THE TOPOGRAPHIC PRIMAL SKETCH

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ABSTRACT

A complete mathematical treatment is given for describing the topographic primal sketch of the underlying grey tone intensity surface of a digital image. Each picture element is independently classified and assigned a unique descriptive label and possibly an orientation direction, both of which are invariant under monotonically increasing gray tone transformations from the set (peak, pit, ridge, ravine, saddle, flat, and hillside), with hillside having subcategories (inflection point, slope, convex hill, concave hill, and saddle hill). The topographic classification is based on the first and second directional derivatives of the estimated image intensity surface. A local, facet model, two-dimensional, cubic polynomial fit is done to estimate the image intensity surface. Zero-crossings of the first or second directional derivative are identified as locations of interest in the image.

1. INTRODUCTION

Representing the fundamental structure of a digital image in a rich and robust way is a primary problem encountered in any general robotics computer-vision system that has to ''understand'' an image. The richness is needed so that shading, highlighting, and shadow information, which are usually present in real manufacturing assembly line situations, are encoded. Richness permits unambiguous object matching to be accomplished. Robustness is needed so that the representation is invariant with respect to monotonically increasing gray tone transformations. Current representations involving edges or the primal sketch as described by Marr (1976; 1980) are impoverished in the sense that they are insufficient for unambiguous matching. They also do not have the required invariance. Basic research is needed to (1) define an appropriate representation, (2) develop a theory that establishes its relationship to properties that three-dimensional objects manifest on the image, and (3) prove its utility in practice. Until this is done, computer-vision research must inevitably be more ad hoc sophistication than science.

The basis of the topographic primal sketch consists of the classification and grouping of the underlying image intensity surface patches according to the categories defined by monotonic, gray tone, invariant functions of directional derivatives. Examples of such categories are peak, pit, ridge, ravine, saddle, flat, and hillside. Associated with the hillside category is the gradient direction. Associated with the ridge and valley category is the direction extremizing curvature. We call this representation the <u>topographic primal sketch</u>.

Why do we believe that this topographic primal sketch can be the basis for computer vision? We believe it because the light-intensity variations on an image are caused by an object's surface orientation, its reflectance, and characteristics of its lighting source. If any of the three-dimensional intrinsic surface characteristics are to be detected, they will be detected owing to the nature of light-intensity variations. Thus, the first step is to discover a robust representation that can encode the nature of these light-intensity variations, a representation that does not change with strength of lighting or with gain settings on the sensing camera. The topographic classification does just that. The basic research issue is to define a set of categories sufficiently complete to form groupings and structures that have strong relationships to the reflectances, surface orientations, and surface positions of the three-dimensional objects viewed in the image.

1.1. The Invariance Requirement

A digital image can be obtained with a variety of sensing-camera gain settings. It can be visually enhanced by an appropriate adjustment of the camera's dynamic range. The gain setting or the enhancing point operator changes the image by some monotonically increasing function that is not necessarily linear. For example, nonlinear enhancing point operators of this type include histogram normalization and equal probability quantization.

In visual perception, exactly the same visual interpretation and understanding of a pictured scene occurs whether the camera's gain setting is

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low or high and whether the image is enhanced or unenhanced. The only difference is that the enhanced image has more contrast, is nicer to look at, and is understood more quickly by the human visual system.

This fact is important because it suggests that many of the current, low-level computer-vision techniques, which are based on edges, cannot ever hope to have the robustness associated with human visual perception. They cannot have the robustness, because they are inherently incapable of invariance under monotonic transformations. For example, edges based on zero-crossings of second derivatives will change in position as the monotonic gray tone transformation changes because convexity of a gray tone intensity surface is not preserved under such transformations. However, the topographic categories peak, pit, ridge, valley, saddle, flat, and hillside with their associated directions do have the required invariance.

1.2. Facet Model

The facet model states that all processing of digital image data has its final authoritative interpretation relative to what the processing does to the underlying gray tone intensity surface. The digital image's pixel values are noisy sampled observations of the underlying surface. Thus, in order to do any processing, we at least have to estimate at each pixel position what this underlying surface is. This requires a model that describes what the general form of the surface would be in the neighborhood of any pixel if there were no noise. To estimate the surface from the neighborhood around a pixel then amounts to estimating the free parameters of the general form. It is important to note that if a different general form is assumed, then a different estimate of the surface is produced. Thus the assumption of a particular general form is necessary and has consequences.

The general form we use is a bivariate cubic. We assume that the neighborhood around each pixel is suitably fit by a bivariate cubic (Haralick 1981;1982). Having estimated this surface around each pixel, the first and second directional derivatives are easily computed by analytic means. The topographic classification of the surface facet is based totally on the first and second directional derivatives. We classify each surface point as peak, pit, ridge, ravine, saddle, flat, or hillside, with hillside being broken down further into the subcategories inflection point, convex hill, concave hill, saddle hill, and slope. Our set of topographic labels is complete in the sense that every combination of values of the first and second directional derivative is uniquely assigned to one of the classes.

1.3. Summary

Our classification approach is based on the estimation of the first-and second-order directional derivatives. Thus, we regard the digital-picture function as a sampling of the underlying function f, where some kind of random noise is added to the true function values. To estimate the first and second partials, we must assume some kind of parametric form for the underlying function f. The classifier must use the sampled brightness values of the digital-picture function to estimate the parameters and then make decisions regarding the locations of relative extrema of partial derivatives based on the estimated values of the parameters.

