c_1, \cdots, u_K, c_K) \in A_{TR}$. Then $(u_1, \cdots, u_K) \in \pi_K(T)$ and there exists a (T,R)-consistent labeling $(l_1, \cdots, l_M)$ of $(1, \cdots, M)$ such that $c_i = l_{u_i}$, $i = 1, \cdots, K$. Since $(l_1, \cdots, l_M)$ is a consistent labeling, for every $(v_1, \cdots, v_N) \in T$, $(v_1, l_{v_1}, \cdots, v_N, l_{v_N})$ is in R. Since $(u_1, \cdots, u_K)$ is in $\pi_K(T)$, there exists $(v_1, \cdots, v_N) \in T$ and a combination $j_1, \cdots, j_K$ of $1, \cdots, N$ such that $u_i = v_{j_1}$, $i = 1, \cdots, K$. But since $(v_1, \cdots, v_N) \in T$, and $(l_1, \cdots, l_M)$ is a consistent labeling, $(v_1, l_{v_1}, \cdots, v_N, l_{v_N}) \in R$, and $(u_1, l_{u_1}, \cdots, u_K, l_{u_K}) \in \pi_K R$. Since $c_i = l_{u_i}$, $i = 1, \cdots, K$, we have $(u_1, c_1, \cdots, u_K, c_K) \in \pi_K R$.

Suppose $A_{TR} \subseteq \Psi_{KP}^m(\pi_K R)$, for some $m \geqslant 0$ and let $(u_1, c_1, \cdots, u_K, c_K) \in A_{TR}$. Then $(u_1, c_1, \cdots, u_K, c_K) \in \Psi_{KP}^m(\pi_K R)$. Let $u_{K+1}, \cdots, u_P \in U$. Since $(u_1, c_1, \cdots, u_K, c_K) \in A_{TR}$, there exists a (Y, R)-consistent labeling $(l_1, \cdots, l_M)$ of $(1, \cdots, M)$ such that $c_i = l_{u_i}$, $i = 1, \cdots, K$. This implies that $(l_{u_1}, \cdots, l_{u_K}, l_{u_{K+1}}, \cdots, l_{u_P})$ is a (T,R)-consistent labeling of $(u_1, \cdots, u_K, u_{K+1}, \cdots, u_P)$. Now suppose that $j_1, \cdots, j_N$ is a combination of $1, \cdots, P$ satisfying $(u_{j_1}, \cdots, u_{j_K}) \in T$. By definition of A_{TR}, we have $(u_{j_1}, l_{j_1}, \cdots, u_{j_K}, l_{j_K}) \in A_{TR}$. But $A_{TR} \subseteq \Psi_{KP}^m(\pi_K R)$ so that $(u_{j_1}, l_{j_1}, \cdots, u_{j_K}, l_{j_K}) \in \Psi_{KP}^m(\pi_K R)$. By definition of Ψ_{KP},

$$(u_1, c_1, \cdots, u_K, c_K) \in \Psi_{KP}(\Psi_{KP}^m(\pi_K R)) = \Psi_{KP}^{m+1}(\pi_K R).$$

Hence, $A_{TR} \subseteq \Psi_{KP}^{m+1}(\pi_K R)$ and by induction

$$A_{TR} \subseteq \Psi_{KP}^m(\pi_K R)$$

for every nonnegative m.

Theorem 10: $\Psi_{KP}(A_{TR}) = A_{TR}$.

Proof: Let $(u_1, c_1, \cdots, u_K, c_K) \in A_{TR}$. Then $(u_1, \cdots, u_K) \in \pi_K T$ and there exists a (T, R)-consistent labeling $(l_1, \cdots, l_M)$ of $1, \cdots, M$ such that $c_i = l_{u_i}$, $i = 1, \cdots, K$. Let $u_{K+1}, \cdots, u_P \in U$ and let $j_1, \cdots, j_K$ be a combination of $1, \cdots, P$. Then $(l_{u_{j_1}}, \cdots, l_{u_{j_K}}, l_{u_{K+1}}, \cdots, l_{u_P})$ is a (T,R)-consistent labeling of $(u_{j_1}, \cdots, u_{j_K}, u_{K+1}, \cdots, u_P)$. Now let $(u_{j_1}, \cdots, u_{j_K}) \in T$. By definition of A_{TR}, $(u_{j_1}, l_{u_{j_1}}, \cdots, u_{j_K}, l_{u_{j_K}}) \in A_{TR}$.

Now by definition of Ψ_{KP}, $(u_{j_1}, l_{u_{j_1}}, \cdots, u_{j_K}, l_{u_{j_K}}) \in \Psi_{KP}(A_{TR})$. Since $1, \cdots, K$ is a combination of $1, \cdots, N$, $(u_1, l_{u_1}, \cdots, u_K, l_{u_K}) \in \Psi_{KP}(A_{TR})$. But $l_{u_i} = c_i$ so that $(u_1, c_1, \cdots, u_K, c_K) \in \Psi_{KP}(A_{TR})$.

If it were the case that $\pi_K \phi_{KP}^\infty R = \Psi_{KP}^\infty(\pi_K R)$ and if it were, therefore, possible to construct $\phi_{KP}^\infty(R)$ from $\Psi_{KP}^\infty(\pi_K R)$, we might choose to do so since it might be more storage efficient to save the projections and work with the Ψ_{KP} operator rather than with the ϕ_{KP} operator.

There is a direct relationship between Ψ_{KP} and ϕ_{KP} which can easily be proven with the help of the definition of a generalized restriction. If $R \subseteq (U \times L)^N$ and $H \subseteq (U \times L)^K$, we define the *restriction of R by H* as

$$R|_H = \{(u_1, l_1, \cdots, u_N, l_N) \in R \mid \pi_K(u_1, l_1, \cdots, u_N, l_N) \in H\}.$$

$R|_H$ is that part of R whose projection is in H. The following properties are immediately derivable from the definition of $R|_H$:

1) $R = R|_H$ implies $\pi_K R \subseteq H$,
2) $R = R|_{\pi_K R}$,
3) $S \subseteq R$ implies $S \subseteq R|_{\pi_K S}$, and
4) $\pi_K R|_H \subseteq H$.

Theorem 11 states that $\phi_{KP} R$ can be recovered from $\Psi_{KP} \pi_K R$ via the restriction: $\phi_{KP} R = R|_{\Psi_{KP}(\pi_K R)}$. Corollary 4 uses this fact together with the properties of the generalized restriction to prove that ϕ_{KP} cannot remove any N-tuples from R if and only if Ψ_{KP} cannot remove anything from R: $R = \phi_{KP} R$ if and only if $\Psi_{KP}(\pi_K R) = \pi_K R$. Thus R is a fixed point of ϕ_{KP} if and only if its projection $\pi_K R$ is a fixed point of Ψ_{KP}.

Theorem 11: $\phi_{KP}R = R|_{\Psi_{KP}(\pi_K R)}$.

