Session IVB: Optic Flow

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Rigid Body Motion and the Optic Flow Image

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ABSTRACT

The purpose of the article is to establish the relationship between Rigid Body Motion and the Optic Flow Image and solve motion parameters from optic flow. An essential equation relating optic flow to rigid body motion which only involves in rotation and translation velocities as well as optic flow is established. A sufficient condition under which mode of motion, rotation velocity, translation velocity orientation, and relative depth (or surface structure) can be uniquely determined by optic flow is set up. A unified and stable scheme is proposed to compute motion parameters and other related information by using a variation of the essential equation no matter whether or not the motion is a pure rotation. The relationship between rigid body motion and optic flow is cleared up:

Rigid Body Motion = Optic Flow + One Spatial Point Motion

I. INTRODUCTION

The purpose of the article is establish the relationship between rigid body motion and its corresponding optic flow perspective projections and solve motion parameters from optic flow. The optic flow perspective projection is contained in the optic flow image which for each pixel (X,Y) contains the projected motion (u,v).

In the article a basic and essential equation relating the optic flow image {[(X,Y),(u,v)]} and the rigid body motion (Ω, k) with Ω being the instantaneous rotation angular velocity and k being the instantaneous translation velocity is established (see section 2). The equation is as follows:

(*)

$$[u,\ v,\ o]\ (k \times \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}) = (k \times \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix})'\ (\Omega \times \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix})$$

or

$$\left| \begin{bmatrix} u \\ v \\ o \end{bmatrix} - \Omega \times \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix},\ \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix},\ k \right| = 0$$

which is a nonlinear equation for (Ω, k).

It is proved (see section III) that under a certain condition the general solution of the nonlinear equation (*) is (Ω, k*) with k* = α k when k $\neq$ 0 or k* an arbitrary vector when k = 0. Moreover, the nonlinear equation (*) is equivalent to the following linear equation

(**) $$\Big(\sum_{[(X,Y),(u,v)]} B'\ B \Big) h = 0$$

where $B = [X^2,\ Y^2,\ 1,\ XY,\ X,\ Y,\ v,\ -u,\ uY-vX]$. The relationship between two general solutions of equations (*) and (**) can be described by

$$\begin{bmatrix} h7 \\ h8 \\ h9 \end{bmatrix} = \begin{bmatrix} k_1^* \\ k_2^* \\ k_3^* \end{bmatrix}$$

and

$$\begin{bmatrix} h1 \\ h2 \\ h3 \\ h4 \\ h5 \\ h6 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_2 & -\omega_3 \\ -\omega_1 & 0 & -\omega_3 \\ -\omega_1 & -\omega_2 & 0 \\ \omega_2 & \omega_1 & 0 \\ \omega_3 & 0 & \omega_1 \\ 0 & \omega_3 & \omega_2 \end{bmatrix} \begin{bmatrix} k_1^* \\ k_2^* \\ k_3^* \end{bmatrix}$$

The relation is invertible if h $\neq$ 0 or equivalently

$$\begin{bmatrix} h_7 \\ h_8 \\ h_9 \end{bmatrix} \neq 0.$$

Concerning equation (**) itself, there holds

$$\text{Rank}\left(\sum_{[(X,Y),(u,v)]} B'B \right) = 8 \quad \text{iff} \quad k \neq 0$$

or

$$\text{Rank}\left(\sum_{[(X,Y),(u,v)]} B'B \right) = 6 \quad \text{iff} \quad k = 0$$

It is also proved (see section III) that under certain conditions Ω, mode of motion (i.e, if it is a pure rotation or not), $\frac{k}{\|k\|}$ (if $k \neq 0$) and the so-called relative depth, i.e. the surface structure, (if $k \neq 0$) can all be uniquely determined by the optic flow image.

In section IV, it is proved that the rigid body motion = the optic flow image + One spatial point motion.

In section V, a related algorithm and experimental results are given.

Section VI is a summary. The topic "Rigid Body Motion and Optic Flow" is a basic concern in the computer vision circles. Many important results have been obtained (see References). However, the article has something new. We develop an essential optic flow equation which does not involve the depth information Uniqueness of motion mode (whether pure rotation or not), rotational velocity, translation orientation and relative depth (when translation velocity $k \neq 0$) from the optic flow are all proved under certain conditions. A unified scheme which is equivalent to solving the smallest eigenvalue-vector of a nonnegative 9 x 9 matrix $\sum_{[(X,Y),(u,v)]} B'B$ (by using, for instance, the Singular Value Decomposition stable computational scheme) is set up no matter whether or not the motion is a pure rotation. The goal to clear up the relationship between rigid body motion and optic flow is achieved:

Rigid Body Motion = Optic Flow + One Spatial Point Motion which means that given optic flow and one spatial point motion the rigid body motion is uniquely determined.