In Section 2, we will discuss the mathematical properties of the topographic structures in terms of the directional derivatives in the continuous surface domain. Because a digital image is a sampled surface and each pixel has an area associated with it, characteristic topographic structures may occur anywhere within a pixel's area. Thus, the implementation of the mathematical topographic definitions is not entirely trivial.

In Section 3, we will discuss the local cubic estimation scheme. In Section 4, we will show the results of the classifier on several test images.

2. THE MATHEMATICAL CLASSIFICATION OF TOPOGRAPHIC STRUCTURES

In this section, we formulate our notion of topographic structures on continuous surfaces and show their invariance under monotonically increasing gray tone transformations. In order to understand the mathematical properties used to define our topographic structures, one must understand the idea of the directional derivative discussed in most advanced calculus books. For completeness, we first give the definition of the directional derivative, then the definitions of the topographic labels. Finally, we show the invariance under monotonically increasing gray tone transformations.

2.1. The Directional Derivative

In two dimensions, the rate of change of a function f depends on direction. We denote the directional derivative of f at the point (r,c) in the direction β by $f'_\beta(r,c)$. It is defined as

$$f'_\beta(r,c) = \lim_{h \to 0} \frac{f(r+h*\sin\beta, c+h*\cos\beta) - f(r,c)}{h}$$

The direction angle β is the clockwise angle from the column axis. It follows directly from this definition that

$$f'_\beta(r,c) = \frac{\partial f}{\partial r}(r,c) * \sin\beta + \frac{\partial f}{\partial c}(r,c) * \cos\beta.$$

We denote the second derivative,,of f at the point (r,c) in the direction β by $f_\beta''(r,c)$ and it follows that

$$f_\beta'' = \frac{\partial^2 f}{\partial r^2} * \sin^2\beta + 2*\frac{\partial^2 f}{\partial r \partial c} * \sin\beta * \cos\beta + \frac{\partial^2 f}{\partial c^2} * \cos^2\beta .$$

The gradient of f is a vector whose magnitude,

$$\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial c}\right)^2 {}^{\frac{1}{2}}$$

at a given point (r,c) is the maximum rate of change of f at that point, and whose direction,

$$\tan^{-1} \frac{\frac{\partial f}{\partial r}}{\frac{\partial f}{\partial c}}$$

is the direction in which the surface has the greatest rate of change.

2.2. The Mathematical Properties

We will use the following notation to describe the mathematical properties of our various topographic categories for continuous surfaces. Let

- ∇f = gradient vector of a function f;
- $||\nabla f||$ = gradient magnitude;
- $\omega^{(1)}$ = unit vector in direction in which second directional derivative has greatest magnitude;
- $\omega^{(2)}$ = unit vector orthogonal to $\omega^{(1)}$;
- λ_1 = value of second directional derivative in the direction of $\omega^{(1)}$;
- λ_2 = value of second directional derivative in the direction of $\omega^{(2)}$;
- $\nabla f \cdot \omega^{(1)}$ = value of first directional derivative in the direction of $\omega^{(1)}$; and
- $\nabla f \cdot \omega^{(2)}$ = value of first directional derivative in the direction of $\omega^{(2)}$.

Without loss of generality, we assume $|\lambda_1| >= |\lambda_2|$.

Each type of topographic structure in our classification scheme is defined in terms of the above quantities. In order to calculate these values, the-first and second-order partials with respect to r and c need to be approximated. These five partials are as follows:

$$\frac{\partial f}{\partial r}, \frac{\partial f}{\partial c}, \frac{\partial^2 f}{\partial r^2}, \frac{\partial^2 f}{\partial c^2}, \frac{\partial^2 f}{\partial r \partial c} .$$

The gradient vector is simply $\left(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial c}\right)$. The second directional derivatives may be calculated by forming the Hessian where the Hessian is a 2*2 matrix defined as

$$H = \begin{vmatrix} \frac{\partial^2 f}{\partial r^2} & \frac{\partial^2 f}{\partial r \partial c} \\ \frac{\partial^2 f}{\partial c \partial r} & \frac{\partial^2 f}{\partial^2 f \partial c^2} \end{vmatrix} .$$

Hessian matrices are used extensively in nonlinear programming. Only three parameters are required to determine the Hessian matrix H, since the order of differentiation of the cross partials may be interchanged. That is

$$\frac{\partial^2 f}{\partial r \partial c} = \frac{\partial^2 f}{\partial c \partial r}$$

The eigenvalues of the Hessian are the values of the extrema of the second directional derivative, and their associated eigenvectors are the directions in which the second directional derivative is extremized. This can easily be seen by rewriting f_β'' as the quadratic form

$$f_\beta'' = (\sin\beta \ \cos\beta) * H * \begin{vmatrix} \sin\beta \\ \cos\beta \end{vmatrix} .$$

Thus,

$$H\omega^{(1)} = \lambda_1 \omega^{(1)} \text{ and } H\omega^{(2)} = \lambda_2 \omega^{(2)} .$$

Furthermore, the two directions represented by the eigenvectors are orthogonal to one another. Since H is a 2*2 symmetric matrix, calculation of the eigenvalues and eigenvectors can be done efficiently and accurately using the method of Rutishauser (1971). We may obtain the values of the first directional derivative in the direction of either extrema of the second directional derivative by simply taking the dot product of the gradient with the appropriate eigenvector:

$$\nabla f \cdot \omega^{(1)}$$
$$\nabla f \cdot \omega^{(2)}$$

There is a direct relationship between the eigenvalues λ_1 and λ_2 and curvature in the directions $\omega^{(1)}$ and $\omega^{(2)}$: When the first directional derivative $\nabla f \cdot \omega^{(i)} = 0$, then $\lambda_i / (1 + (\nabla f \cdot \omega^{(i)})^2)^{1.5}$ curvature in the direction $\omega^{(i)}$, i = 1 or 2.

