Site Model Construction Using Geometric Constrained Optimization

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Abstract

This paper presents an optimization approach for estimating the 3D parameters of buildings from multiple images by fitting the observations to the geometric building models. Through multi-image traingulation, the coordinates of the building corners and their covariances are estimated from the located image corresponding points. Then by a constrained optimization the estimated point positions and the partial models of the buildings are integrated. Numerical methods are used to solve the constrained optimization problem by iteratively updating an approximate solution. To provide an initial solution, the maximum likelihood estimators for the parameters of the planes and the lines are used. The covariances of the estimated parameters can be determined by propagating the input covariance matrix through the linearized optimization model. Experimental results on the RADIUS model boards are then given.

1 Introduction

1.1 Site Model Construction

In this paper we discuss how we used photogrammetry to establish a ground truth site model for the RADIUS Model Board Data Set made available by the University of Washington. There are two RADIUS model boards. Also given are the 3D coordinates of some building vertices. The procedure of constructing 3D building models from given images usually involves the following steps: *Feature Extraction, Correspondence*, and *3D Inference*. At first the 2D features such as edges, corners, lines are detected on the images. These 2D features are matched to 3D features. Based on the corresponding 2D features, the 3D object model and the camera model, the 3D inference procedure estimate the unknown parameters, and the covariance matrices of the 3D models.

Since most of the buildings on the two model boards are polyhedron, geometrical relations between linear elements on the buildings provide useful information for 3D model construction. And it is this kind of information which constitutes what we call a partial building model and is employed in the constrained optimization. The constrained optimization procedure takes the derived 3D points and their covariance matrices as observations. It uses the partial models of the buildings to enforce constraints on the building parameters. To estimate the optimal 3D parameters that satisfy the relations in the partial models. By solving the optimization problem and propagating the errors we can obtain the 3D parameters and the associated covariance matrix.

1.2 Previous Work

Using geometric information to infer 3D parameters from the observed image features attracted many researchers in vision or related areas. Bopp and Krauss (1978) studied the relationship between 2D image points and their corresponding 3D points. Haralick (1980) presented a set of perspective projection properties for points, lines and planes. Förstner (1985) applied reliability theory to estimate 3D points from block triangulation in aerial images. The angles between multiple lines provides important cues of 3D structure. For parallel lines, Barnard (1983), Mitiche and Habelrih (1989) described techniques for deriving the orientation of two or more parallel lines from their perspective projections. Mitiche and Habelrih also described an algorithm to infer object orientation from a set of orthogonal lines. Kanatani (1988) presented a technique for the derivation of the orientation of lines from a rectangular corner and from a corner with two right angles. Kawabata (1989) described techniques for deriving the orientation and position of quadrilaterals from its image by using angle relationships and distance relations. Haralick and Shapiro (1990) discussed important issues in analytic photogrammetry and in perspective projection analysis.

Most of the previous algorithms on inverse perspective projection analysis are for some special cases. It is difficult to directly apply those algorithms to con-

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struct complex building models, which may involve a large amount of features and complicated geometric relations between 3D features. To solve a 3D parameter estimation problem with general geometric constraints we developed an optimization framework to formulate and to solve the problem.

2 Triangulation Model

2.1 Problem Statement

Suppose that a set of 3D points are projected to multiple images. In the multi-image triangulation procedure, we assume that the image perturbation on the perspective projection of these 3D points are independent.

A world coordinate system x-y-z is specified. A 3D point with unknown world coordinates $(x, y, z)^T$ is mapped to J images through perspective projection camera models.

Let the camera coordinate system of the jth image be denoted as x_j-y_j-z_j. The rotation matrix from x-y-z to x_j-y_j-z_j is R_j and the projectivity center of the jth camera in the world coordinate system is $\mathbf{p}_j = (x_j^c, y_j^c, z_j^c)$. Both R_j and $\mathbf{p}_j$ are known. If the parameters in R_j and $\mathbf{p}_j$ contains random perturbation, the covariance matrix of these parameters are provided. Without loss of generality, we assume that these cameras have been calibrated and their focal lengths and principal points are given.

Let the unperturbed projection on the images be denoted by $(u_1, v_1), ..., (u_J, v_J)$ and the noise perturbed observations be denoted by $(\hat{u}_1, \hat{v}_1), ..., (\hat{u}_J, \hat{v}_J)$.

Using these notations, the point estimation problem can be formally stated as follows.

- Given
  - J camera models that include the interior parameters and the exterior orientation parameters $\{R_j, \mathbf{p}_j, j = 1, ..., J\}$,
  - the observed projection of an unknown 3D point on the J images $\{\hat{u}_j, \hat{v}_j, j = 1, ..., J\}$,
  - an optimality criterion.
- Goal
  - determine the optimal 3D point coordinates under the given criterion,
  - estimate the covariance matrix of the derived coordinates.

The optimality criterion is selected to be the maximum posterior probability. The noise is assumed to be Gaussian distribution with zero mean.

2.2 Perturbation Model

The mapping from the 3D point coordinates to the the unperturbed projection on an image is described by a perspective transform. It consist of an Euclidean transform from the world coordinate system to the camera centered coordinate system, and a perspective projection from the 3D camera coordinate system to the 2D image coordinate system.

Let us consider the Euclidean transform first. Given a point with world coordinates $\mathbf{x} = (x, y, z)^T$, its coordinates in the jth camera coordinate system $\mathbf{x}_j = (x_j, y_j, z_j)^T$ is an affine function of the world coordinates.

$$\begin{pmatrix} x_j \\ y_j \\ z_j \end{pmatrix} = R_j \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_j^c \\ y_j^c \\ z_j^c \end{pmatrix} \right)$$

Let $-R_j(x_j^c, y_j^c, z_j^c)^T$ be denoted by $\mathbf{t}_j = (t_{x,j}, t_{y,j}, t_{z,j})^T$, the transform can be represented as

$$\mathbf{x}_j = R_j \mathbf{x} + \mathbf{t}_j \tag{1}$$

In the jth camera coordinate system suppose the optic axis of the camera is defined as the z_j axis and the projectivity center is defined as the origin. Through perspective projection, the 3D point with coordinates $(x_j, y_j, z_j)^T$ is mapped to an image point with coordinates $(u_j, v_j)^T$, where

$$\begin{pmatrix} u_j \\ v_j \end{pmatrix} = f_j \begin{pmatrix} x_j/z_j \\ y_j/z_j \end{pmatrix} \tag{2}$$

Through equations (1) and (2) the true projection coordinates are defined as the functions of the world coordinates.

The observed projection coordinates are perturbed by random noise $(\delta u_j, \delta v_j)^T \sim \mathcal{N}(0, \Sigma_j)$.

$$\begin{pmatrix} u_j' \\ v_j' \end{pmatrix} = \begin{pmatrix} u_j \\ v_j \end{pmatrix} + \begin{pmatrix} \delta u_j \\ \delta v_j \end{pmatrix}$$

Let $\mathbf{u}_j'$ denote $(u_j', v_j')^T$ and $\mathbf{u}_j$ denote $(u_j, v_j)^T$. The observation $\mathbf{u}_j'$ has distribution $\mathcal{N}(\mathbf{u}_j, \Sigma_j)$

2.3 Optimization Model

As mentioned in the problem statement the maximum posterior probability criterion is applied to the triangulation. The observations of the projections of a 3D point on multi-images and their covariances are given. The parameters to be estimated are the 3D point coordinates. In the current stage the given camera models are treated as fixed parameters. By Bayes formula, maximizing the posterior probability

$$P(x, y, z \mid \{u_j', v_j',\ j = 1, ..., J\})$$

is equivalent to

$$\max_{x,y,z} P(u_j', v_j',\ j = 1, ..., J \mid x, y, z) P(x, y, z)$$

If $(x, y, z)^T$ is uniformly distributed in the sample space, the problem becomes a maximum likelihood problem,

$$\max_{x,y,z} P(u_j', v_j',\ j = 1, ..., J \mid x, y, z)$$

With a normal noise model, the maximum likelihood is proportional to

$$\min_{x,y,z} \sum_{j=1}^{J} (\mathbf{u}'_j - \mathbf{u}_j)^T \Sigma_j^{-1} (\mathbf{u}'_j - \mathbf{u}_j) \qquad (3)$$

where $\mathbf{u}_j = (f_j x_j / z_j, f_j y_j / z_j)^T$ is a function of unknown parameters x, y, z.