Proof: First we show $\phi_{KP}R \subseteq R|_{\Psi_{KP}(\pi_K R)}$. Suppose $(u_1, l_1, \cdots, u_N, l_N) \in \phi_{KP}R$. Then $(u_1, l_1, \cdots, u_N, l_N) \in R$. Let $j_1, \cdots, j_K$ be a combination of $1, \cdots, N$. Since $(u_1, l_1, \cdots, u_N, l_N) \in R$, $(u_{j_1}, l_{j_1}, \cdots, u_{j_K}, l_{j_K}) \in \pi_K R$. To complete the first demonstration, we must show that $(u_{j_1}, l_{j_1}, \cdots, u_{j_K}, l_{j_K}) \in \Psi_K(\pi_K R)$. Let $u'_{K+1}, \cdots, u'_P \in U$. Since $j_1, \cdots, j_K$ is a combination of $1, \cdots, N$, $u'_{K+1}, \cdots, u'_P \in U$, and $(u_1, l_1, \cdots, u_N, l_N) \in \phi_{KP}R$, by definition of ϕ_{KP}, there exist labels $l'_{K+1}, \cdots, l'_P \in L$ such that $(l_{j_1}, \cdots, l_{j_K}, l'_{K+1}, \cdots, l'_P)$ is a (T, R)-consistent labeling of $(u_{j_1}, \cdots, u_{j_K}, u'_{K+1}, \cdots, u'_P)$.

To simplify the notation, let

$$v_p = \begin{cases} u_{j_p} & 1 \leq p \leq K \\ u'_p & K < p \leq P \end{cases}$$

$$l^*_p = \begin{cases} l_{j_p} & 1 \leq p \leq K \\ l'_p & K < p \leq P. \end{cases}$$

Then we have $(l^*_1, \cdots, l^*_P)$ is a (T, R)-consistent labeling of $(v_1, \cdots, v_P)$. To complete the proof that $(u_{j_1}, l_{j_1}, \cdots, u_{j_K}, l_{j_K}) \in \Psi_{KP}(\pi_K R)$, we must show that for every combination $i_1, \cdots, i_N$ of $1, \cdots, P$ satisfying $(v_{i_1}, \cdots, v_{i_N}) \in T$, $\pi_K(v_{i_1}, l^*_{i_1}, \cdots, v_{i_N}, l^*_{i_N}) \in \pi_K R$.

Let $i_1, \cdots, i_N$ be a combination of $1, \cdots, P$ satisfying $(v_{i_1}, \cdots, v_{i_N}) \in T$. Since $(l^*_1, \cdots, l^*_P)$ is a (T, R)-consistent labeling of $(v_1, \cdots, v_P)$ and $(v_{i_1}, \cdots, v_{i_N}) \in T$, $(v_{i_1}, l^*_{i_1}, \cdots, v_{i_N}, l^*_{i_N}) \in R$. Thus $\pi_K(v_{i_1}, l^*_{i_1}, \cdots, v_{i_N}, l^*_{i_N}) \in \pi_K R$. Therefore, $(u_{j_1}, l_{j_1}, \cdots, u_{j_K}, l_{j_K}) \in \Psi_{KP}(\pi_K R)$, $(u_1, l_1, \cdots, u_N, l_N) \in R|_{\Psi_{KP}(\pi_K R)}$, and $\phi_{KP}R \subseteq R|_{\Psi_{KP}(\pi_K R)}$.

Suppose $(u_1, l_1, \cdots, u_N, l_N) \in R|_{\Psi_{KP}(\pi_K R)}$. Let $j_1, \cdots, j_K$ be a combination of $1, \cdots, N$ and let $u'_{K+1}, \cdots, u'_P \in U$. Since $(u_1, l_1, \cdots, u_N, l_N) \in R|_{\Psi_{KP}(\pi_K R)}$, $(u_{j_1}, l_{j_1}, \cdots, u_{j_K}, l_{j_K}) \in \Psi_{KP}(\pi_K R)$. By definition of Ψ_{KP}, there exist labels $l'_{K+1}, \cdots, l'_P \in L$ such that $(l_{j_1}, \cdots, l_{j_K}, l'_{K+1}, \cdots, l'_P)$ is a (T, R)-consistent labeling of $(u_{j_1}, \cdots, u_{j_K}, u'_{K+1}, \cdots, u'_P)$. Thus $(u_1, l_1, \cdots, u_N, l_N) \in \phi_{KP}R$ and $R|_{\Psi_{KP}(\pi_K R)} \subseteq \phi_{KP}R$.

Corollary 4: $\phi_{KP}R = R$ if and only if $\Psi_{KP}(\pi_K R) = \pi_K R$.

Proof: Suppose $\phi_{KP}R = R$. By the theorem, $\phi_{KP}R = R|_{\Psi_{KP}(\pi_K R)}$. Hence, $R = R|_{\Psi_{KP}(\pi_K R)}$. But by property 1) $R = R|_{\Psi_{KP}(\pi_K R)}$ implies $\pi_K R \subseteq \Psi_{KP}(\pi_K R)$. Since by definition of Ψ_{KP}, $\Psi_{KP}(\pi_K R) \subseteq \pi_K R$, we obtain that $\pi_K R = \Psi_{KP}(\pi_K R)$.

Suppose $\Psi_{KP}(\pi_K R) = \pi_K R$. Then $R|_{\Psi_{KP}(\pi_K R)} = R|_{\pi_K R}$. But by the theorem, $\phi_{KP}R = R|_{\Psi_{KP}(\pi_K R)}$. Since by property 2) $R|_{\pi_K R} = R$, we obtain $\phi_{KP}R = R$.

In order to derive the relationship between $\phi^\infty_{KP}R$ and $\Psi^\infty_{KP}(\pi_K R)$, we will first have to have some characterization of $\phi^\infty_{KP}R$ and $\Psi^\infty_{KP}(\pi_K R)$. The first one we prove (Theorem 12) says that R restricted to $\Psi^\infty_{KP}(\pi_K R)$ cannot achieve any further reductions than Ψ^∞_{KP} itself can. The mild condition between R and T required is one that we have used before: R has no "excess baggage"; $(u_1, l_1, \cdots, u_N, l_N) \in R$ implies $(u_1, \cdots, u_N) \in T$. It will follow immediately from this abovementioned result $\pi_K R|_{\Psi^\infty_{KP}(\pi_K R)} = \Psi^\infty_{KP}(\pi_K R)$ that

1) $\pi_K R|_{\Psi^\infty_{KP}(\pi_K R)}$ is a fixed point under Ψ_{KP} (Corollary 5),

2) $R|_{\Psi^\infty_{KP}(\pi_K R)}$ is a fixed point under ϕ_{KP} (Corollary 6),

and

3) $R|_{\Psi^\infty_{KP}(\pi_K R)} \subseteq \phi^\infty_{KP}R$. (Corollary 7).

Theorem 12: Let $R \subseteq (U \times L)^N$ and $T \subseteq U^N$ satisfy $(u_1, l_1, \cdots, u_N, l_N) \in R$ implies $(u_1, \cdots, u_N) \in T$. Let $K \leq N \leq P$ and $K < P$. Then $\pi_K R|_{\Psi^\infty_{KP}(\pi_K R)} = \Psi^\infty_{KP}(\pi_K R)$.