Having the gradient magnitude and direction and the eigenvalues and eigenvectors of the Hessian, we can describe the topographic classification scheme.

2.2.1. Peak

A peak (knob) occurs where there is a local maxima in all directions. In other words, we are on a peak if, no matter what direction we look in, we see no point that is as high as the one we are on (Fig. 1). The curvature is downward in all directions. At a peak the gradient is zero, and the second directional derivative is negative in all directions. To test whether the second directional derivative is negative in all directions, we just have to examine the value of the second directional derivative in the directions that make it smallest and largest. A point is therefore classified as a peak if it satisfies the following conditions:

$$||\nabla f|| = 0, \lambda_1 < 0, \lambda_2 < 0.$$

2.2.2. Pit

A pit (sink, bowl) is identical to a peak except that it is a local minima in all directions rather than a local maxima. At a pit the gradient is zero, and the second directional derivative is positive in all directions. A point is classified as a pit if it satisfies the following conditions:

$$||\nabla f|| = 0, \lambda_1 > 0, \lambda_2 > 0.$$

2.2.3. Ridge

A ridge occurs on a ridge-line, a curve consisting of a series of ridge points. As we walk along the ridge-line, the points to the right and left of us are lower than the ones we are on. Furthermore, the ridge-line may be flat, slope upward, slope downward, curve upward, or curve downward. A ridge occurs where there is a local maximum in one direction, as illustrated in Fig. 2. Therefore, it must have negative second-directional derivative in the direction across the ridge and also a zero first-directional derivative in that same direction. The direction in which the local maximum occurs may correspond to either of the directions in which the curvature is ''extremized'', since the ridge itself may be curved. For nonflat ridges, this leads to the first two cases below for ridge characterization. If the ridge is flat, then the ridge-line is horizontal and the gradient is zero along it. This corresponds to the third case. The defining characteristic is that the second directional derivative in the direction of the ridge-line is zero, while the second directional derivative across the ridge-line is negative. A point is therefore classified as a ridge if it satisfies any one of the following three sets of conditions:

$$||\nabla f|| \neq 0, \lambda_1 < 0, \nabla f \cdot \omega^{(1)} = 0$$

or

$$||\nabla f|| \neq 0, \lambda_2 < 0, \nabla f \cdot \omega^{(2)} = 0$$

or

$$||\nabla f|| = 0, \lambda_1 < 0, \lambda_2 = 0.$$

A geometric way of thinking about the definition for ridge is to realize that the condition $\nabla f \cdot \omega^{(i)} = 0$ means that the gradient direction (which is defined for nonzero gradients) is orthogonal to the direction $\omega^{(i)}$ of extremized curvature.

2.2.4. Ravine

A ravine (valley) is identical to a ridge except that it is a local minimum rather than maximum in one direction. As we walk along the ravine-line, the points to the right and left of us are higher than the one we are on (see Fig. 2). A point is classified as a ravine if it satisfies any one of the following three sets of conditions:

$$||\nabla f|| \neq 0, \lambda_1 > 0, \nabla f \cdot \omega^{(1)} = 0$$

or

$$||\nabla f|| \neq 0, \lambda_2 > 0, \nabla f \cdot \omega^{(2)} = 0$$

or

$$||\nabla f|| = 0, \lambda_1 > 0, \lambda_2 = 0.$$

2.2.5. Saddle

A saddle occurs where there is a local maximum in one direction and a local minimum in a perpendicular direction, as illustrated in Fig. 2. A saddle must therefore have positive curvature in one direction and negative curvature in a perpendicular direction. At a saddle, the gradient magnitude must be zero and the extrema of the second directional derivative must have opposite signs. A point is classified as a saddle if it satisifies the following conditions:

$$||\nabla f|| = 0, \lambda_1 * \lambda_2 < 0.$$

2.2.6. Flat

A flat (plain) is a simple, horizontal surface, as illustrated in Fig. 3. It, therefore, must have zero gradient and no curvature. A point is classified as a flat if it satisfies the following conditions:

$$||\nabla f|| = 0, \lambda_1 = 0, \lambda_2 = 0.$$

Given that the above conditions are true, a flat may be further classified as a <u>foot</u> or <u>shoulder</u>. A foot occurs at that point where the flat just begins to turn up into a hill. At this point, the third directional derivative in the direction toward the hill will be nonzero, and the surface increases in this direction. The shoulder is an analogous case and occurs where the flat is ending and turning down into a hill. At this point, the maximum magnitude of the third directional derivative is nonzero, and the surface decreases in the direction toward the hill. If the third directional derivative is zero in all directions, then we are on a flat, not near a hill. Thus a flat may be further qualified as being a foot or shoulder, or not qualified at all.