The objective function in (3) can be rewritten into full matrix format. We can define the residuals ξ_j, η_j and $\mathbf{g}$ as

$$\begin{aligned} \xi_j &= u'_j - u_j(x, y, z) \\ \eta_j &= v'_j - v_j(x, y, z) \\ \mathbf{g} &= (g_1, ..., g_n)^T = (\xi_1, \eta_1, \xi_2, \eta_2, ..., \xi_J, \eta_J)^T \end{aligned}$$

where $n = 2J$. Let Θ denote the unknown parameters, Σ denote total covariance matrix

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 & . & . & 0 \\ 0 & \Sigma_2 & . & . & 0 \\ . & . & & & . \\ 0 & . & . & . & \Sigma_J \end{bmatrix}$$

and Q denote Σ^{-1}. The optimization problem can be reformulated as a general least squares problem,

$$\min_{\Theta} \left\{ f(\Theta) = \frac{1}{2} \mathbf{g}(\Theta)^T Q \mathbf{g}(\Theta) \right\} \qquad (4)$$

The general least squares problem can be solved by using a numerical optimization method, such as Newton's method. Starting from an approximate solution, a numerical method iteratively update the current value by using the gradient direction or the Hessian matrix. Please refer to [15], [6] for details.

Once the iteration converge to a local minimum point, we take the minimum point as the solution. At this minimum point, we can derive the error propagation matrix and use it to propagate the input error covariance to the output [15], [16]. Hence we can get both the 3D point coordinates and their covariance matrix.

3 Geometric Constrained Optimization

In the last section we discussed how to used the triangulation method to derive the 3D points and the associated covariance matrices. These data are used by the geometric construction procedure to determine the building parameters by fitting the 3D point observations with building models. We assume that the correspondence between the observations and the points in the building models are established (either from hypothesis or from ground-truth).

Starting from a problem statement, we develop the optimization model based on analyzing the building partial models and the perturbation model. In order to reduce the computational complexity we decompose a big least squares problem into a set of small least squares problems based on the independence between the buildings. A problem in formulating building parameter estimation is that some of the building vertices may not be observed. To deal with this problem we select the parameters that are estimable from the given observations. To produce a good initial solution for the numerical optimization methods, the maximum likelihood estimators of the linear object parameters are studied. The sequential quadratic programming method and the computation of the linear model are discussed in this section.

3.1 Problem Statement

The problem can be described as follows.

- Given
  - partial models of the polygon buildings that includes 3D linear objects (points, lines, planes) and the geometric relations between them.
  - observations of a set of corresponding points. The observations contain the 3D coordinates and the associated covariance matrices.
- Goal :
  - estimate the building parameters (point coordinates, plane and line location and orientation) that satisfy the partial model and are optimal under a given optimality criterion.
  - determine the covariance matrix of the estimated parameters.

To transform this problem into an optimization framework, we need to have the mathematical model that constrains the unknown 3D parameters and the model that links the unknown parameters to the observations. They are the partial model and the perturbation model.

3.2 Partial Building Models

3.2.1 Geometric Relations in Partial Building Models

A site model currently consists of a group of polyhedron building models. Corresponding to the planar surfaces, edges, vertices on a building surface, each building model consists of a set of linear objects (planes, lines and points) and their geometric relations. Sometimes a ground normal vector and its relations with a subset of linear objects are also given.

point: The parameters related to a point are its coordinates, denoted by (x, y, z) or in vector form $\mathbf{x}$.

line: A line is defined by a direction cosines $\mathbf{e} = (e_x, e_y, e_z)$ and a reference point $\mathbf{b} = (b_x, b_y, b_z)$ on the line. We choose the unique $\mathbf{b}$ that satisfies $\mathbf{e} \cdot \mathbf{b} = \mathbf{e}^T \mathbf{b} = 0$.

plane: A plane is specified by a normal vector $\mathbf{v} = (\alpha, \beta, \gamma)$ and a directed distance constant d.

A partial building model consists of five groups of relations. The first two specify the locations of the linear objects. The other three specify the angles between linear objects and the normal length conditions. Figure 1 and figure 2 illustrates these relations.

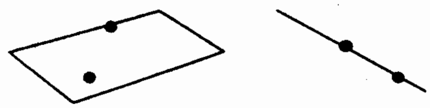


Figure 1: Position relations in a partial model

The first group defines which points are on each plane. Each ***point-plane relation*** can be described by a planar equation in canonical form.

$$\mathbf{x}^T\mathbf{v} + d = x\alpha + y\beta + z\gamma + d = 0; \qquad (5)$$

The same point may occur on more than one plane.

The second group specifies which points are on each line. Each ***point-line relation*** can be described by a line equation either in the explicit (parametric) form

$$\mathbf{x} = \eta\mathbf{e} + \mathbf{b}, \qquad \eta \in \mathcal{R}$$

or in the implicit form

$$(I - \mathbf{e}\mathbf{e}^T)(\mathbf{x} - \mathbf{b}) = 0. \qquad (6)$$

The same point may occur on more than one line.

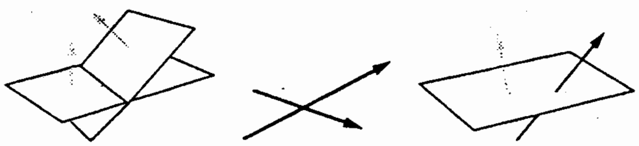


Figure 2: Angle relations in a partial model

The third group, the ***plane-angle-plane relations***, contains the inner products of the normal vectors of the planar surfaces. A zero inner product means two perpendicular planes, while 1 or -1 means two parallel planes. This group may also contain relations between a given vector $(\alpha_0, \beta_0, \gamma_0)$ and a subset of planar normal vectors.

The fourth group, called the ***line-angle-line relations***, contains the inner products of the direction cosines of the lines. A zero inner product means two perpendicular lines, while 1 or -1 means two parallel lines.

The fifth group, ***plane-angle-line***, contains the inner products of the normal vectors of the planes and the direction cosines of the lines. A zero inner product means that a line is parallel to a plane. While a value 1 or -1 means that a line is perpendicular to a plane.

3.2.2 Partial Model Database

Using the described relation set, we can create the partial model for a given polyhedron building. The simplest model is a cubic block. It contains 6 planes, 12 lines and 8 points. Figure 3(a) and (b) show the point-plane relations and the point-line relations. Table 1 give a list of the relations.

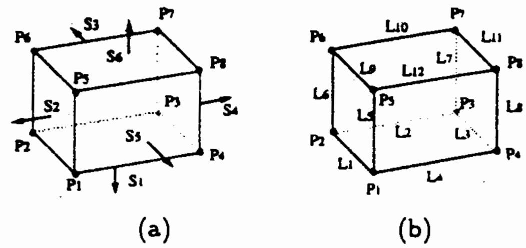


Figure 3: angle relations in flat roof model
(a) point-plane relations, (b) point-line relations

Table 1: position relations in flat roof model

plane	points			
S_1	P_1,	P_2,	P_3,	P_4
S_2	P_1,	P_2,	P_5,	P_6
S_3	P_2,	P_3,	P_6,	P_7
S_4	P_3,	P_4,	P_7,	P_8
S_5	P_1,	P_4,	P_5,	P_8
S_6	P_5,	P_6,	P_7,	P_8

line	points	
L_1	P_1,	P_2
L_2	P_2,	P_3
L_3	P_3,	P_4
L_4	P_4,	P_1
L_5	P_1,	P_5
L_6	P_2,	P_6
L_7	P_3,	P_7
...	...	...