Proof: By property 4), $\pi_K R|_{\Psi^\infty_{KP}(\pi_K R)} \subseteq \Psi^\infty_{KP}(\pi_K R)$. Suppose $(u_1, l_1, \cdots, u_K, l_K) \in \Psi^\infty_{KP}(\pi_K R)$. Then $(u_1, l_1, \cdots, u_K, l_K) \in \pi_K R$. Since $(u_1, l_1, \cdots, u_K, l_K) \in \pi_K R$, there exists $u_{K+1}, \cdots, u_N \in U$ and $l_{K+1}, \cdots, l_N \in L$ such that $(u_1, l_1, \cdots, u_N, l_N) \in R$. Now let $u_{N+1}, \cdots, u_P \in U$. Because $(u_1, l_1, \cdots, u_K, l_K) \in \Psi^\infty_{KP}(\pi_K R)$ and $u_{K+1}, \cdots, u_P \in U$, there exist $l'_{K+1}, \cdots, l'_P \in L$ such that $(l_1, \cdots, l_K, l'_{K+1}, \cdots, l'_P)$ is a (T, R)-consistent labeling of $(u_1, \cdots, u_K, u_{K+1}, \cdots, u_P)$ and $i_1, \cdots, i_N$ a combination of $1, \cdots, P$ satisfying $(u_{i_1}, \cdots, u_{i_N}) \in T$ implies $\pi_K(u_{i_1}, l_{i_1}, \cdots, u_{i_N}, l_{i_N}) \in \Psi^\infty_{KP}(\pi_K R)$. But by hypothesis $(u_1, l_1, \cdots, u_N, l_N) \in R$ implies $(u_1, \cdots, u_N) \in T$. Hence, $\pi_K(u_1, l_1, \cdots, u_N, l_N) \subseteq \Psi^\infty_{KP}(\pi_K R)$. Now, by definition, $(u_1, l_1, \cdots, u_N, l_N) \in R$ and $\pi_K(u_1, l_1, \cdots, u_N, l_N) \subseteq \Psi^\infty_{KP}(\pi_K R)$ imply $(u_1, l_1, \cdots, u_N, l_N) \in R|_{\Psi^\infty_{KP}(\pi_K R)}$ so that

$$(u_1, l_1, \cdots, u_K, l_K) \in \pi_K R|_{\Psi^\infty_{KP}(\pi_K R)}.$$

Therefore, $\Psi^\infty_{KP}(\pi_K R) \subseteq \pi_K R|_{\Psi^\infty_{KP}(\pi_K R)}$.

Corollary 5: $\Psi_{KP}(\pi_K R|_{\Psi^\infty_{KP}(\pi_K R)}) = \pi_K R|_{\Psi^\infty_{KP}(\pi_K R)}$.

Proof:

$$\Psi_{KP}(\pi_K R|_{\Psi^\infty_{KP}(\pi_K R)}) = \Psi_{KP}(\Psi^\infty_{KP}(\pi_K R)) = \Psi^\infty_{KP}(\pi_K R)$$

$$= \pi_K R|_{\Psi^\infty_{KP}(\pi_K R)}.$$

Corollary 6: $\phi_{KP}(R|_{\Psi^\infty_{KP}(\pi_K R)}) = R|_{\Psi^\infty_{KP}(\pi_K R)}$.

Proof: By Corollary 4,

$$\Psi_{KP}(\pi_K R|_{\Psi^\infty_{KP}(\pi_K R)}) = \pi_K R|_{\Psi^\infty_{KP}(\pi_K R)}.$$

By Corollary 3, $\Psi_{KP}(\pi_K S) = \pi_K S$ if and only if $\phi_{KP}S = S$. Hence $\phi_{KP}(R|_{\Psi^\infty_{KP}(\pi_K R)}) = R|_{\Psi^\infty_{KP}(\pi_K R)}$.

Corollary 7: Let $R \subseteq (U \times L)^N$, $T \subseteq U^N$, $K \leq N \leq P$, and $K < P$. If $(u_1, l_1, \cdots, u_N, l_N) \in R$ implies $(u_1, \cdots, u_N) \in T$, then $R|_{\Psi^\infty_{KP}(\pi_K R)} \subseteq \phi^\infty_{KP}R$.

Proof: Certainly $R|_{\Psi^\infty_{KP}(\pi_K R)} \subseteq R$. From this a simple induction will show that $\phi^\infty_{KP}(R|_{\Psi^\infty_{KP}(\pi_K R)}) \subseteq \phi^\infty_{KP}R$. But by Corollary 6, $\phi_{KP}(R|_{\Psi^\infty_{KP}(\pi_K R)}) = R|_{\Psi^\infty_{KP}(\pi_K R)}$. From this a simple induction will yield $R|_{\Psi^\infty_{KP}(\pi_K R)} = \phi^\infty_{KP}(R|_{\Psi^\infty_{KP}(\pi_K R)})$. Hence, we obtain $R|_{\Psi^\infty_{KP}(\pi_K R)} \subseteq \phi^\infty_{KP}R$.

In Theorem 13 we prove that the projection of the fixed point under repeated application of ϕ_{KP} must be contained in the fixed point under repeated application of Ψ_{KP} on the projection. That is, $\pi_K \phi_{KP}^{\infty} R \subseteq \Psi_{KP}^{\infty}(\pi_K R)$. Combining this result with the fact that $R|_{\Psi_{KP}^{\infty}(\pi_K R)} \subseteq \phi_{KP}^{\infty} R$ from Corollary 7 yields $\pi_K \phi_{KP}^{\infty} R = \Psi_{KP}^{\infty}(\pi_K R)$ (Corollary 8). Using the fact that $R|_{\pi_K \phi_{KP}^{\infty} R}$ is a fixed point under ϕ_{KP} (Corollary 6) and $\pi_K \phi_{KP}^{\infty} R = \Psi_{KP}^{\infty}(\pi_K R)$ yields that $R|_{\pi_K \phi_{KP}^{\infty} R}$ is a fixed point under ϕ_{KP} (Corollary 9). Finally, in Corollary 10, we prove the equivalence of the fixed-point power of ϕ_{KP} and Ψ_{KP}: $\phi_{KP}^{\infty} R = R|_{\Psi_{KP}^{\infty}(\pi_K R)}$. This means that we may choose to repeatedly apply ϕ_{KP} to a fixed point or Ψ_{KP} to a fixed point and from one, we can obtain the other via the relations

1) $\Psi_{KP}^{\infty}(\pi_K R) = \pi_K \phi_{KP}^{\infty} R$ and
2) $\phi_{KP}^{\infty} R = R|_{\Psi_{KP}^{\infty}(\pi_K R)}$.

Therefore, since ϕ_{KP} and Ψ_{KP} take the same amount of work to apply, the power of ϕ_{KP} and Ψ_{KP} is the same. The only difference is an implementation one concerning space and speed of convergence to the fixed point.