II. Basic Relation Between Rigid Body Motion and Optic Flow Image

Suppose a rigid body is in motion in the half space $\{z < 0\}$. The motion can be uniquely represented by a translation vector $T_o(t)$ and a rotation matrix $R_o(t)$ (i.e. an orthonormal matrix of the first kind $R_o R_o' = I_3$, $\det(R_o) = 1$) as follows:

$$P(t) = R_o(t)P(o) + T_o(t) \qquad (1)$$

where $P(t)$ represents the position vector of object point at the time t.

Let $(X(t), Y(t))$ be central projective coordinates of $P(t)$ visible from the origin O onto the plane $z = +1$. Let $(u(t), v(t))$ represent the instantaneous velocity of $(X(t), Y(t))$. Thus, $\{[(X(t), Y(t)), (u(t), v(t))]\}$ represents a frame of the optic flow image.

What kind of relations exists between the optic flow image and the motion parameters T_o, R_o? It can be understood that the problem has the basic importance. Before answering the problem, we need to establish a descriptive form of instantaneous motion. We can proceed as follows:

$$P(t+\Delta t) = R_o(t+\Delta t)P(O) + T_o(t+\Delta t) \quad (\text{by}(1)) \qquad (2)$$

$P(t+\Delta t) = R(t,\Delta t)P(t) + T(t,\Delta t)$ (from t to t + Δt, it is assumed that the rigid body motion is specified by $T(t,\Delta t), R(t,\Delta t)$) (3)

It is obvious that

$$R(t,0) = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(t,0) = 0 \qquad (4)$$

Thus, there are a matrix $S(t)$ and a vector $k(t)$ so that

$$R(t,\Delta t) = I_3 + S(t)\Delta t$$

$$T(t,\Delta t) = k(t)\Delta t \qquad (5)$$

Replacing $P(t)$ in (3) by $P(t) = R_o(t)P(o) + T_o(t)$, it follows

$$P(t+\Delta t) = R(t,\Delta t)R_o(t)P(o) + T(t,\Delta t) + R(t,\Delta t)T_o(t) \qquad (6)$$

Since $P(o)$ can be any point on the rigid body, it follows in comparison with (2)

$$R_o(t+\Delta t) = R(t,\Delta t)R_o(t) \qquad (7)$$

$$T_o(t+\Delta t) = T(t,\Delta t) + R(t,\Delta t)T_o(t)$$

or

$$R(t,\Delta t) = R_o(t+\Delta t)R_o'(t)$$

$$T(t,\Delta t) = T_o(t+\Delta t) - R(t,\Delta t)T_o(t) \qquad (8)$$

$$= T_o(t+\Delta t) - R_o(t+\Delta t)R_o'(t)T_o(t)$$

Thus, noticing (5), it follows

$$R_o(t+\Delta t)R_o'(t) \doteq I_3 + S(t)\Delta t \quad (9)$$
$$T_o(t+\Delta t) - R_o(t+\Delta t)R_o(t)T_o(t) = k(t)\Delta t$$

or

$$S(t) \doteq \frac{R_o(t+\Delta t) - R_o(t)}{\Delta t} \cdot R_o'(t)$$
$$k(t) \doteq \frac{T_o(t+\Delta t) - T_o(t)}{\Delta t} - S(t)T_o(t) \quad (10)$$

As a result,

$$S(t) = \dot{R}_o(t)R_o'(t) = -R_o(t)\dot{R}_o'(t)$$
$$k(t) = \dot{T}_o(t) - S(t)T_o(t) \quad (11)$$