The plane-plane, line-line and plane-line relations in the cubic model are illustrated in figure 4(a), (b), (c). The associated table 2 lists some of these relations.

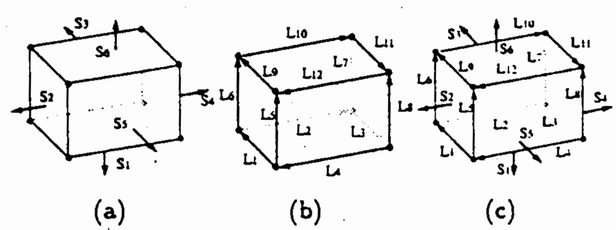


Figure 4: angle relations in flat roof model
(a) plane-angle-plane (b) line-angle-line
(c) plane-angle-line

The second example is a model of house with peaked roof. It contains 7 planar surfaces, 15 lines and 10 points. The position relations in the model are illustrated in figure 5. Figure 6 give the angles relations.

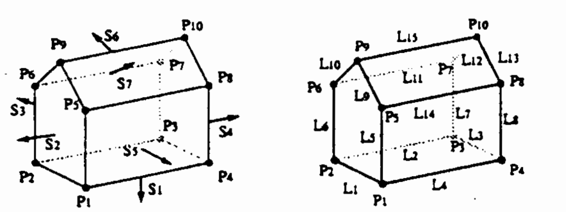
Table 2: angle relations in flat roof model

plane-plane angles			line-line angles		
plane	plane	cosine of angle	line	line	cosine of angle
S_1	S_1	1	L_1	L_1	1
S_1	S_2	0	L_1	L_2	0
S_1	S_3	0	L_1	L_3	-1
S_1	S_4	0	L_1	L_4	0
S_1	S_5	0	L_1	L_5	0
S_1	S_6	-1	L_1	L_6	0
S_2	S_2	1	L_1	L_7	0
...	...	...	...	...	...

plane-line relations		
plane	line	cosine of angle
S_1	L_1	0
S_1	L_2	0
S_1	L_3	0
S_1	L_4	0
S_1	L_5	-1
S_1	L_6	-1
S_1	L_7	-1
...	...	...

The third example is a hip roof model which has a roof with sloping ends and sides. It contains 9 planes, 17 lines and 10 points. Figure 7 shows the position relations in the model and figure 8 shows the angle relations.

In a real site model some buildings may have very complicated structures. Dozens, or even hundred of linear objects may be involved in a complex building. The partial model of model board 1 includes about two thousand linear objects and thirty thousand geometric relations. The partial model of model board 2 contains about fifteen hundred linear objects and twenty thousand geometric relations.

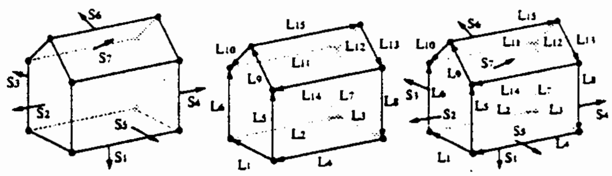


(a) (b)

Figure 5: position relations in peaked roof model
(a) point-plane relations, (b) point-line relations

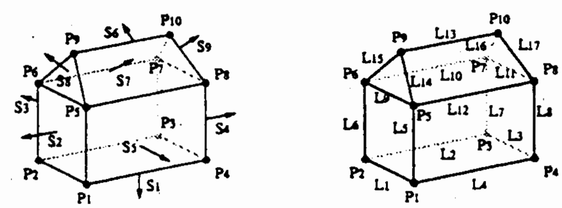
3.3 Optimization Framework

The observed 3D points and the associated covariance matrix Σ are obtained from triangulation. Hav-



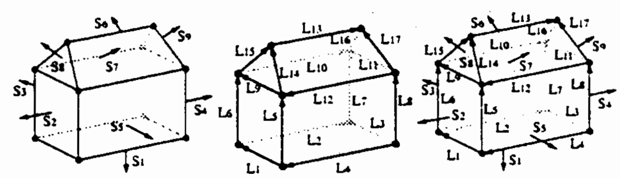
(a) (b) (c)

Figure 6: angle relations in peaked roof model
(a) plane-angle-plane (b) line-angle-line
(c) plane-angle-line



(a) (b)

Figure 7: position relations in hip roof model
(a) point-plane relations, (b) point-line relations



(a) (b) (c)

Figure 8: angle relations in hip roof model
(a) plane-angle-plane (b) line-angle-line
(c) plane-angle-line

ing the partial object model and the perturbation model, we can define the estimation problem. Let $\Theta \in I\!R^m$ denote the parameters, $X' \in I\!R^m$ denote the observations, and $p(X' \mid \Theta)$ denote the likelihood function. In the building estimation problem, the parameters are the coordinates of the points, the normal vectors and distance constants of the planes, and the direction cosines and reference points of the lines.

Assume that the given optimality criterion is the maximum posterior probability, a Bayesian approach can be used to transform the problem into a maximum likelihood problem with constraints. Let the constraints be denoted by $\Theta \in C_\Theta \subset I\!R^m$. The problem can be expressed as a constrained optimization problem.

$$\min\{-p(X' \mid \Theta) \mid \Theta \in C_\Theta\}$$

The problem can be reformulated by taking logarithm of the probability function. Under the assumption of Gaussian noise, we obtain a least squares model. The objective function is the sum of squared errors between the estimated point positions and the observed points.

$$\begin{array}{ll} \min\limits_{\Theta} & \{f(\Theta) := (X' - X)^T \Sigma^{-1} (X' - X)\} \quad (7) \\ \text{subject to} & \Theta \in C_\Theta \end{array}$$

where X denotes the unknown 3D points, and the feasible set C_Θ is determined by the partial model and the unit length of the directional vectors.

If the noise effecting different 3D points are independent, the objective function can be rewritten as

$$f(\Theta) = \sum_{i=1}^{K} (\mathbf{x}'_i - \mathbf{x}_i)^T \Sigma_i^{-1} (\mathbf{x}'_i - \mathbf{x}_i)$$

where Σ_i is the covariance matrix of the ith point, and K is the number of observed points.

3.4 Constraints

The constraints come from the relations in the partial model and the unit vector length requirements. We list the relations included in the building models as follows.

- A point-on-plane relation gives constraint

$$\mathbf{x}^T \mathbf{v} + d = 0$$

- A point-on-line relation provides

$$(I - \mathbf{e}\mathbf{e}^T)(\mathbf{x} - \mathbf{b}) = 0$$

  which contains three equations. It can be proved that two linearly independent equations can be obtained from them.

- A plane-angle-plane relation provides one equation,

$$\mathbf{v}_i \cdot \mathbf{v}_j - \rho_{ij} = \mathbf{v}_i^T \mathbf{v}_j - \rho_{ij} = 0$$

  where ρ_{ij} is the cosines of the angle between the two planes.

- A line-angle-line relation gives

$$\mathbf{e}_i \cdot \mathbf{e}_j - \tau_{ij} = \mathbf{e}_i^T \mathbf{e}_j - \tau_{ij} = 0$$

  where τ_{ij} is the cosines of the angle between the two lines.

- A plane-angle-line relation gives

$$\mathbf{v}_i \cdot \mathbf{e}_j - \psi_{ij} = \mathbf{v}_i^T \mathbf{e}_j - \psi_{ij} = 0$$

  where ψ_{ij} is the cosines of the angle between the normal vector of plane i and direction cosines of line j.