Theorem 13: Let $R \subseteq (U \times L)^N$ and $T \subseteq U^N$. Let $K \leqslant N \leqslant P$ and $K < P$. Then $\pi_K \phi_{KP}^{\infty} R \subseteq \Psi_{KP}^{\infty} \pi_K R$.

Proof: The proof is by induction. Let $(u_1, l_1, \cdots, u_K, l_K) \in \pi_K \phi_{KP}^{\infty} R$. Since $\phi_{KP}^{\infty} R \subseteq R$, then $\pi_K \phi_{KP}^{\infty} R \subseteq \pi_K R$. Hence, $(u_1, l_1, \cdots, u_K, l_K) \in \pi_K R = \Psi_{KP}^{\infty}(\pi_K R)$.

Suppose $(u_1, l_1, \cdots, u_K, l_K) \in \pi_K \phi_{KP}^{\infty} R$ implies $(u_1, l_1, \cdots, u_K, l_K) \in \Psi_{KP}^{m} \pi_K R$ for some $m \geqslant 0$. Then if $(v_1, l_1^*, \cdots, v_N, l_N^*) \in \phi_{KP}^{\infty} R$, $\pi_K(v_1, l_1^*, \cdots, v_N, l_N^*) \in \Psi_{KP}^{m} \pi_K R$. Let $(u_1, l_1, \cdots, u_K, l_K) \in \pi_K \phi_{KP}^{\infty} R$. Then there exists $(v_1, l_1^*, \cdots, v_N, l_N^*) \in \phi_{KP}^{\infty} R$ satisfying $(u_1, l_1, \cdots, u_K, l_K) \in \pi_K(v_1, l_1^*, \cdots, v_N, l_N^*)$. Let $u_{K+1}, \cdots, u_P \in U$. By definition of ϕ_{KP}, $(v_1, l_1^*, \cdots, v_N, l_N^*) \in \phi_{KP}^{\infty} R$, $(u_1, l_1, \cdots, u_K, l_K) \in \pi_K(v_1, l_1^*, \cdots, v_N, l_N^*)$ and $u_{K+1}, \cdots, u_P \in U$ imply there exist labels $l_{K+1}, \cdots, l_P$ such that $(l_1, \cdots, l_P)$ is a $(T, \phi_{KP}^{\infty} R)$-consistent labeling of $(u_1, \cdots, u_P)$.

Now let $j_1, \cdots, j_N$ be a combination of $1, \cdots, P$ satisfying $(u_{j_1}, \cdots, u_{j_N}) \in T$. Since $(l_1, \cdots, l_P)$ is a $(T, \phi_{KP}^{\infty} R)$-consistent labeling of $(u_1, \cdots, u_P)$ and $(u_{j_1}, \cdots, u_{j_N}) \in T$, then by definition of $(T, \phi_{KP}^{\infty} R)$-consistency, $(u_{j_1}, l_{j_1}, \cdots, u_{j_N}, l_{j_N}) \in \phi_{KP}^{\infty} R$. But by supposition, $(u_{j_1}, l_{j_1}, \cdots, u_{j_N}, l_{j_N}) \in \phi_{KP}^{\infty} R$ implies $\pi_K(u_{j_1}, l_{j_1}, \cdots, u_{j_N}, l_{j_N}) \in \Psi_{KP}^{m} \pi_K R$.

By definition of Ψ_{KP}, $(u_1, l_1, \cdots, u_K, l_K) \in \Psi_{KP}^{m+1} \pi_K R$. Hence, by induction $\pi_K \phi_{KP}^{\infty} R \subseteq \Psi_{KP}^{m} \pi_K R$ for every $m \geqslant 0$. Then by definition of Ψ_{KP}^{∞}, $\pi_K \phi_{KP}^{\infty} R \subseteq \Psi_{KP}^{\infty} \pi_K R$.

Corollary 8: Suppose $(u_1, l_1, \cdots, u_N, l_N) \in R$ implies $(u_1, \cdots, u_N) \in T$. Then $\Psi_{KP}^{\infty}(\pi_K R) = \pi_K \phi_{KP}^{\infty} R$.

Proof: By Corollary 7, $R|_{\Psi_{KP}^{\infty}(\pi_K R)} \subseteq \phi_{KP}^{\infty} R$. Hence, $\pi_K R|_{\Psi_{KP}^{\infty}(\pi_K R)} \subseteq \pi_K \phi_{KP}^{\infty} R$. But by Theorem 12,

$$\pi_K R|_{\Psi_{KP}^{\infty}(\pi_K R)} = \Psi_{KP}^{\infty}(\pi_K R).$$

Hence, $\Psi_{KP}^{\infty}(\pi_K R) \subseteq \pi_K \phi_{KP}^{\infty} R$. Now by this theorem

$$\pi_K \phi_{KP}^{\infty} R \subseteq \Psi_{KP}^{\infty}(\pi_K R).$$

Therefore, $\Psi_{KP}^{\infty}(\pi_K R) = \pi_K \phi_{KP}^{\infty} R$.

Corollary 9: Suppose $(u_1, l_1, \cdots, u_N, l_N) \in R$ implies $(u_1, \cdots, u_N) \in T$. Then $\phi_{KP}(R|_{\pi_K \phi_{KP}^{\infty} R}) = R|_{\pi_K \phi_{KP}^{\infty} R}$.

Proof: By Corollary 6, $\phi_{KP}(R|_{\Psi_{KP}^{\infty}(\pi_K R)}) = R|_{\Psi_{KP}^{\infty}(\pi_K R)}$. By Corollary 8, $\Psi_{KP}^{\infty}(\pi_K R) = \pi_K \phi_{KP}^{\infty} R$. Simple substitution of this into the above relation yields $\phi_{KP}(R|_{\pi_K \phi_{KP}^{\infty} R}) = R|_{\pi_K \phi_{KP}^{\infty} R}$.

Corollary 10: Suppose $(u_1, l_1, \cdots, u_N, l_N) \in R$ implies $(u_1, \cdots, u_N) \in T$. Then $\phi_{KP}^{\infty} R = R|_{\Psi_{KP}^{\infty}(\pi_K R)}$.

Proof: Certainly, $R|_{\pi_K \phi_{KP}^{\infty} R} \subseteq R$. Using induction upon repeated application of ϕ_{KP} yields $\phi_{KP}^{\infty}(R|_{\pi_K \phi_{KP}^{\infty} R}) \subseteq \phi_{KP}^{\infty} R$. But by property 3), $\phi_{KP}^{\infty} R \subseteq R|_{\pi_K \phi_{KP}^{\infty} R}$. Now using induction upon repeated application of ϕ_{KP} yields

$$\phi_{KP}^{\infty} R \subseteq \phi_{KP}^{\infty}(R|_{\pi_K \phi_{KP}^{\infty} R}).$$

Therefore, $\phi_{KP}^{\infty} R = \phi_{KP}^{\infty}(R|_{\pi_K \phi_{KP}^{\infty} R})$. By Corollary 9,

$$R|_{\pi_K \phi_{KP}^{\infty} R}$$

is a fixed point under ϕ, hence $\phi_{KP}^{\infty}(R|_{\pi_K \phi_{KP}^{\infty} R}) = R|_{\pi_K \phi_{KP}^{\infty} R}$. By Corollary 8, $\Psi_{KP}^{\infty}(\pi_K R) = \pi_K \phi_{KP}^{\infty} R$. Therefore, $\phi_{KP}^{\infty} R = R|_{\Psi_{KP}^{\infty}(\pi_K R)}$.