Notice that $S'(t) = -S(t)$. From (5), it is clear that $S(t)$ represents the instantaneous rotation velocity and $k(t)$ represents the instantaneous translation velocity. We let

$$S(t) = \begin{bmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{bmatrix} \quad (12)$$

and $\Omega(t) = \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \end{bmatrix}$ From (3) and (5), we have

$$P(t+\Delta t) \doteq (I_3+S(t)\Delta t)P(t) + k(t)\Delta t \quad (13)$$

and hence

$$\dot{P}(t) = S(t)P(t) + k(t) \quad (13)'$$

It could be verified that

$$S(t)P(t) = \Omega(t) \times P(t) \quad (14)$$

Therefore,

$$\dot{P}(t) = \Omega(t) \times P(t) + k(t) \quad (15)$$

which is the ordinary instantaneous motion

<u>Lemma 1</u>. $R_o(t) = \exp(\int_0^t S(\tau)d\tau)$

$$T_o(t) = \int_0^t \exp(\int_\tau^t S(\xi)d\xi)k(\tau)d\tau \quad (16)$$

Proof. We can rewrite (11) as follows

$$\begin{cases} \dot{R}_o(t) - S(t)R_o(t) = 0 \\ \dot{T}_o(t) - S(t)T_o(t) - k(t) = 0 \end{cases} \quad (17)$$

From (1), we have the following initial

$$\begin{cases} R_o(0) = I_3 \\ T_o(0) = 0 \end{cases} \quad (18)$$

It is classical in the theory of ordinary differential equations that $R_o(t)$ and $T_o(t)$ are uniquely determined by (16). □

Lemma 1 and (11) indicate that two descriptions (T_o,R_o) and (k,S) of rigid body motion are equivalent to each other. Thus, exploring the relationship between the optic flow image $\{[(X,Y),(u,v)]\}$ and the motion (T_o,R_o) is equivalent to exploring the one between the optic flow image and the same motion (k,S). It is worth noting that $k(t)\neq\dot{T}_o(t)$ in general. A interesting fact which seems to conflict the superficial intuition.

From (13)' it follows

$$\dot{P}(t) = \frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \frac{d}{dt}\left\{z(t)\begin{bmatrix} X(t) \\ Y(t) \\ 1 \end{bmatrix}\right\} = \dot{z}(t)\begin{bmatrix} X(t) \\ Y(t) \\ 1 \end{bmatrix} + z(t)\begin{bmatrix} u(t) \\ v(t) \\ 0 \end{bmatrix}$$
$$= z(t)\,S(t)\begin{bmatrix} X(t) \\ Y(t) \\ 1 \end{bmatrix} + k(t) = z(t)\Omega(t) \times \begin{bmatrix} X(t) \\ Y(t) \\ 1 \end{bmatrix} + k(t) \quad (19)$$

Thus

$$z(t)\left\{\begin{bmatrix} u(t) \\ v(t) \\ 0 \end{bmatrix} - \Omega(t) \times \begin{bmatrix} X(t) \\ Y(t) \\ 1 \end{bmatrix}\right\} + \dot{z}(t)\begin{bmatrix} X(t) \\ Y(t) \\ 1 \end{bmatrix} = k(t) \quad (20)$$

The equality (19) implies necessarily

$$\left|\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} - \Omega \times \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}, \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}, k\right| = 0 \quad (21)$$

$$[u,v,0]\left(k \times \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}\right) = \left(k \times \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}\right)' \left(\Omega \times \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}\right) \quad (22)$$

which is the basic relation between the optic flow and the motion parameters k, S.

III. From Optic Image to the Rotation Ω, the Translation Orientation $k/\|k\|$ when $k \neq 0$ and the Relative Depth (i.e. Surface Structure) when $k \neq 0$

We have established a basic equation which the motion parameters k, Ω satisfy. How about uniqueness? Suppose (k^*, Ω^*) is a solution of (22). That is

$$[u,v,0]\left(k^* \times \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}\right) = \left(k^* \times \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}\right)' \left(\Omega^* \times \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}\right) \quad (23)$$

What is the relationship between (k^*, Ω^*) and the motion parameters (k, Ω)? From (20), we have

$$z \begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = z\,\Omega \times \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} - \dot{z} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} + k \quad (24)$$

which is the equation the true motion parameters (k, Ω) have to satisfy with.
Replacing $[u,v,0]$ in (23) by (24), it follows

$$(\Omega x P)'(k^* x P) - \frac{\dot{z}}{z} P' (k^* x P) + k' (k^* x P)$$

$$= (k^* x P)' (\Omega^* x P)$$

$$[(\Omega x P)' - (\Omega^* x P)'](k^* x P) + k' (k^* x P) = 0$$

$$[(\Omega - \Omega^*) x P]' (k^* x P) + k' (k^* x P) = 0 \quad (25)$$

Let $K(t) = \begin{bmatrix} 0 & -k_3(t) & k_2(t) \\ k_3(t) & 0 & -k_1(t) \\ -k_2(t) & k_1(t) & 0 \end{bmatrix}$,