2.2.7. Hillside

A hillside point is anything not covered by the previous categories. It has a nonzero gradient and no strict extrema in the directions of maximum and minimum second directional derivative. If the hill is simply a tilted flat (i.e., has constant gradient), we call it a slope. If its curvature is positive (upward), we call it a convex hill. If its curvature is negative (downward), we call it a concave hill. If the curvature is up in one direction and down in a perpendicular direction, we call it a saddle hill. A saddle hill is illustrated in Fig. 2, and the slope, convex hill, and concave hill are illustrated in Fig. 3.

A point on a hillside is an inflection point if it has a zero-crossing of the second directional derivative taken in the direction of the gradient. The inflection-point class is the same as the step edge defined by Haralick (1982), who classifies a pixel as a step edge if there is some point in the pixel's area having a zero-crossing of the second directional derivative taken in the direction of the gradient.

To determine whether a point is a hillside, we just take the complement of the disjunction of the conditions given for all the previous classes. Thus if there is no curvature, then the gradient must be non zero. If there is curvature, then the point must not be a relative extremum. Therefore, a point is classified as a hillside if all three sets of the following conditions are true ('->' represents the operation of logical implication):

$$\lambda_1 = \lambda_2 = 0 \rightarrow ||\nabla f|| \neq 0,$$

and

$$\lambda_1 \neq 0 \rightarrow \nabla f \cdot \omega^{(1)} \neq 0,$$

and

$$\lambda_2 \neq 0 \rightarrow \nabla f \cdot \omega^{(2)} \neq 0.$$

Rewritten as a disjunction of clauses rather than a conjunction of clauses, a point is classified as a hillside if any one of the following four sets of conditions are true:

$$\nabla f \cdot \omega^{(1)} \neq 0, \nabla f \cdot \omega^{(2)} \neq 0$$

or

$$\nabla f \cdot \omega^{(1)} \neq 0, \lambda_2 = 0$$

or

$$\nabla f \cdot \omega^{(2)} \neq 0, \lambda_1 = 0$$

or

$$||\nabla f|| \neq 0, \lambda_1 = 0, \lambda_2 = 0.$$

We can differentiate between different classes of hillsides by the values of the second directional derivative. The distinction can be made as follows:

SLOPE if $\lambda_1 = \lambda_2 = 0$

CONVEX if $\lambda_1 >= \lambda_2 >= 0$, $\lambda_1 \neq 0$

CONCAVE if $\lambda_1 <= \lambda_2 <= 0$, $\lambda_1 \neq 0$

SADDLE HILL if $\lambda_1 * \lambda_2 < 0$

A slope, convex, concave, or saddle hill is classified as an inflection point if there is a zero-crossing of the second directional derivative in the direction of maximum first directional derivative (i.e., the gradient).

2.2.8. Summary of the Topographic Categories

A summary of the mathematical properties of our topographic structures on continuous surfaces can be found in Table 1. The table exhaustively defines the topographic classes by their gradient magnitude, second directional derivative extrema values, and the first directional derivatives taken in the directions which extremize second directional derivatives. Each entry in the table is either 0, +, -, or *. The 0 means not significantly different from zero; + means significantly different from zero on the positive side; - means significantly different from zero on the negative side, and '*' means it does not matter. The label ''Cannot Occur'' means that it is impossible for the gradient to be nonzero and the first directional derivative to be zero in two orthogonal directions.

From the table, one can see that our classification scheme is complete. All possible combinations of first and second directional derivatives have a corresponding entry in the table. Each topographic category has a set of mathematical properties that uniquely determines it.

(Note: Special attention is required for the degenerate case $\lambda_1 = \lambda_2 \neq 0$, where $\omega^{(1)}$ and $\omega^{(2)}$ can be any two orthogonal directions. In this case, there always exists an extreme direction ω which is orthogonal to ∇f, and thus the first directional derivative $\nabla f \cdot \omega$ is always zero in an extreme direction. To avoid spurious zero directional derivatives, we choose $\omega^{(1)}$ and $\omega^{(2)}$ such that $\nabla f \cdot \omega^{(1)} \neq 0$ and $\nabla f \cdot \omega^{(2)} \neq 0$, unless the gradient is zero.)

3.0 SURFACE ESTIMATION

In this section we discuss the estimation of the parameters required by the topographic classification scheme of Section 2 using the local cubic facet model (Haralick 1981). It is important to note that the classification scheme of Section 2 and the algorithm of Section 3 are independent of the method used to estimate the first-and second-order partials of the underlying digital image-intensity surface at each sampled point. Although we are currently using the cubic model and discuss it here, we expect that a spline-based estimation scheme or a discrete-cosines estimation scheme may, in fact, provide better estimates.

4.1. Local Cubic Facet Model

In order to estimate the required partial derivatives, we perform a least-squares fit with a two-dimensional surface, f, to a neighborhood of each pixel. It is required that the function f be continuous and have continuous first-and second-order partial derivatives with respect to r and c in a neighborhood around each pixel in the rc plane.

We choose f to be a cubic polynomial in r and c expressed as a combination of discrete orthogonal polynomials. The function f is the best discrete least-squares polynomial approximation to the image data in each pixel's neighborhood. More details can be found in Haralick's paper (1981), in which each coefficient of the cubic polynomial is evaluated as a linear combination of the pixels in the fitting neighborhood.