- In the plane-angle-plane and line-angle-line relations, we get the unit vector length constraints when $i = j$, i.e.,

$$\begin{aligned} \mathbf{v}_i \cdot \mathbf{v}_i &= \|\mathbf{v}_i\|_2 = 1; \\ \mathbf{e}_i \cdot \mathbf{e}_i &= \|\mathbf{e}_i\|_2 = 1; \end{aligned}$$

- The uniqueness of the reference point on a line requires

$$\mathbf{e}_j \cdot \mathbf{b}_j = 0$$

If a constant vector $\mathbf{v}_0$ for the site is available, then the angle relations related to $\mathbf{v}_0$ provide more constraints.

$$\begin{aligned} \mathbf{v}_0 \cdot \mathbf{v}_j - \rho_{0j} &= \mathbf{v}_0^T \mathbf{v}_j - \rho_{0j} = 0 \\ \mathbf{v}_0 \cdot \mathbf{e}_j - \tau_{0j} &= \mathbf{v}_0^T \mathbf{e}_j - \tau_{0j} = 0 \end{aligned}$$

More relations can be included in the building models if further information is available. For example, if the distance between two points $\mathbf{x}_i$ and $\mathbf{x}_j$ is given as r_{ij}, a constraint can be obtained,

$$\|\mathbf{x}_i - \mathbf{x}_j\|_2 - r_{ij} = 0$$

Using the constraints equations to express the feasible set C_Θ, the optimization problem (7) can be written as

$$\begin{array}{ll} \min\limits_{\Theta} & \{f(\Theta) := (X' - X)^T \Sigma^{-1} (X' - X)\} \\ \text{subject to} & h_i(\Theta) = 0, \quad i = 1, \ldots, r \end{array}$$

3.5 Initial Guess

After setting up the constrained nonlinear least squares problem, we can use a numerical method to compute its solution. The method requires an initial guess. Some numerical methods, like Newton's method, are particularly sensitive to the quality of initial choice because of its inherent assumption that the initial value is fairly close to the final solution.

The initial guess of the point coordinates directly comes from the observations. The initial guess of the direction and position of a plane or a line is computed from the associated points by using the maximum likelihood estimators. In the following discussion about the initial value, we use the hat accent ^ to denote an estimator.

3.5.1 A Plane

Suppose that we observed n points, $\mathbf{x}_1, .., , \mathbf{x}_n$, from a plane. Assume that the observation noise is independent and identically distributed as $\mathcal{N}(0, \sigma^2 I)$. The maximum likelihood estimator of the plane normal vector is the solution to the following problem.

$$\min_{\mathbf{v}} \quad \mathbf{v}^T \left(\sum_{i=1}^{n} (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^T \right) \mathbf{v}$$

$$\text{subject to} \quad \mathbf{v}^T \mathbf{v} = 1;$$

where $\bar{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i$ is the mean of the related observations. Using singular value decomposition the matrix $\sum_{i=1}^{n} (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^T$ can be decomposed into

$$\sum_{i=1}^{n} (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^T = UWU^T$$

where U is an orthonormal matrix satisfying $U^T = U^{-1}$, and W is a diagonal matrix containing singular values. The minimum function value equals the smallest singular value in W and the estimate of the normal vector is the column vector in U that corresponds to the smallest singular value. The estimator of the distance variable d is $d = -\mathbf{v}^T \bar{\mathbf{x}}$.

3.5.2 A Line

Suppose that we have observations of n points, $\mathbf{x}_1, .., , \mathbf{x}_n$, on a line. Assume that the observation noise is independent and identically distributed as $\mathcal{N}(0, \sigma^2 I)$. The maximum likelihood estimator of the line direction cosines is the solution to the following problem.

$$\min_{\mathbf{v}} \quad -\mathbf{v}^T \left(\sum_{i=1}^{n} (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^T \right) \mathbf{v}$$

$$\text{subject to} \quad \mathbf{v}^T \mathbf{v} = 1;$$

where $\bar{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i$ is the mean of the related observations. Similar to the plane case, we can use singular value decomposition to decompose the matrix $\sum_{i=1}^{n} (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^T$ into UWU^T. Thus the objective function becomes $-\mathbf{v}UWU^T\mathbf{v}$. To minimize the function we choose the column vector in U that corresponds to the largest singular value in W as $\mathbf{v}$. The estimator of reference point $\mathbf{b}$ is determined by $\mathbf{b} = (I - \mathbf{v}\mathbf{v}^T)\bar{\mathbf{x}}$.

3.5.3 Parallel Planes

Suppose that we observed K sets of points $\{\{\mathbf{x}_{i,k},\ i = 1, ..., n(k)\},\ k = 1, ..., K\}$ from K parallel planes, where $\{\mathbf{x}_{i,k},\ i = 1, ..., n(k)\}$ corresponds to the kth plane. Assume that the observation noise is independent and identically distributed as $\mathcal{N}(0, \sigma^2 I)$. The maximum likelihood estimator of the normal vector of the planes is the solution to the following problem.

$$\min_{\mathbf{v}} \quad \mathbf{v}^T \left(\sum_{k=1}^{K} \sum_{i=1}^{n} (\mathbf{x}_{i,k} - \bar{\mathbf{x}}_k)(\mathbf{x}_{i,k} - \bar{\mathbf{x}}_k)^T \right) \mathbf{v}$$

$$\text{subject to} \quad \mathbf{v}^T \mathbf{v} = 1;$$

where $\bar{\mathbf{x}}_k = \frac{1}{n(k)} \sum_{i=1}^{n(k)} \mathbf{x}_{i,k}$ is the mean of the points on the kth plane. Using singular value decomposition the matrix $\sum_{k=1}^{K} \sum_{i=1}^{n(k)} (\mathbf{x}_{i,k} - \bar{\mathbf{x}}_k)(\mathbf{x}_{i,k} - \bar{\mathbf{x}}_k)^T$ can be decomposed into UWU^T. The estimate of the normal vector is the column vector in U that corresponds to the smallest singular value in W. The estimator of the distance variable d_k for the kth plane is $d_k = -\mathbf{v}^T \bar{\mathbf{x}}_k$.

3.5.4 Parallel Lines

Suppose that we observed K sets of points $\{\{\mathbf{x}_{i,k},\ i = 1, ..., n(k)\},\ k = 1, ..., K\}$ from K parallel lines, where point set $\{\mathbf{x}_{i,k},\ i = 1, ..., n(k)\}$ corresponds to the kth line. Assume that the observation noise is independent and identically distributed as $\mathcal{N}(0, \sigma^2 I)$. The maximum likelihood estimator of the line direction cosines can be obtained by solving

$$\min_{\mathbf{v}} \quad -\mathbf{v}^T \left(\sum_{k=1}^{K} \sum_{i=1}^{n(k)} (\mathbf{x}_{i,k} - \bar{\mathbf{x}}_k)(\mathbf{x}_{i,k} - \bar{\mathbf{x}}_k)^T \right) \mathbf{v}$$

$$\text{subject to} \quad \mathbf{v}^T \mathbf{v} = 1;$$

where $\bar{\mathbf{x}}_k = \frac{1}{n(k)} \sum_{i=1}^{n(k)} \mathbf{x}_{i,k}$ is the mean of the points corresponding to the kth line. Similar to the parallel plane case, we can use singular value decomposition to decompose the matrix $\sum_{k=1}^{K} \sum_{i=1}^{n(k)} (\mathbf{x}_{i,k} - \bar{\mathbf{x}}_k)(\mathbf{x}_{i,k} - \bar{\mathbf{x}}_k)^T$ into UWU^T. The estimate of the direction cosines is the column vector in U corresponding to the largest singular value in W. The estimator of the reference point $\mathbf{b}_k$ for the kth line is determined by