$$K^*(t) = \begin{bmatrix} 0 & -k_3^*(t) & k_2^*(t) \\ k_3^*(t) & 0 & -k_1^*(t) \\ -k_2^*(t) & k_1^*(t) & 0 \end{bmatrix}$$

$$\Omega^*(t) = \begin{bmatrix} \omega_1^*(t) \\ \omega_2^*(t) \\ \omega_3^*(t) \end{bmatrix}, \quad S^*(t) = \begin{bmatrix} 0 & -\omega_3^*(t) & \omega_2^*(t) \\ \omega_3^*(t) & 0 & -\omega_1^*(t) \\ -\omega_2^*(t) & \omega_1^*(t) & 0 \end{bmatrix}$$

and remember

$$\Omega(t) = \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \end{bmatrix}, \quad S(t) = \begin{bmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{bmatrix}.$$

We can rewrite (25) as follows

$$[(S-S^*)P]' K^*P + k' K^*P = 0$$

or

$$P'(S^*-S)K^*P + k'K^*P = 0 \quad (26)$$

Lemma 2. Assume that the part on the rigid body producing the optic flow image can not be contained in a quadratic surface containing the origin O. Then, (k^*, Ω^*) is a solution of (22) iff

$$k'K^* = 0$$
and
$$(S^*-S)K^* + K^*(S^*-S) = 0 \quad (27)$$

Proof. From the previous reasoning, it is clear that (k^*, Ω^*) is a solution of (22) iff (k^*, Ω^*) satisfies (26) with (k, Ω) being the genuine motion parameters. Under the assumption of lemma, (k^*, Ω^*) satisfies (26) for all points P's which produce optic flow image iff the coefficients of both the first term and the second term in (26) should be zero. That means

$$k' K^* = 0$$
and
$$(S^*-S)K^* + [(S^*-S)K^*]' = 0$$
the latter leads to (27). □

Lemma 3. $k' K^* = 0$ iff $k^* = \alpha k$ with α any real number when $k \neq 0$ or any vector (K^* any skew-symmetric matrix) when $k=0$.

Proof. Since $k' K^* = (k \times k^*)'$, the conclusion is immediate. □

Lemma 4. (26) holds iff $K^* = 0$ or $K^* \neq 0$ and $S^* = S$.

Proof. (26) holds iff $(S^*-S)K^*$ is skew-symmetric.

Let $S^*-S = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$. Then

$$(S^*-S)K^* = \begin{bmatrix} -\omega_3 k_3^* - \omega_2 k_2^* & \omega_2 k_1^* & \omega_3 k_1^* \\ \omega_1 k_2^* & -\omega_3 k_3^* - \omega_1 k_1^* & \omega_3 k_2^* \\ \omega_1 k_3^* & \omega_2 k_3^* & -\omega_2 k_2^* - \omega_1 k_1^* \end{bmatrix}$$

Thus

$(S^*-S)K^*$ is skew-symmetric iff

$$\omega_3 k_3^* + \omega_2 k_2^* = \omega_3 k_3^* + \omega_1 k_1^* = \omega_2 k_2^* + \omega_1 k_1^* = 0 \quad (28)$$

$$\omega_2 k_1^* + \omega_1 k_2^* = \omega_1 k_3^* + \omega_3 k_1^* = \omega_2 k_3^* + \omega_3 k_2^* = 0 \quad (29)$$

It is easy to see that (28) is equivalent to

$$\omega_1 k_1^* = \omega_2 k_2^* = \omega_3 k_3^* = 0 \quad (30)$$

Suppose $k_1^* \neq 0$. Then, (30) implies $\omega_1=0$ and the first two equalities in (29) lead to $\omega_2 = \omega_3 = 0$. Similar treatment applied to $k_2^* \neq 0$ or $k_3^* \neq 0$. Therefore, $(S^*-S)K^*$ is skew-symmetric iff $K^* = 0$ or $K^* \neq 0$ and $S^*-S = 0$. □