To express the procedure precisely and without reference to a particular set of polynomials tied to neighborhood size, we will canonically write the fitted bicubic surface for each fitting neighborhood as

$$f(r,c) = k_1 + k_2 r + k_3 c + k_4 r^2 + k_5 rc + k_6 c^2 + k_7 r^3 + k_8 r^2 c + k_9 rc^2 + k_{10} c^3,$$

where the center of the fitting neighborhood is taken as the origin. It quickly follows that the needed partials evaluated at local coordinates (r,c) are

$$\frac{\partial f}{\partial r} = k_2 + 2k_4 r + k_5 c + 3k_7 r^2 + 2k_8 rc + k_9 c^2$$

$$\frac{\partial f}{\partial c} = k_3 + k_5 r + 2k_6 c + k_8 r^2 + 2k_9 rc + 3k_{10} c^2$$

$$\frac{\partial^2 f}{\partial r^2} = 2k_4 + 6k_7 r + 2k_8 c$$

$$\frac{\partial^2 f}{\partial c^2} = 2k_6 + 2k_9 r + 6k_{10} c$$

$$\frac{\partial^2 f}{\partial r \partial c} = k_5 + 2k_8 r + 2k_9 c$$

It is easy to see that if the above quantities are evaluated at the center of the pixel where local coordinates (r,c) = (0,0), only the constant terms will be of significance. If the partials need to be evaluated at an arbitrary point in a pixel's area, then a linear or quadratic polynomial value must be computed.

4. Test Examples

In this section, we show the results of the classifier on three images. The results for a chair are shown in figures 4,5,6,7. For a set of screws, it is shown in figures 8,9,10, and 11. For some machine parts in figures 12,13, and 14. Notice how the highlighting can occur depending on the positioning of the parts. The ridge labels are quite useful for determining where the highlighting occurs.

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Table 1. Mathematical Properties of Topographic Structures

$\|\nabla f\|$	λ_1	λ_2	$\nabla f \cdot \omega^{(1)}$	$\nabla f \cdot \omega^{(2)}$	Label
0	−	−	0	0	Peak
0	−	0	0	0	Ridge
0	−	+	0	0	Saddle
0	0	0	0	0	Flat
0	+	−	0	0	Saddle
0	+	0	0	0	Ravine
0	+	+	0	0	Pit
+	−	−	−,+	−,+	Hillside
+	−	*	0	*	Ridge
+	*	−	*	0	Ridge
+	−	0	−,+	*	Hillside
+	−	+	−,+	−,+	Hillside
+	0	0	*	*	Hillside
+	+	−	−,+	−,+	Hillside
+	+	0	−,+	*	Hillside
+	+	*	0	*	Ravine
+	*	+	*	0	Ravine
+	+	+	−,+	−,+	Hillside
+	*	*	0	0	Cannot Occur