We close this section by noting that the original Waltz filtering algorithm [11] is a sequential version of Ψ_{12}. The Rosenfeld scene labeling algorithm [9] is a parallel version exactly like Ψ_{12}. The Ψ_P operator described by Haralick *et al.* [4] is the Ψ_{1P} operator of this paper and the refinement procedure of Ullman [10] used to find subgraph isomorphisms is the Ψ_{12} operator of this paper.

IV. Complexity

In this section we show the computational importance of removing from R all those N-tuples of unit-label pairs which do not contribute to any (T, R)-consistent labelings. We show that a standard tree search for finding all (T, R)-consistent labelings of a unit-label relation R having only N-tuples of unit-label pairs which contribute to (T, R)-consistent labelings can be done quicker for smaller relations than larger ones. This motivates the empirically good results that Waltz [11] obtained: evidently for scene edge labeling problems, the Ψ_{12} look-ahead operator is powerful enough to reduce the unit-label constraint relation to a minimal or near minimal one at each node in the tree so that the tree search becomes easy to do.

Our first task will be to define the concept of minimal relation. Let $T \subseteq U^N$ and $R \subseteq (U \times L)^N$. R is *minimal* with respect to T if and only if $Q \subseteq R$ and $\mathcal{L}(T, Q) = \mathcal{L}(T, R)$ imply $Q = R$. It is immediate from the definition of S_{TR} (the minimal relation with respect to T and R) in Section IV that S_{TR} is a minimal relation with respect to T and that R minimal with respect to T implies R minimal with respect to T and R. Hence, by Theorem 4 of Section II:

1) all N-tuples of unit-label pairs in a minimal R participate in a (T, R)-consistent labeling and
2) unit-labeling constraint relations in which all N-tuples of

unit-label pairs participate in a (T, R)-consistent labeling and in which $(u_1, l_1, \cdots, u_N, l_N) \in R$ implies $(u_1, \cdots, u_N) \in T$ must be minimal relations with respect to T.

Therefore, we may take Theorem 4 as a characterization of minimal relations: if $(u_1, l_1, \cdots, u_N, l_N) \in R$ implies $(u_1, \cdots, u_N) \in T$, then R is minimal with respect to T if and only if all N-tuples of unit-label pairs in R contribute to a (T, R)-consistent labeling.

Now consider the tree search required to find consistent labelings. At each node in the tree search we must restrict some unit to a particular label. This restriction is actually applied to R by throwing out any N-tuples of unit-label pairs in which the specified unit does not have the designated label. Such a restriction in R, of course, restricts in a similar manner the set of consistent labelings that can be found below this node in the tree search.

In order to be able to discuss these kinds of restrictions, we need to adopt a convenient notation. Let $(l_1, \cdots, l_P)$ be a labeling of $(u_1, \cdots, u_P)$. We define the restriction of R to this labeling by

$$\mathcal{L}(T, R)|_{u_1 l_1, \cdots, u_P l_P} = \{(l_1', \cdots, l_M') \in \mathcal{L}(T, R) \mid l_{u_p}' = l_p, \; p = 1, \cdots, P\}.$$

We should then expect that

$$\mathcal{L}(T, R)|_{u_1 l_1, \cdots, u_P l_P} = \mathcal{L}(T, R|_{u_1 l_1, \cdots, u_P l_P}).$$

This is not true unconditionally. Proposition 8 proves that $\mathcal{L}(T, R)|_{u_1 l_1, \cdots, u_P l_P} \subseteq \mathcal{L}(T, R|_{u_1 l_1, \cdots, u_P l_P})$. However, because the only N-tuples of unit-label pairs that have to be checked for consistency are those whose N-tuples of units are in T, it is not necessarily true that $\mathcal{L}(T, R|_{u_1 l_1, \cdots, u_P l_P}) \subseteq \mathcal{L}(T, R)|_{u_1 l_1, \cdots, u_P l_P}$. In order for this to be true, T must include at least one N-tuple of units involving each unit in U. We say that T *covers* U if and only if for every $u \in U$, there exists an N-tuple $(u_1, \cdots, u_N) \in T$ satisfying $u = u_n$ for some $n \in \{1, \cdots, N\}$. When T covers U, Proposition 9 proves that $\mathcal{L}(T, R|_{u_1 l_1, \cdots, u_P l_P}) \subseteq \mathcal{L}(T, R)|_{u_1 l_1, \cdots, u_P l_P}$. In summary, therefore, we have Theorem 13 which states that if T covers U, then $\mathcal{L}(T, R)|_{u_1 l_1, \cdots, u_P l_P} = \mathcal{L}(T, R|_{u_1 l_1, \cdots, u_P l_P})$.

Proposition 8: Let $T \subseteq U^N$, $R \subseteq (U \times L)^N$ and $P > 0$. Suppose $(l_1, \cdots, l_P)$ is a labeling of $(u_1, \cdots, u_P)$. Then $\mathcal{L}(T, R)|_{u_1 l_1, \cdots, u_P l_P} \subseteq \mathcal{L}(T, R|_{u_1 l_1, \cdots, u_P l_P})$.

Proof: Let $(l_1', \cdots, l_M') \in \mathcal{L}(T, R)|_{u_1 l_1, \cdots, u_P l_P}$. Then $l_{u_p}' = l_p$, $p = 1, \cdots, P$. Suppose $(v_1, \cdots, v_N) \in T$. Then by definition of $\mathcal{L}$, $(v_1, l_{v_1}', \cdots, v_N, l_{v_N}') \in R$. If for some m and p, $v_m = u_p$, then $l_p = l_{u_p}' = l_{v_m}'$. Thus, by definition of $R|_{u_1 l_1, \cdots, u_P l_P}$, $(v_1, l_{v_1}', \cdots, v_N, l_{v_N}') \in R|_{u_1 l_1, \cdots, u_P l_P}$. Now by definition of $\mathcal{L}$, $(l_1', \cdots, l_M') \in \mathcal{L}(T, R|_{u_1 l_1, \cdots, u_P l_P})$.

Proposition 9: Let $T \subseteq U^N$ cover U, $R \subseteq (U \times L)^N$ and $P > 0$. Suppose $(l_1, \cdots, l_P)$ is a labeling of $(u_1, \cdots, u_P)$. Then $\mathcal{L}(T, R|_{u_1 l_1, \cdots, u_P l_P}) \subseteq \mathcal{L}(T, R)|_{u_1 l_1, \cdots, u_P l_P}$.