<u>Theorem 1</u>. Under the same assumption as Lemma 2, the basic equation (22) has the general solution $(\alpha k, S)$ (α any real number) when $k \neq 0$ or (k^*, S) (k^* any real vector) when $k = 0$.
Proof. Lemma 2-4 imply Theorem 1. □

Rewrite (22) as follows:

$$[u,v,0]\ K \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = [X,Y,1]\ K\ S \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \quad (31)$$

Suppose K^*, L^* satisfy

$$[u,v,0]\ K^* \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = [X,Y,1]\ L^* \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \quad (32)$$

for all optic flow image points $\{[(X,Y),(u,v)]\}$. What is the relationship between the motion (K,S) (i.e. (k,Ω)) and (K^*, L^*)? Similar to the previous arguments, it is clear that (32) holds iff

$$k'\ K^* = 0 \quad (33)$$

and

$$L^* + L^{*'} = S\ K^* + K^* S \quad (34)$$

when the assumption in the Lemma 2 holds.
<u>Theorem 2</u>. Under the same assumption as Lemma 2, given a solution (K^*, L^*) of the equation (32) with $K^* \neq 0$, the rotation S is uniquely determined by (K^*, L^*) and the translation k is parallel to k^*.
Proof. Under the assumptions of Theorem 2, the relations (33) and (34) hold. (33) implies that the translation k is parallel to k^* (here hence K^* must be skew-symmetric). (34) is equivalent to

$$\begin{cases} -2(\omega_3 k_3^* + \omega_2 k_2^*) = 2\ l_{11}^* \\ -2(\omega_3 k_3^* + \omega_1 k_1^*) = 2\ l_{22}^* \\ -2(\omega_2 k_2^* + \omega_1 k_1^*) = 2\ l_{33}^* \end{cases} \quad (35)$$

$$\begin{cases} \omega_2 k_1^* + \omega_1 k_2^* = l_{12}^* + l_{21}^* \\ \omega_1 k_3^* + \omega_3 k_1^* = l_{13}^* + l_{31}^* \\ \omega_2 k_3^* + \omega_3 k_2^* = l_{23}^* + l_{32}^* \end{cases} \quad (36)$$

It is easy to see that (35) is equivalent to

$$\begin{cases} \omega_1 k_1^* = \dfrac{l_{11}^* - l_{22}^* - l_{33}^*}{2} \\ \omega_2 k_2^* = \dfrac{l_{22}^* - l_{11}^* - l_{33}^*}{2} \\ \omega_3 k_3^* = \dfrac{l_{33}^* - l_{11}^* - l_{22}^*}{2} \end{cases} \quad (37)$$

If $k_1^* \neq 0$, then

$$\omega_1 = \frac{l_{11}^* - l_{22}^* - l_{33}^*}{2k_1^*} \quad (38)$$

From (36) we obtain

$$\begin{cases} \omega_2 = \dfrac{l_{12}^* + l_{21}^* - \omega_1 k_2^*}{k_1^*} \\ \omega_3 = \dfrac{l_{13}^* + l_{31}^* - \omega_1 k_3^*}{k_1^*} \end{cases} \quad (39)$$

Furthermore, if $k_2^* \neq 0$, we also can use the following formula to compute ω_2:

$$\omega_2 = \frac{l_{22}^* - l_{11}^* - l_{33}^*}{2k_2^*} \quad (40)$$

which must in agreement with (39) since the rotation S exists objectively, satisfies (34), and both (34) and (40) uniquely determine ω_2. □

Because of Theorem 2, we attempt to solve the following equation (41) instead of (22) for (k^*, L^*)

$$[u,v,0]\begin{bmatrix} 0 & -k_3^* & k_2^* \\ k_3^* & 0 & -k_1^* \\ -k_2^* & k_1^* & 0 \end{bmatrix}\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \quad (41)$$

$$[X,Y,1]\begin{bmatrix} l_{11}^* & l_{12}^* & l_{13}^* \\ l_{21}^* & l_{22}^* & l_{23}^* \\ l_{31}^* & l_{32}^* & l_{33}^* \end{bmatrix}\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Expanding and rearranging (41) leads to

$$[X^2, Y^2, 1, XY, X, Y, v, -u, uY-vX]\begin{bmatrix} l_{11}^* \\ l_{22}^* \\ l_{33}^* \\ l_{12}^*+l_{21}^* \\ l_{13}^*+l_{31}^* \\ l_{23}^*+l_{32}^* \\ k_1^* \\ k_2^* \\ k_3^* \end{bmatrix} = 0 \quad (42)$$