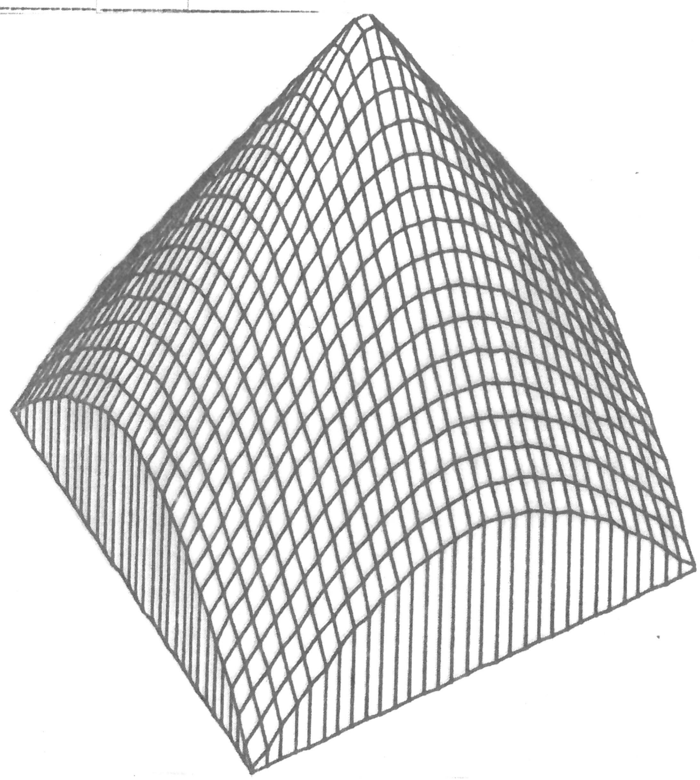

Figure 1: Right Circular Cone

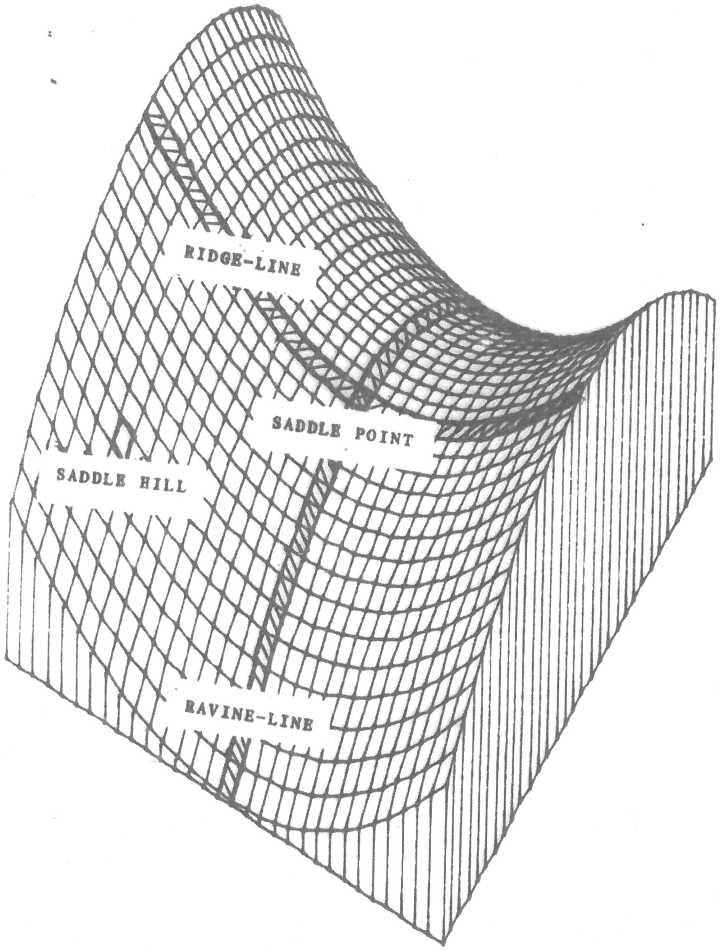

Figure 2: Saddle Surface

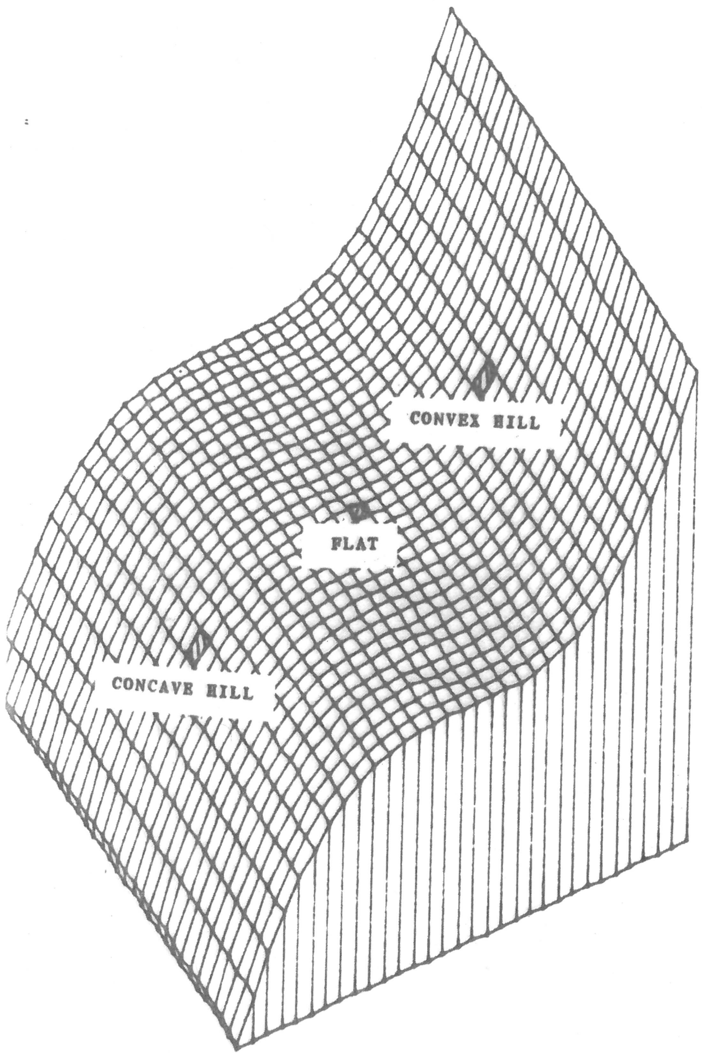

Figure 3 : Hillside

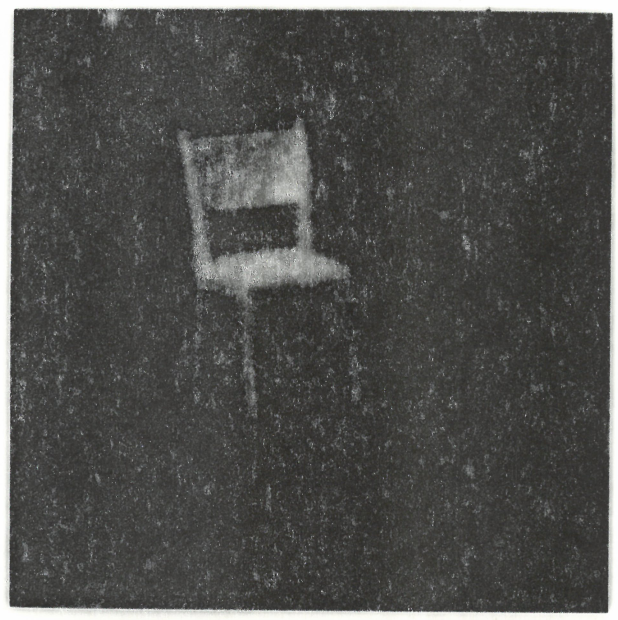

Figure 4 Chair

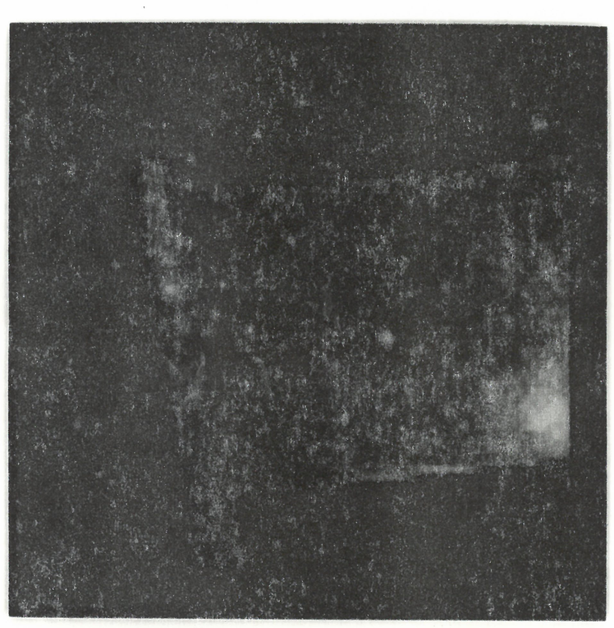