Proof: Let $(l_1', \cdots, l_M') \in \mathcal{L}(T, R|_{u_1 l_1, \cdots, u_P l_P})$. Since $R|_{u_1 l_1, \cdots, u_P l_P} \subseteq R$, $\mathcal{L}(T, R|_{u_1 l_1, \cdots, u_P l_P}) \subseteq \mathcal{L}(T, R)$. Hence, $(l_1', \cdots, l_M') \in \mathcal{L}(T, R)$. Now we only have to show that $l_{u_p}' = l_p, p = 1, \cdots, P$.

Since T covers U, for every $u \in U$, there exists some $(v_1, \cdots, v_N) \in T$ satisfying $v_m = u$ for some $m = 1, \cdots, N$. So let $p \in \{1, \cdots, P\}$ be given. Then there exists $(v_1, \cdots, v_N) \in T$ satisfying $v_m = u_p$ for some $m \in \{1, \cdots, N\}$. Now $(v_1, \cdots, v_N) \in T$ and $(l_1', \cdots, l_M') \in \mathcal{L}(T, R|_{u_1 l_1, \cdots, u_P l_P})$ implies $(v_1, l_{v_1}', \cdots, v_N, l_{v_N}') \in R|_{u_1 l_1, \cdots, u_P l_P}$. But $v_m = u_p$ so that $l_{v_m}' = l_p$. Thus $l_{u_p}' = l_{v_m}' = l_p$. Therefore $(l_1', \cdots, l_M') \in \mathcal{L}(T, R)|_{u_1 l_1, \cdots, u_P l_P}$.

Theorem 14: Let $T \subseteq U^N$ cover U, $R \subseteq (U \times L)^N$, and $P > 0$. Suppose $(l_1, \cdots, l_P)$ is a labeling of $(u_1, \cdots, u_P)$. Then $\mathcal{L}(T, R|_{u_1 l_1, \cdots, u_P l_P}) = \mathcal{L}(T, R)|_{u_1 l_1, \cdots, u_P l_P}$.

Proof: Propositions 8 and 9.

Now suppose that we start a tree search with a nonempty minimal R. Then every N-tuple of unit-label pairs in R is extendable to a (T, R)-consistent labeling and the set $L_u = \{l \in L \mid (u, l) \in \pi_1 R\}$ is nonempty for every unit u. In branching to lower nodes by restricting unit u to each of the labels in L_u, we are guaranteed that below every node to which we branch, we will be able to find at least one (T, R)-consistent labeling. Thus, there will not have to be any backtracking from these nodes due to a failure to find consistent labelings. Unfortunately, as Fig. 3 shows, it is the case that the restricted relations at each of the resulting nodes to which we branch are not necessarily minimal. Furthermore, as Fig. 4 shows, even application of ϕ_{NN+1} to a fixed point of a minimal relation that has been restricted does not necessarily produce minimal relations. Hence, this property of not having to backtrack cannot be guaranteed to propagate.

Although the properties of minimal relations are not as good as we might hope for, we can show that the closer a unit-label constraint relation is to being minimal, the less work the tree search will have. The analysis proceeds as follows.

The work at level $n + 1$ of a basic tree search consists of selecting for the next unit u_{n+1} all the labels that can be assigned to that unit which extend the (T, R)-consistent labeling $(l_1, \cdots, l_n)$ of units $(u_1, \cdots, u_n)$ above that level to a (T, R)-consistent labeling of units $(u_1, \cdots, u_{n+1})$. One not very efficient procedure is to try out each label l_{n+1} in L and check whether $(l_1, \cdots, l_{n+1})$ is a (T, R)-consistent labeling of $(u_1, \cdots, u_{n+1})$. Since we assume that $(l_1, \cdots, l_n)$ is already a (T, R)-consistent labeling of $(u_1, \cdots, u_n)$, we first need to find those N-tuples $(u_{i_1}, \cdots, u_{i_N}) \in T$ for which $i_1, \cdots, i_N$ is a combination of $1, \cdots, n + 1$ and for some k, $i_k = n + 1$. For each of these N-tuples the test of checking whether $(u_{i_1}, l_{i_1}, \cdots, u_{i_N}, l_{i_N}) \in R$ must be performed. The number of such tests cannot be greater than $\#T$. Hence at each node there can be at most $\#L \; \#T$ tests.

This procedure can be easily improved since we can determine with very little computation that certain labels cannot possibly contribute to a consistent labeling. For example, the only labels that could contribute to a consistent labeling are those labels which are associated with unit u_{n+1} in some N-tuple of $R|_{u_1 l_1, \cdots, u_n l_n}$. Let us call this set of labels $Q(u_{n+1} \mid u_1 l_1, \cdots, u_n l_n)$. The set is defined by

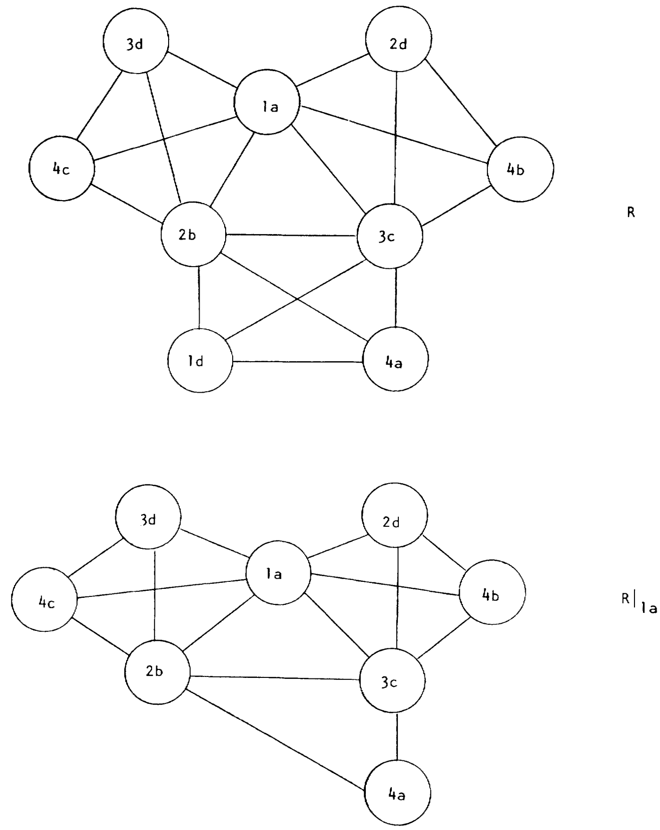

Fig. 3. Illustrates a minimal order 2 unit-label constraint relation R. Notice that $R|_{1a}$ is not minimal since $(2b, 3c)$, $(2b, 4a)$, and $(3c, 4a)$ are in $R|_{1a}$ but are not involved in any consistent labeling of $R|_{1a}$.

$$Q(u_{n+1} | u_1 l_1, \cdots, u_n l_n) = \{l \in L | \text{ for some } (v_1, c_1, \cdots, v_N, c_N) \in R|_{u_1 l_1, \cdots, u_n l_n}, v_k = u_{n+1} \text{ for some } k = 1, \cdots, N \text{ and } l = c_k\}.$$