$$\mathbf{b}_k = (I - \mathbf{v}\mathbf{v}^T)\bar{\mathbf{x}}_k$$

3.5.5 Parallel Planes and Parallel Lines

Suppose that we observed K sets of points $\{\{\mathbf{x}_{i,k},\ i = 1, ..., n(k)\},\ k = 1, ..., K\}$ from K parallel planes, where point set $\{\mathbf{x}_{i,k},\ i = 1, ..., n(k)\}$ corresponds to the kth plane. Suppose that we also observed L sets of points $\{\{\mathbf{x}'_{i,l}\ i = 1, ..., m(l)\},\ l = 1, ..., L\}$ from L parallel lines, where point set $\{\mathbf{x}'_{i,l},\ i = 1, ..., m(l)\}$ corresponds to the lth line. Assume that the normal vectors of the planes and the direction cosines are parallel and that the observation noise is independent and identically distributed as $\mathcal{N}(0, \sigma^2 I)$. The maximum likelihood estimator of the normal vector can be derived by solving

$$\min_{\mathbf{v}} \quad \mathbf{v}^T \left(\sum_{k=1}^{K} \sum_{i=1}^{n(k)} (\mathbf{x}_{i,k} - \bar{\mathbf{x}}_k)(\mathbf{x}_{i,k} - \bar{\mathbf{x}}_k)^T - \sum_{l=1}^{L} \sum_{i=1}^{m(l)} (\mathbf{x}'_{i,l} - \bar{\mathbf{x}}'_l)(\mathbf{x}'_{i,l} - \bar{\mathbf{x}}'_l)^T \right) \mathbf{v}$$

$$\text{subject to} \quad \mathbf{v}^T\mathbf{v} = 1;$$

where $\bar{\mathbf{x}}_k = \frac{1}{n(k)} \sum_{i=1}^{n(k)} \mathbf{x}_{i,k}$ is the mean of the points corresponding to the kth plane and $\bar{\mathbf{x}}'_l = \frac{1}{m(l)} \sum_{i=1}^{m(l)} \mathbf{x}_{i,l}$ is the mean of the points corresponding to the lth line. Similar to the parallel plane case, we can apply singular value decomposition to decompose the matrix,

$$UWU^T = \sum_{k=1}^{K} \sum_{i=1}^{n(k)} (\mathbf{x}_{i,k} - \bar{\mathbf{x}}_k)(\mathbf{x}_{i,k} - \bar{\mathbf{x}}_k)^T - \sum_{l=1}^{L} \sum_{i=1}^{n(l)} (\mathbf{x}_{i,l} - \bar{\mathbf{x}}_l)(\mathbf{x}_{i,l} - \bar{\mathbf{x}}_l)^T$$

The estimate of the normal vector is the column vector in U corresponding to the smallest singular value in W. The estimator of the distance constant d_k for the kth plane can be determined by $d_k = -\mathbf{v}^T\bar{\mathbf{x}}_k$. The estimator of the reference point $\mathbf{b}_l$ for the lth line can be determined by $\mathbf{b}_l = (I - \mathbf{v}\mathbf{v}^T)\bar{\mathbf{x}}'_l$.

3.6 Optimization

Once we have an initial guess, iterative optimization methods such as the reduced gradient method [9], sequential quadratic programming [6], or augmented Lagrangian method [2] can be used. At each iteration a local model around the current value is derived. The iterative methods search for the optimal solution of the local model by using the feasible gradient direction or the reduced Hessian matrix and update the current value.

Our software for partial model fitting used the sequential quadratic programming algorithm with trust region method. Given an equality constrained optimization problem

$$\min_{\theta} \quad f(\Theta) \qquad \text{subject to} \quad \mathbf{h}(\Theta) = 0 \tag{8}$$

At the kth iteration the standard sequential quadratic programming method approximates the objective function with a quadratic function $q_k(\Delta\Theta)$ and the constraints with a system of linear equations. Thus the local model can be represented by

$$\min_{\Delta\theta} \quad q_k(\Delta\Theta) \qquad \text{subject to} \quad H\Delta\Theta - h_0 = 0 \tag{9}$$

where q_k is

$$q_k(\Delta\Theta) = \frac{1}{2}(\Delta\Theta)^T Q \Delta\Theta + (\nabla f)^T \Delta\Theta$$

Q is the Hessian matrix of $\mathcal{L}(\Theta)$ at kth iteration

$$Q = \nabla_\theta^2 \mathcal{L}(\Theta^k, \lambda^k)$$

and $\mathcal{L}(\Theta, \lambda)$ is the Lagrangian function of problem (8). The matrix H contains the first order derivatives of the constraints.

$$H = \begin{pmatrix} \frac{\partial h_1}{\partial \Theta} \\ \cdot \\ \cdot \\ \frac{\partial h_r}{\partial \Theta} \end{pmatrix}$$

Let the quadratic subproblem (9) be rewritten as

$$\min_{\Delta\theta} \quad \frac{1}{2}(\Delta\Theta)^T Q(\Delta\Theta) + c^T\Delta\Theta \qquad \text{subject to} \quad H\Delta\Theta - b = 0 \tag{10}$$

where c denotes ∇f and b denotes $-\mathbf{h}(\Theta_{k-1})$.

This problem can be solved by using ***range space method*** if Q is positive definite and the constraint qualification is satisfied. The Lagrangian function of (10) can be constructed as

$$L = \frac{1}{2}(\Delta\Theta)^T Q(\Delta\Theta) + c^T\Delta\Theta + \lambda^T(H\Delta\Theta - b)$$

From the first order conditions for a minimum point $(\Delta\Theta^*, \lambda^*)$, $\nabla_{\Delta\theta} L = 0$ and $\nabla_\lambda L = 0$, we have

$$Q\Delta\Theta^* + c + H^T\lambda^* = 0 \tag{11}$$

$$H\Delta\Theta^* - b = 0 \tag{12}$$

Left multiplying (11) with HQ^{-1} gives

$$H\Delta\Theta^* + HQ^{-1}c + HQ^{-1}H^T\lambda^* = 0 \tag{13}$$

Substituting (12) into this equation leads to

$$\lambda^* = -(HQ^{-1}H^T)^{-1}(b + HQ^{-1}c) \tag{14}$$

The value $\Delta\Theta^*$ can then be derived from (13),

$$\Delta\Theta^* = -Q^{-1}(c + H^T\lambda^*) \tag{15}$$

If the condition for range-space method is not satisfied, ***null space method*** can be used. At first, we use orthogonal factorization to derive a matrix Z that contains a basis of the null space of H. Then we compute a particular solution $\Delta\Theta_0$ to the equation

$$H\Delta\Theta - b = 0.$$

The general feasible set can be expressed as

$$\Delta\Theta = \Delta\Theta_0 + Zw \tag{16}$$

By solving the problem

$$\min_{w} q_k(\Delta\Theta_0 + Zw)$$

we can obtain the optimal value of w,

$$w = -(Z^TQZ)^{-1}Z^T(Q\Delta\Theta_0 + c)$$

Replacing this result into (16) gives

$$\Delta\Theta^* = \Delta\Theta_0 - Z(Z^TQZ)^{-1}Z^T(Q\Delta\Theta_0 + c) \quad (17)$$

The Lagrange multipliers can be computed from $\Delta\Theta^*$ by solving

$$Q\Delta\Theta^* + c + H^T\lambda^* = 0$$

If the constraint qualification holds, the unique solution of the multipliers is

$$\lambda^* = -(HH^T)^{-1}H(Q\Delta\Theta^* + c) \quad (18)$$

The motivation for using trust region method is that the quadratic model at a given point is an adequate approximation of (8) in some region around that point. Usually a trust region at the kth iteration is defined by

$$\Omega^{(k)} = \{\Theta^k + \Delta\Theta \mid \|\Delta\Theta\| \leq \tau_k \quad (19)$$

where $\tau_k \in I\!R_+$. Without specification the default norm in the rest of the subsection is L_2 norm, which is most important in defining the trust region.