Figure 5 Upper left corner of chair

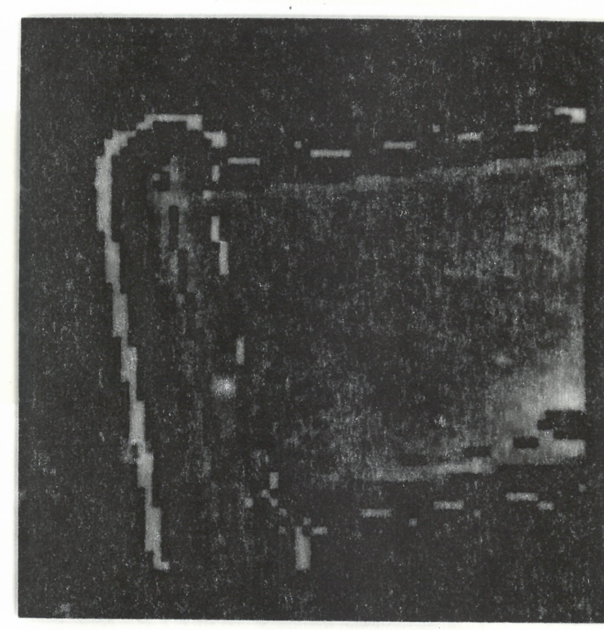

Figure 6 Ridges (black) ravines (white)

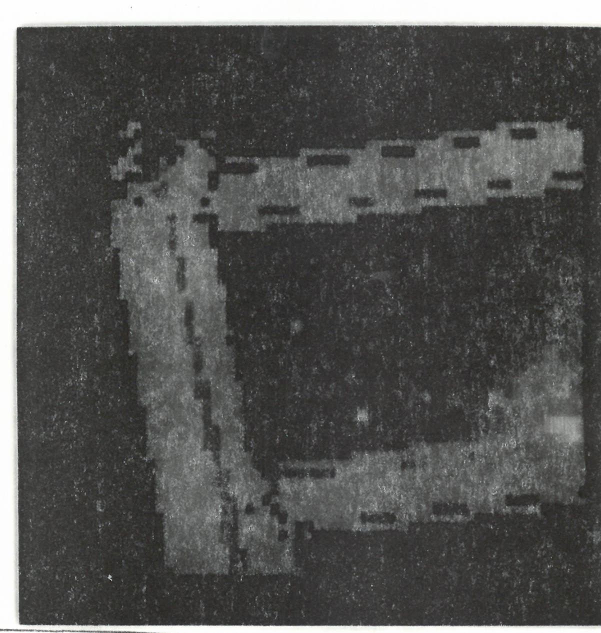

Figure 7 Hillside (white)

Figure 8 Screw

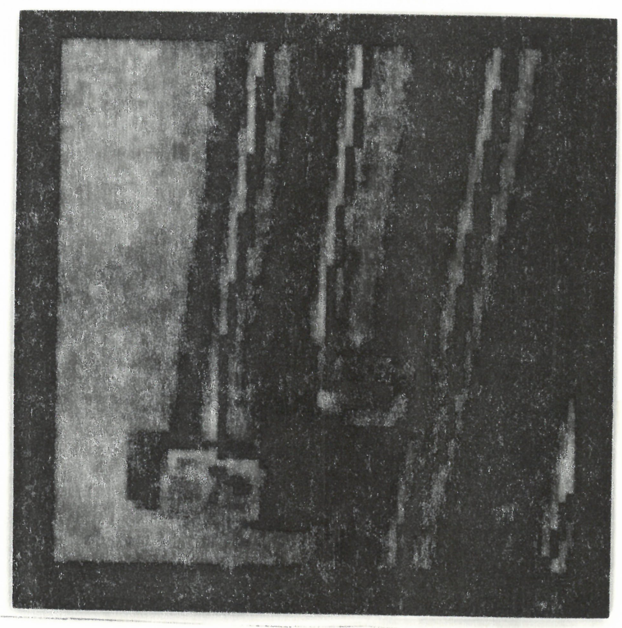

Figure 9 Ridges (black)