The number of nodes in the tree search at level n can be written as

$$P_n = \sum_{l_1 \in Q(u_1)} \sum_{l_2 \in Q(u_2 | u_1 l_1)} \cdots \sum_{l_n \in Q_n(u_n | u_1 l_1, \cdots, u_{n-1}, l_{n-1})} 1$$

Hence, the total number of nodes visited in the tree is $\sum_{m=1}^{M} P_m$ and this times the upper bound $\#T$ of the number of tests which must be performed at each node yields $\#T \sum_{m=1}^{M} P_m$ as an upper bound for the number of tests required in the tree search. It is clear from the definition of Q that if $S \subseteq R$, then the Q associated with S at every step of the tree search will be smaller than the Q associated with R and the number of N-tuples from T that have to be tested will also be smaller. Therefore, the total number of tests for the smaller relation S will be fewer than for R. Since any S satisfying $\mathcal{L}(T, R) = \mathcal{L}(T, S)$ will yield the same consistent labelings, the smallest number of tests will be done for that relation S_{TR} which is minimal with respect to T and R. This result also holds if the look-ahead operators ϕ_{KP} or Ψ_{KP} are used at each node in the tree search since $S \subseteq R$ implies $\phi_{KP}(S) \subseteq \phi_{KP}(R)$ and $\Psi_{KP}(\pi_K S) \subseteq \Psi_{KP}(\pi_K R)$.

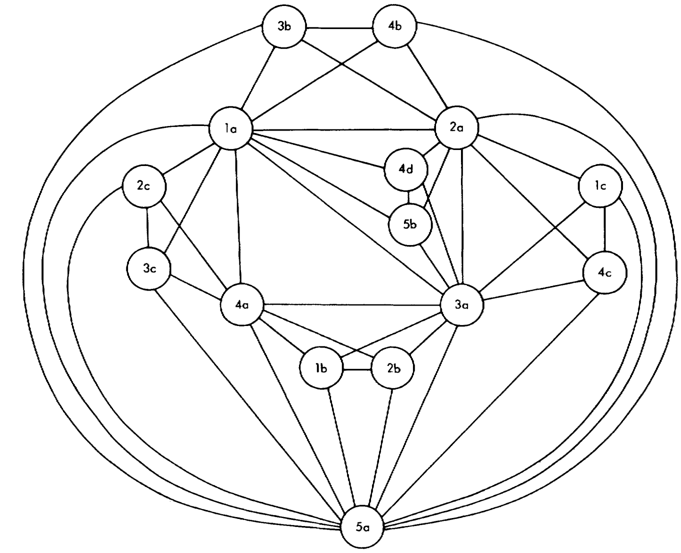

Fig. 4. Shows a minimal relation R having the property that $R|_{5a}$ is not minimal and $\phi_{NN+1}^{\infty}(R|_{5a})$ is not minimal. The unit-label pair $(1, a, 3, a)$ is in $\phi_{NN+1}^{\infty}(R|_{5a})$ but it does not participate in any $(T, R|_{1a})$-consistent labeling, where $T = U \times U$.

The use of the ϕ_{KP} or Ψ_{KP} operator at each node of the tree search has a chance to be effective only because it is a computationally cheap way of determining a smaller set of labels which are guaranteed to contain all the labels that might contribute to a consistent labeling. In other words, a small computational expense early has the opportunity to save us from a large computational expense later. However, the *NP* completeness of the labeling problem suggests that there are always pathologic cases which will force as complete a tree search as possible with failure to find consistent labelings regardless of the use of ϕ_{KP} or Ψ_{KP}. Therefore, a worst case complexity analysis will not really show any better results with the use of the look-ahead operators than without the use of the look-ahead operators.

This worst case complexity analysis is discouraging. The only positive fact that can be brought out is the extent to which a compatibility relation must be pathologically packed in order to render ϕ_{KP} or Ψ_{KP} ineffective. Fig. 5 shows the smallest order two unit-label constraint for which no consistent labelings exist and for which ϕ_{23} is ineffective. However, as soon as this relation is restricted in any way by the removal of even one pair from R, the ϕ_{23} operator reduces the restricted relation to nothing.

In order to understand what kind of situations ϕ_{KP} and Ψ_{KP} will not be effective in, we must examine properties of relations which can be fixed points of ϕ_{KP} or Ψ_{KP}. We proceed from the definition of ϕ_{KP} or Ψ_{KP}. Thus, $R = \phi_{KP} R$ if and only if $(u_1, l_1, \cdots, u_N, l_N) \in R$ implies that for every combination $i_1, \cdots, i_K$ of $1, \cdots, N$ and for every $u'_{K+1}, \cdots, u'_P \in U$, there exists labels $l'_{K+1}, \cdots, l'_P \in L$ such that for every combination $j_1, \cdots, j_N$ of $1, \cdots, P$ satisfying $(u'_{j_1}, \cdots, u'_{j_N}) \in T$, we must have $(u'_{j_1}, l'_{j_1}, \cdots, u'_{j_N}, l'_{j_N}) \in R$, where $u'_k = u_{i_k}$, $k = 1, \cdots, K$. Hence, if $N = 2$, $K = 2$, $P = 3$, and $T = U \times U$, then $(u_1, l_1, u_2, l_2) \in R$ and $R = \phi_{23} R$ implies that for every $u_3 \in U$ there exists a $l_3 \in L$ such that $(u_1, l_1,$

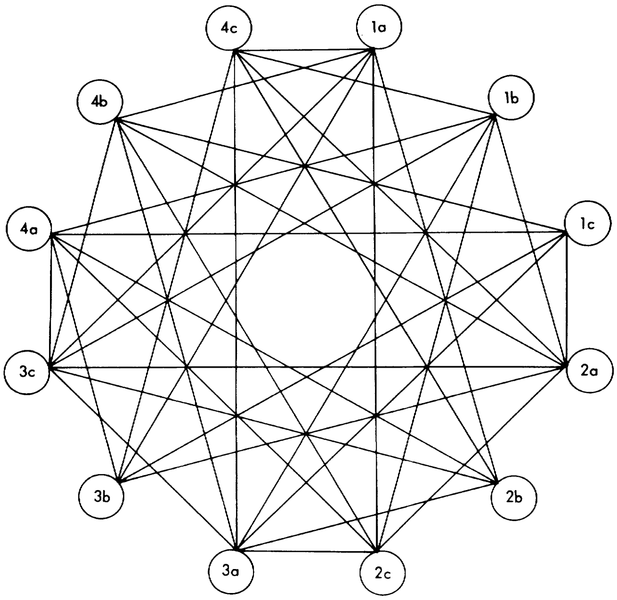

Fig. 5. Illustrates the smallest unit-label constraint relation R for which there are no (T, R)-consistent labelings and for which $\phi_{23}R = R$. For this situation $U = \{1, 2, 3, 4\}$, $L = \{a, b, c\}$, and $T = U \times U$.