Let

$$B = [X^2, Y^2, 1, XY, X, Y, v, -u, uY-vX], \quad (43)$$

$$h = \begin{bmatrix} l_{11}^* \\ l_{22}^* \\ l_{33}^* \\ l_{12}^*+l_{21}^* \\ l_{13}^*+l_{31}^* \\ l_{23}^*+l_{32}^* \\ k_1^* \\ k_2^* \\ k_3^* \end{bmatrix}. \quad (44)$$

It is obvious that

$$\{ \sum_{[(X,Y),(u,v)]} B' \; B \} h = 0 \quad (45)$$

Let

$$W = \sum_{[(X,Y),(u,v)]} B' B \quad (46)$$

It is clear that $W \geq 0$ and h is an eigenvector corresponding to the smallest eigenvalue (i.e. zero) of W.

From (35) and (36) it is clear that within l_{11}^*, l_{22}^*, l_{33}^*, $l_{12}^*+l_{21}^*$, $l_{13}^*+l_{31}^*$, $l_{23}^*+l_{32}^*$, each one is a linear homogeneous function of k_1^*, k_2^*, k_3^*. Thus, relating to Theorem 1, it follows

Theorem 3. The general solution h of (45) is one-parameter iff $k \neq 0$, is three-parameter iff $k = 0$. And

$$\text{Rank}(W) = 8 \quad \text{iff } k \neq 0 \quad (47)$$

$$\text{Rank}(W) = 6 \quad \text{iff } k = 0 \quad (48)$$

Theorem 4. $k=0$ iff for two optic flow image points $\{[(X_i, Y_i), (u_i, v_i)] \; i=1,2\}$ there holds

$$\begin{vmatrix} \omega_1 & \omega_2 \\ X_i & Y_i \end{vmatrix}\left(\begin{bmatrix} X_i \\ Y_i \end{bmatrix} + \begin{bmatrix} u_i \\ v_i \end{bmatrix}\right) = \begin{bmatrix} \begin{vmatrix} \omega_2 & \omega_3 \\ Y_i & 1 \end{vmatrix} \\ \begin{vmatrix} \omega_3 & \omega_2 \\ 1 & X_i \end{vmatrix} \end{bmatrix}, \quad i=1,2. \quad (49)$$

Proof.

From $\dot{z}\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} + z\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = z\,\Omega x\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} + k,$

it follows

$$\dot{z} = z\begin{vmatrix} \omega_1 & \omega_2 \\ X & Y \end{vmatrix} + k3 \quad (50)$$

and

$$z\begin{vmatrix} \omega_1 & \omega_2 \\ X & Y \end{vmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} + k_3\begin{bmatrix} X \\ Y \end{bmatrix} + z\begin{bmatrix} u \\ v \end{bmatrix} = z\begin{bmatrix} \begin{vmatrix} \omega_2 & \omega_3 \\ Y & 1 \end{vmatrix} \\ \begin{vmatrix} \omega_3 & \omega_1 \\ 1 & X \end{vmatrix} \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \quad (51)$$

Noticing $z < 0$, it is clear that $k = 0$ if and only if for two optic flow image points $\{[X_i, Y_i), (u_i, v_i)] \; i=1,2\}$ it holds

$$\begin{vmatrix} \omega_1 & \omega_2 \\ X_i & Y_i \end{vmatrix}\begin{bmatrix} X_i \\ Y_i \end{bmatrix} + \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} \omega_2 & \omega_3 \\ Y_i & 1 \end{vmatrix} \\ \begin{vmatrix} \omega_3 & \omega_2 \\ 1 & X_i \end{vmatrix} \end{bmatrix}, \quad i= 1,2. \quad (52)$$

□

Theorem 5. Keep the same assumptions of Lemma 2. Assume that (k^*, Ω) is a solution of (23) with $k^* \neq 0$ and $\begin{bmatrix} X \\ Y \end{bmatrix}$ is linear independent with $\begin{bmatrix} k_1^* \\ k_2^* \end{bmatrix}$.