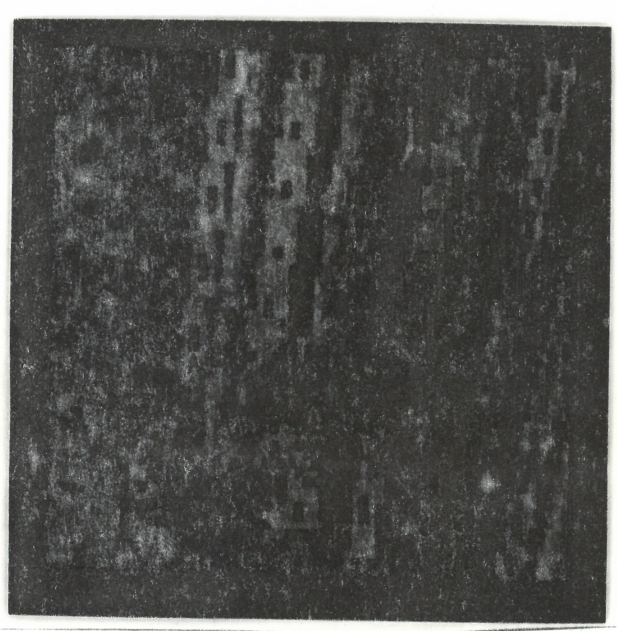

Figure 10 Convex Hillside (white)

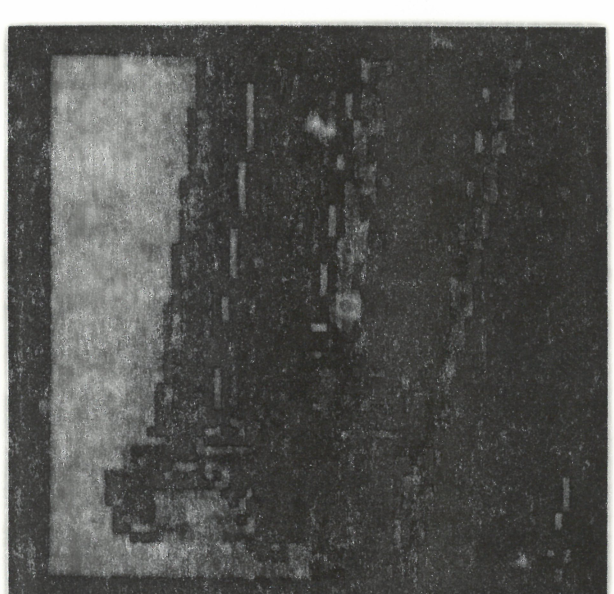

Figure 11 Concave Hillside (black)

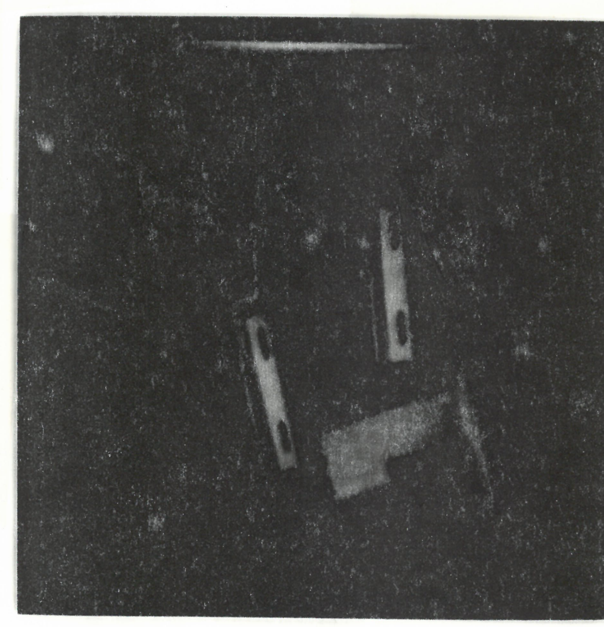

Figure 12 Machine parts

Figure 13 Upper left corner subimage ridges (black) ravines (white)

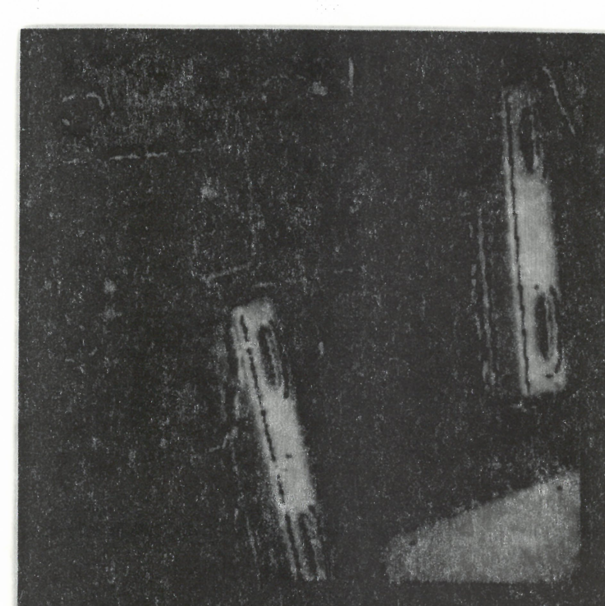

Figure 14 Center subimage ridges (black) ravines (white)