$u_3, l_3) \in R$ and $(u_2, l_2, u_3, l_3) \in R$. Pictorially, if two given nodes are connected by an edge in the graph of R, then for each specified unit n, there must exist a node (u, l) which has edges that will form a triangle with the edge connecting the given two nodes. In practice, testing for this condition turns out to be a strong enough test that most graphs that do not contain consistent labelings are rapidly reduced to the empty set. Therefore, we will finish the complexity analysis not by a worst case analysis but by what we expect is a representative case analysis.

A. Representative Complexity Analysis

In our discussion the term "branch" will mean a sequence of edges in the search tree from the root to a terminal node. We assume that in a representative case analysis the number b of branches traversed by a tree search which uses the ϕ_{NP} operator is a parameter α times one plus the number of consistent labelings, $[b = \alpha(1 + \#\mathcal{L}(T, R))]$. The parameter α indicates the difficulty of finding the consistent labelings for the given T and R relations.

Each iteration of the ϕ_{NP} operator requires the generation of at most $\#R({}^{M}_{P-N})\ \#L^{P-N}$ labelings which must be tested for consistency. To test a labeling for consistency requires checking at most $\#T$ N-tuples of unit-label pairs. Thus, if an operation is to check if an N-tuple of unit-label pairs is in R, then each iteration of ϕ_{NP} cannot take more than $\#R({}^{M}_{P-N})\ \#L^{P-N}\#T$ operations.

In any branch of the tree search, there can be at most $(M + 1)$ iterations of applying ϕ_{NP} to a fixed point since there are at most M nodes in a branch and we can apply ϕ_{NP} to the initial constraint relation. In addition, there cannot be a combined total of any more than $\#R$ more iterations in the branch, since each iteration of ϕ_{NP} not reaching a fixed point takes at least one N-tuple of unit-label pairs out of R. Hence, the number of operations in any branch is at most $(\#R + M + 1)\ \#R({}^{M}_{P-N})\ \#L^{P-N}\#T$. Since we have assumed that the number of branches is $\alpha[1 + \#\mathcal{L}(T, R)]$, we obtain that the number of operations required to find all (T, R)-consistent labelings is $\alpha[1 + \#\mathcal{L}(T, R)]\ (\#R + M + 1)\ \#R({}^{M}_{P-N})\ \#L^{P-N}\#T$.

This polynomial expression is multiplied by the parameter α which, in worst cases, could be exponential in M. However, practical problems seem to have parameter values for α which must be low order polynomials in M. This behavior in practice is similar to the behavior of other algorithms that solve problems of exponential complexity. For example, the simplex algorithm for linear programming hardly ever exhibits the worst case behavior in practice [12]. Therefore, in the next section we discuss a way of grading constraint relations so that although the computational complexity changes by grade, α is essentially for each grade.

B. Grading Unit-Label Constraint Relations

The fact that minimal constraint relations are well behaved for the first level of the tree search suggests the idea that we ought to give a name to those relations which are well behaved at every level of the tree search. Therefore, we will say that the relation pair (T, R) has complexity P if and only if P is the smallest integer such that for every labeling $(l_1, \cdots, l_n)$ of units $(u_1, \cdots, u_n)$ $\phi_{NP}^{\infty}\{\cdots \phi_{NP}^{\infty}[\phi_{NP}^{\infty}(R)|_{u_1 l_1}]|_{u_2 l_2} \cdots\}$ is minimal. We should expect that a tree search using ϕ_{NP} for a pair of relations having complexity P can be done in polynomial time since there will be no backtracking. To prove this we only need to recall that a relation which is minimal and nonempty can be restricted by any unit-label pair appearing in one of its N-tuples and the resulting restricted relation has a nonempty set of consistent labelings. Now suppose we begin the tree search with an arbitrary (T, R) of complexity P. Hence, $\phi_{NP}^{\infty}(R)$ is minimal. There are then two cases: either $\phi_{NP}^{\infty}(R) = \phi$ or $\phi_{NP}^{\infty}(R) \neq \phi$. If $\phi_{NP}^{\infty}(R) = \phi$, then $\mathcal{L}(T, R) = \phi$ and the tree search has terminated. If $\phi_{NP}^{\infty}(R) \neq \phi$, then $\phi_{NP}^{\infty}(R)$ minimal implies that no matter how we wish to restrict $\phi_{NP}^{\infty}(R)$ so long as we choose a unit-label pair (u, l) that appears in one of the N-tuples in $\phi_{NP}^{\infty}(R)$ the set

$$\mathcal{L}(T, \phi_{NP}^{\infty}(R))|_{ul} \neq \phi.$$

Since (T, R) has complexity P and $\phi_{NP}^{\infty}(\phi_{NP}^{\infty}(R)|_{ul})$ is minimal, then by Theorem 13 and Proposition 3, $\mathcal{L}(T, \phi_{NP}^{\infty}(R))|_{ul} = \mathcal{L}(T, \phi_{NP}^{\infty}(R)|_{ul}) = \mathcal{L}(T, \phi_{NP}^{\infty}(\phi_{NP}^{\infty}(R)|_{ul}))$. Hence,

$$\mathcal{L}(T, \phi_{NP}^{\infty}(R))|_{ul} \neq \phi$$

implies $\phi_{NP}^{\infty}(\phi_{NP}^{\infty}(R)|_{ul}) \neq \phi$. Thus, after restricting and iterating ϕ_{NP} to a fixed point, the resulting relation is minimal and nonempty, thereby allowing the tree search to proceed in the same well behaved manner it started.

By the analysis of the previous section, the number of operations required to do the tree search for a relation pair of complexity P will be $\#\mathcal{L}(T, R)\ (\#R + M + 1)\ \#R({}^{M}_{P-N})\ \#L^{P-N}\#T$ which is a polynomial in M of order $(P - N + 1)$.

Our conjecture as to the reason why look-ahead operators have been computationally efficient when incorporated into the tree search is that the relation pairs (T, R) which investigators have been handling in conjunction with artificial intelligence or computer vision tasks have been of low complexity. Furthermore, we conjecture that if we could determine the proportion of relation pairs having complexity equal to P for relation pairs where the label and unit set size is held constant, we would find many more relation pairs having small complexity than large complexity. In other words, the hard problems and the pathologic problems are rarities. But since we do not know how to do the required counting, we cannot verify the conjecture at this time. Hence, we leave it as an open problem.

V. Conclusions

We have described the consistent labeling problem and the use of look-ahead operators in solving the consistent labeling problem. The look-ahead operators ϕ_{KP} and Ψ_{KP} are generalizations of the operators used by various researchers to help eliminate backtracking in tree searching. We have shown that the fixed-point power of ϕ_{KP} and Ψ_{KP} is the same and that the standard tree search for finding consistent labeling can be quickest for what we defined to be minimal unit-label constraint relations. Finally, we have suggested a way to grade compatibility unit-label constraint relations and have shown that for relations of complexity P, all consistent labelings can be found in polynomial time where the order of the polynomial is $(P - N + 1)$.

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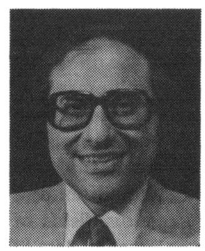

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Linda G. Shapiro, for a photograph and biography, see p. 126 of the March 1980 issue of this Transactions.