A important issue in numerical optimization is scaling [4]. If some variables vary greatly in magnitude, the roundoff errors of the variables with large magnitude may overshadow the variables with small magnitude. For example, if the point observations are in the range of $[10^3, 10^4]$ mm and the components of the normal vectors are in the range of $[10^{-2}, 10^{-1}]$, the singular value decomposition of the constraint matrix may be unstable. Thus, some angle constraints may not be satisfied. An obvious remedy is to rescale the independent variables such that the magnitudes are comparable. After finding the optimal solution, the variables are scaled back to the original scales.

4 Error Propagation

Suppose that the iteration converges to a local minimum point. To know the reliability of the result, we use the error propagation approach [15] to transform the input error covariance matrix to the output covariance matrix.

4.1 Linear Model

In the building estimation problem, we have the optimization model

$$\min_{\theta} \quad f(\Theta)$$
$$\text{subject to} \quad \mathbf{h}(\Theta) = 0$$

where f is the sum of squared errors between the estimated 3D points and the observed 3D points.

The Lagrangian function is

$$L(X', \Theta, \Lambda) = f(X', \Theta) + \Lambda^T\mathbf{h}(\Theta)$$

Suppose that $(\tilde{X}, \tilde{\Theta}, \tilde{\Lambda})$ is a optimal point. From the necessary conditions of a local minimum point, the linearized model at the optimal point can be obtained by solving [6] [16]

$$\begin{pmatrix} \tilde{\mathbf{Q}} & \tilde{\mathbf{H}}^T \\ \tilde{\mathbf{H}} & 0 \end{pmatrix} \begin{pmatrix} \Delta\Theta \\ \Delta\Lambda \end{pmatrix} = \begin{pmatrix} -\tilde{\mathbf{B}}\Delta X \\ 0 \end{pmatrix} \quad (20)$$

The Lagrangian matrix at the point of $(\tilde{X}, \tilde{\Theta}, \tilde{\Lambda})$ can be approximated by the Lagrangian matrix at the minimum if the error is small. Hence the linear model can be approximated by

$$\begin{pmatrix} \mathbf{Q}^* & (\mathbf{H}^*)^T \\ \mathbf{H}^* & 0 \end{pmatrix} \begin{pmatrix} \Delta\Theta \\ \Delta\Lambda \end{pmatrix} = \begin{pmatrix} -\mathbf{B}^*\Delta X \\ 0 \end{pmatrix}$$

where

$$\begin{aligned} \mathbf{Q}^* &= \nabla^2_{\theta\theta}\mathcal{L}(X', \hat{\Theta}, \hat{\Lambda}) \\ &= \nabla^2 f(X', \hat{\Theta}) + \sum_{j=1}^{r} \hat{\lambda}_j \nabla^2 h_j(\hat{\Theta}) \\ \mathbf{B}^* &= \nabla^2_{\theta\mathbf{X}}\mathcal{L}(X', \hat{\Theta}, \hat{\Lambda}) \\ &= \nabla^2_{\theta\mathbf{X}} f(X', \hat{\Theta}) \\ \mathbf{H}^* &= \nabla\mathbf{h}(\hat{\Theta}) \end{aligned}$$

Assume that the constraints are linearly independent. Then the row vectors in matrix H^* are linearly independent. We can use the null space method to compute the error propagation matrix J [6] [16].

Once the error propagation matrix is obtained, we can propagate the covariance matrix of the observations Σ to the output. The covariance matrix of the estimated parameters, Σ_Θ, can be approximated by $\Sigma_\Theta = J\Sigma J^T$.

5 Experimental Results

There are 38 images of RADIUS model board 1. One of them is shown in figure 9. Through multi-image triangulation, the 3D position of the building vertices were estimated. Based on these derived point coordinates, a ground map and a roof map of model board 1 can be drawn. Figure 10 shows the roof map. The shapes of some buildings are significantly distorted. Using optimization method we fit the triangulation result to the partial model of model board 1. The optimization result is shown in figure 11. The geometrical structures of the buildings are recovered through the constrained optimization.

Results on RADIUS Model Board 2 images are comparable.

To verify the result we project the estimated 3D building models onto the images of model board 2. Figure 12 shows the overlay of the projected 3D models on one of the images.