Then, the following equation (53) has a solution for α iff $k \neq 0$.

$$\alpha \begin{vmatrix} \omega_1 & \omega_2 \\ X & Y \end{vmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + k_3^* \begin{bmatrix} X \\ Y \end{bmatrix} + \alpha \begin{bmatrix} u \\ v \end{bmatrix} = \alpha \begin{bmatrix} \begin{vmatrix} \omega_2 & \omega_3 \\ Y & 1 \end{vmatrix} \\ \begin{vmatrix} \omega_3 & \omega_1 \\ 1 & X \end{vmatrix} \end{bmatrix} + \begin{bmatrix} k_1^* \\ k_2^* \end{bmatrix} \quad (53)$$

Moreover, when (53) is solvable for α, the solution is unique and

$$\frac{k}{\|k\|} = -\text{sign}(\alpha) \frac{k^*}{\|k^*\|} \quad (54)$$

Proof. (If) $k \neq 0$ implies $k^* = \beta k$ with some real number β by Theorem 1. Thus, $\alpha = z\beta$ will be a solution of (53).
(Only if) Suppose that α is a solution of (53) and $k = 0$. In this case, applying (52), it follows

$$k_3^* \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} k_1^* \\ k_2^* \end{bmatrix} \quad (55)$$

which contradicts the assumption that $\begin{bmatrix} X \\ Y \end{bmatrix}$ and $\begin{bmatrix} k_1^* \\ k_2^* \end{bmatrix}$ are linear independent.

Concerning the latter part of Theorem 5, we point out that the existence of two solutions, α_1 and α_2, would lead to

$$(\alpha_1-\alpha_2)k_3^* \begin{bmatrix} X \\ Y \end{bmatrix} = (\alpha_1-\alpha_2) \begin{bmatrix} k_1^* \\ k_2^* \end{bmatrix} \quad (56)$$

a contradiction.

Using $k^* = \beta k$ and $\alpha = z\beta$, it is obvious that

$$\frac{k^*}{\|k^*\|} = \text{sign}(\beta) \frac{k}{\|k\|} = -\text{sign}(\alpha) \frac{k}{\|k\|}$$

since $z < 0$. □

Corollary. If $k = 0$ and $\begin{bmatrix} X_i \\ Y_i \end{bmatrix}$ $(i = 1, 2)$ is linear independent with k^*, then

$$\alpha_i = z_i \beta, \quad i = 1, 2$$

As a result

$$\frac{z_1}{z_2} = \frac{\alpha_1}{\alpha_2} \quad (57)$$

Theorem 6. Assume $k \neq 0$. Then, except at most one image point, the relative depth

$$\frac{z}{\|k\|} = -\frac{1}{\|k^*\|} \cdot \frac{\|b\|}{\|a\|} \quad (58)$$

where

$$a = \begin{vmatrix} \omega_1 & \omega_2 \\ X & Y \end{vmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} - \left[\begin{vmatrix} \omega_2 & \omega_3 \\ Y & 1 \end{vmatrix}, \begin{vmatrix} \omega_3 & \omega_1 \\ 1 & X \end{vmatrix} \right]' \quad (59)$$

$$b = \begin{bmatrix} k_1^* \\ k_2^* \end{bmatrix} - k_3^* \begin{bmatrix} X \\ Y \end{bmatrix} \quad (60)$$

Proof. $k \neq 0$ implies $k^* = k$ and also $a \neq 0$ except for at most one image point by Theorem 4. The equation $\alpha a = b$ has a solution $\alpha = z\beta$ as clear by the argument in Theorem 5. Thus, except for at most one image point, the following equality holds:

$$\frac{\|b\|}{\|a\|} = |z| \cdot |\beta| = |z| \cdot \frac{\|k^*\|}{\|k\|}$$

or

$$\frac{|z|}{\|k\|} = \frac{\|b\|}{\|k^*\| \cdot \|a\|}$$

Noticing $z < 0$, it follows

$$\frac{z}{\|k\|} = -\frac{\|b\|}{\|k^*\| \cdot \|a\|}. \quad \square$$

Theorem 7. Assume $k \neq 0$. Then,

$\frac{k}{\|k\|} = \pm \frac{k^*}{\|k^*\|}$ holds if and only if there are three optic flow image points so that

$$a\ \|b\| \pm b\ \|a\| = 0 \quad (61)$$

Proof. $k \neq 0$ implies

$$k^* = \beta k$$

and

$$(z\beta)\ a = b$$

Thus, the fact that k has the same orientation as $\pm k^*$ implies that a has the same orientation as $\pm b$ since z is always negative. Thus,