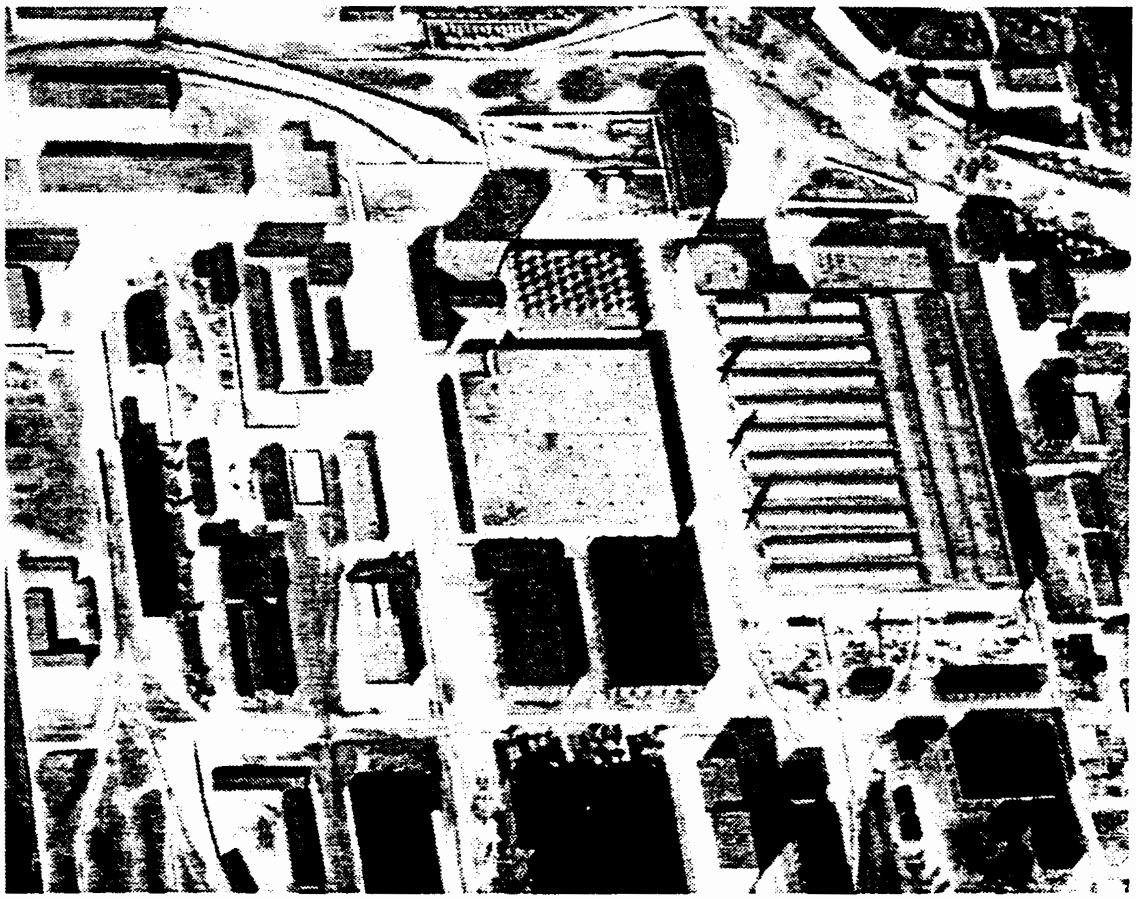


Figure 9: An image of RADIUS model board 1

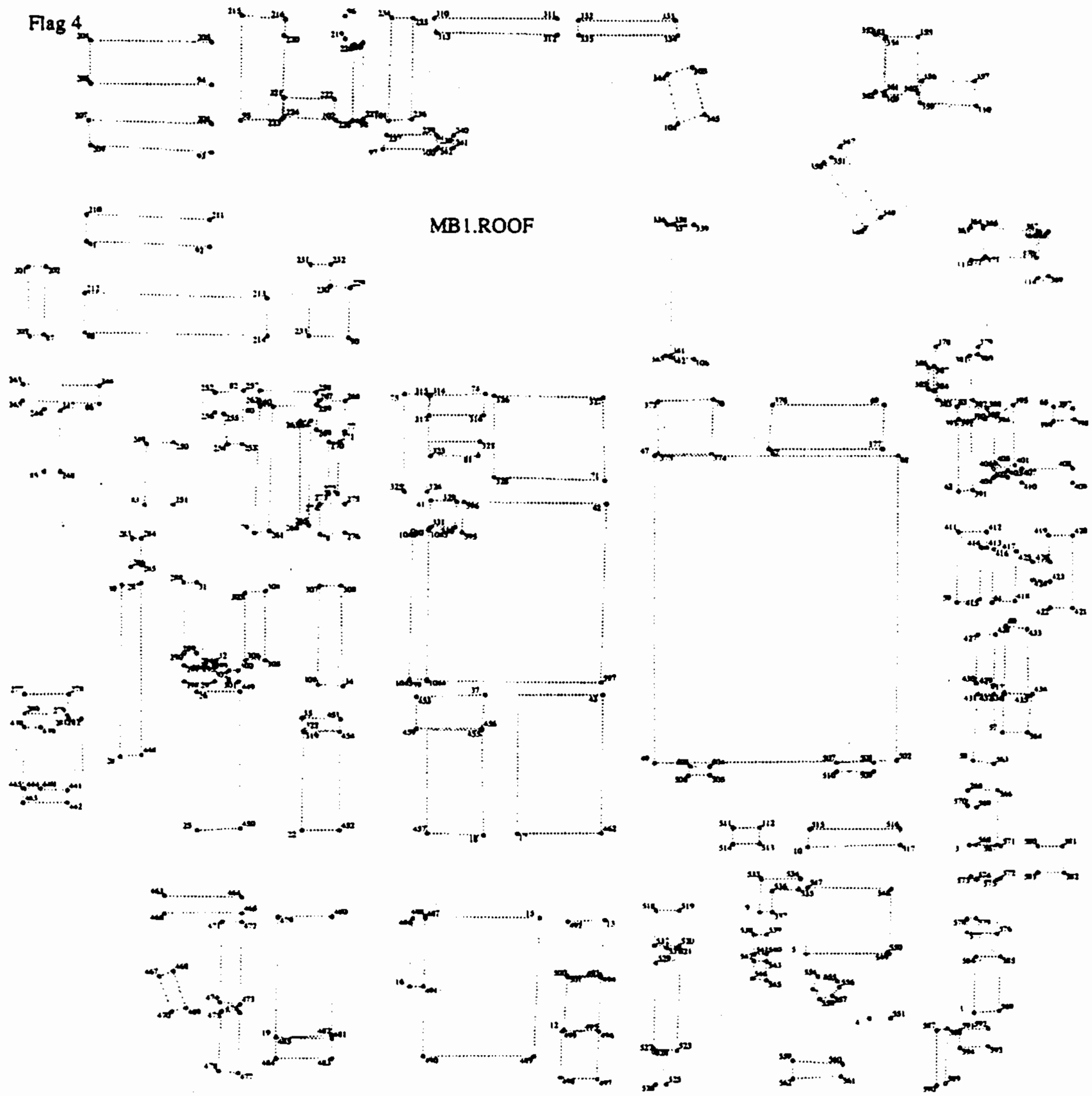


Figure 10: Roof map of Model Board 1 from triangulation

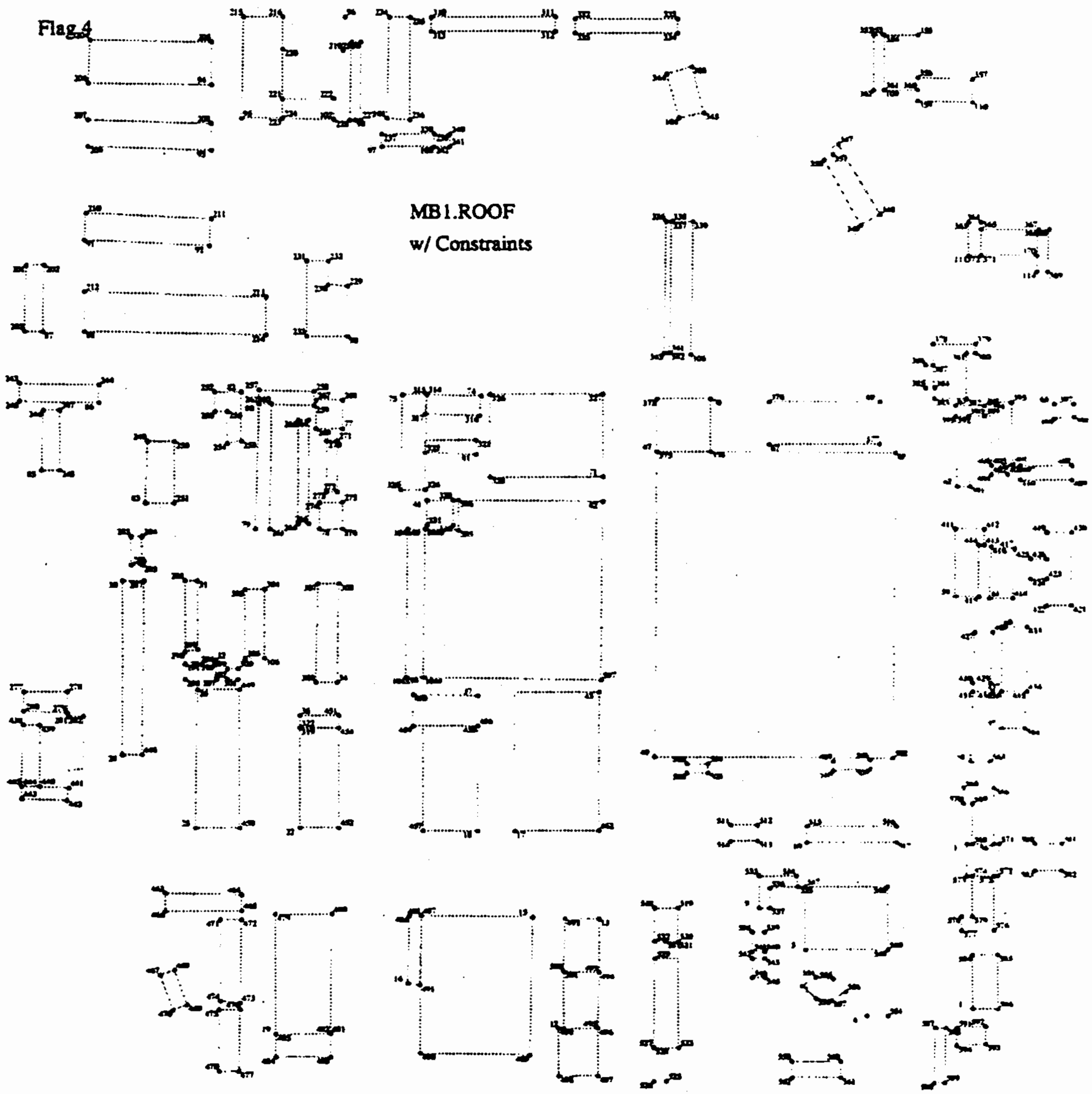


Figure 11: Roof map of Model Board 1 with geometrical constraints

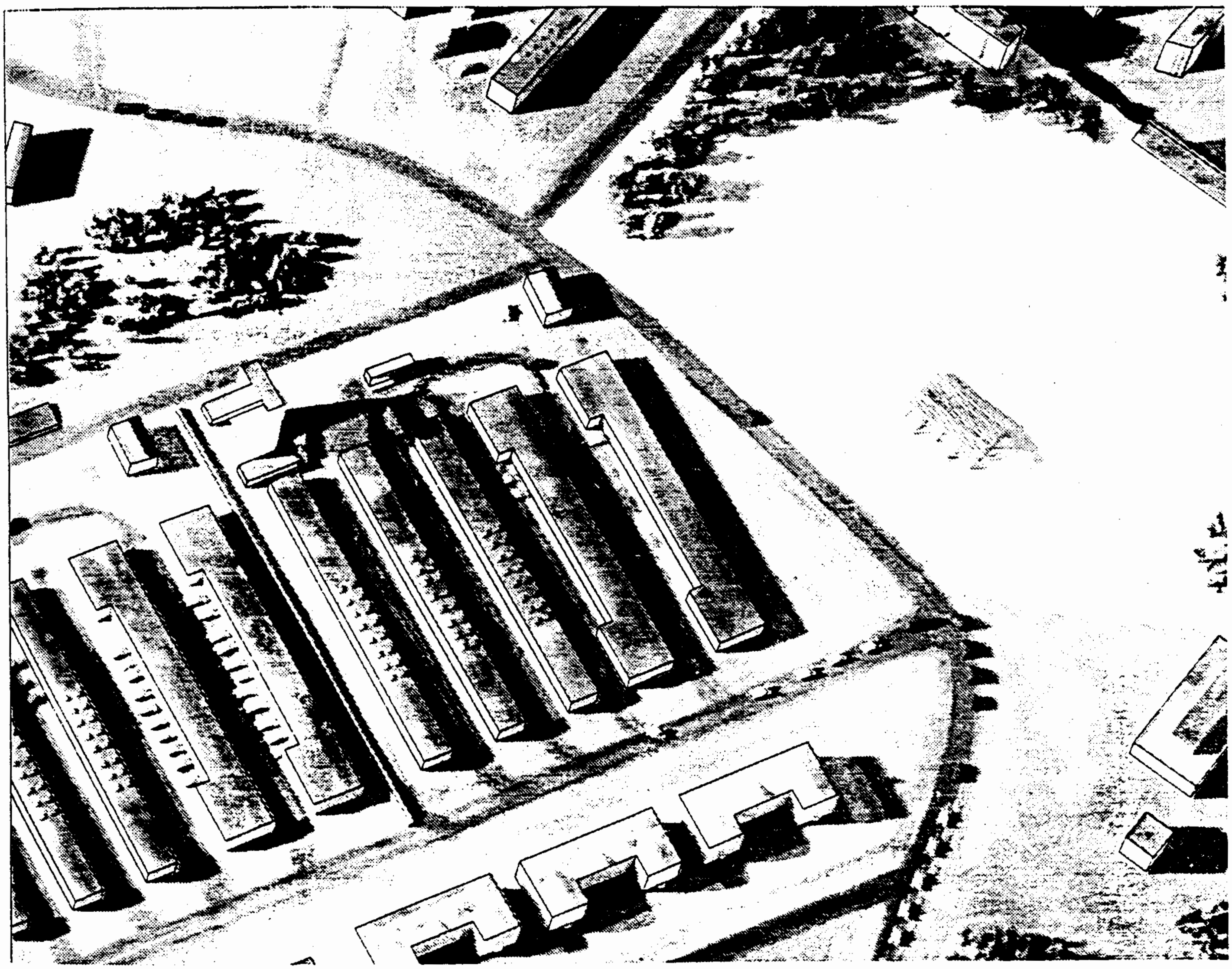


Figure 12: Projected building models on an image of model board 2

5.1 Comparison of Covariance Matrices

After obtaining an optimum solution for the real data, we can compute the error propagation model at the optimum point. By propagating the input error covariance matrix through the error propagation model, we can approximately determine the output error covariance matrix. Since the constrained optimization process integrates the input point data with the information of the geometric relations, the resultant point data should be more accurate that the input, i.e., the covariances of the output points should be smaller than that of the input points. For both the input data and the output data, we calculated the trace of the covariance matrix of each point and computed the distributions of these traces. The resultant distributions are illustrated in figure 13 and 14. The x axis in the figures is set to $10log_{10}(trace)$ for visualizing the results. In the input distribution the small traces around -60 are related to the ground-truth points where covariance matrices were set small (variances $\sigma^2 = 10^{-6}$) and the others are related to the triangulated points. In the output, the traces of the triangulated points are significantly improved. Most of them decrease about 10 dB.

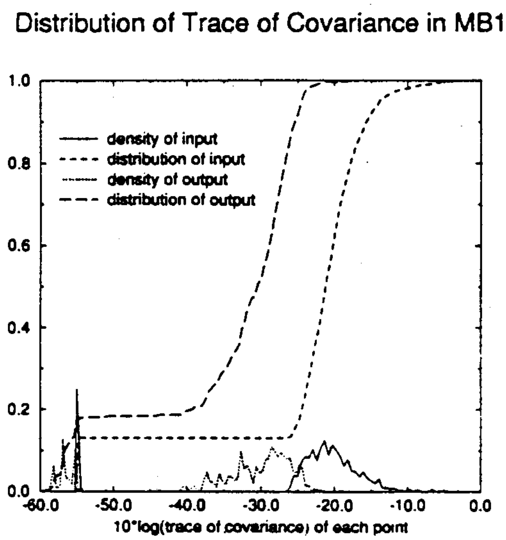


Figure 13: Distribution of trace of covariance matrix for model board 1

6 Conclusion

In this paper, we presented an optimization approach for the modeling and the performance characterization of 3D parameter estimation problems. This approach was applied to the building model construction problem to estimate the optimal parameters and their statistics.

The modeling is based on the partial models of the 3D objects, the projection model, the perturbation model and the observed data. The partial model of a

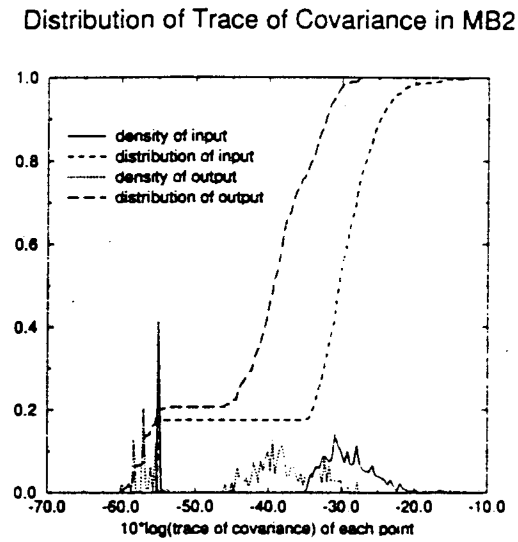


Figure 14: Distribution of trace of covariance matrix for model board 2

3D object consists of the geometric relations of the basic 3D elements (features) of the object. The observed image features are perturbed by random noise. The maximum posterior estimation problem can be framed as a constrained optimization problem. When the perturbation has a Gaussian distribution $\mathcal{N}(0, \Sigma)$ the optimization problem becomes a least squares problem.

The optimization approach was applied to the building model reconstruction problem in RADIUS project. Model boards 1 and 2 were processed and the resulting ground truth is available from the University of Washington on a set of CDROMS.

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