$\frac{k}{\|k\|} = \pm \frac{k^*}{\|k^*\|}$ implies (61)

Conversely, assume that there are three optic flow image points. Since a is not zero except for at most one image point by Theorem 4 and b is not zero except for at most one image point as easily seen from (60). Thus, there is at least one within the three points such that both a and b are not zero. In that case, the equality (61) for that point implies

$$\frac{a}{\|a\|} = \mp \frac{b}{\|b\|}$$

where "-" implies $\beta > 0$ and hence $\frac{k}{\|k\|} = \frac{k^*}{\|k^*\|}$ and "+" implies $\beta < 0$ and hence $\frac{k}{\|k\|} = -\frac{k^*}{\|k^*\|}$. □

Remark. For the case of k being nonzero, at least eight image points are needed since Rank $(\Sigma A'A)=8$. As a result, there are at least six image points where both a and b are not zero.

IV. Rigid Body Motion = Optic Flow Image + One Spatial Point Motion

Suppose $k_3 = 0$ (for instance, when $t \geq \tau$). Then, from (50) it follows

$$\dot{z} = z \begin{vmatrix} \omega_1 & \omega_2 \\ X & Y \end{vmatrix} \qquad (t \geq \tau)$$

and

$$z(t) = z(\tau) \exp\left(\int_\tau^t \begin{vmatrix} \omega_1(\xi) & \omega_2(\xi) \\ X(\xi) & Y(\xi) \end{vmatrix} d\xi\right) \qquad (62)$$

That is, the absolute depth $z(t)$ can be uniquely determined given the absolute depth $z(\tau)$ at the time τ. By the way,

$$P(t) = z(t) \begin{bmatrix} X(t) \\ Y(t) \\ 1 \end{bmatrix} \text{ and } \dot{P}(t) = z(t) \begin{vmatrix} \omega_1(t) & \omega_2(t) \\ X(t) & Y(t) \end{vmatrix} \begin{bmatrix} X(t) \\ Y(t) \\ 1 \end{bmatrix} + z(t) \begin{bmatrix} u(t) \\ v(t) \\ 0 \end{bmatrix}$$

are uniquely determined. Thus, $k = \dot{P}(t) - \Omega(t) \times P(t)$ is uniquely determined.

In general, we have

Theorem 8. The rigid body motion (k,Ω) = the optic flow image + one spatial point motion.

Proof. The optic flow image uniquely determines the rotation $\Omega(t)$. The rotation $\Omega(t)$ plus one spatial pont motion $\{P(t),\dot{P}(t)\}$ uniquely determines the translation by

$$k(t) = \dot{P}(t) - \Omega(t) \times P(t)$$

□

Example 1. $\Omega = 0$. Then (25) reduces to

$$[u,v,0] \left(k \times \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}\right) = 0 \qquad (63)$$

or equivalently,

$$[u,v] \begin{bmatrix} \begin{vmatrix} k_2 & k_3 \\ Y & 1 \end{vmatrix} \\ \begin{vmatrix} k_3 & k_1 \\ 1 & X \end{vmatrix} \end{bmatrix} = 0 \qquad (64)$$

or still equivalently,

$$u[k_2 - k_3 Y] + v[k_3 X - k_1] = 0 \qquad (65)$$

Lemma 5. Assume $\Omega = 0$. Then $k = 0$ iff $u = 0$, $v = 0$ for all optic flow image points.

Proof. $k = 0$ iff any vector k^* satisfies

$$u[k_2^* - k_3^* Y] + v[k_3^* X - k_1^*] = 0 \qquad (66)$$

by Theorem 1 and hence $u = 0$, $v = 0$. Also see Theorem 4. □

Suppose k_0^* is the solution of minimization problem

$$\min_{\|k^*\|=1} \Sigma \, [u\,(k_2^* - k_3^* Y) + v(k_3^* X - k_1^*)]^2 \qquad (67)$$

and α_0 is the solution of following equation

$$k_{03}^* \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} + \alpha_0 \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} k_{01}^* \\ k_{02}^* \end{bmatrix} \qquad (68)$$

for some $\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$ linear independent with $\begin{bmatrix} k_{01}^* \\ k_{02}^* \end{bmatrix}$. Then, $k = 0$ and $\frac{k}{\|k\|} = -\,\text{sign}(\alpha_0)\frac{k_0^*}{\|k_0^*\|}$ by Theorem 5.

If the absolute depth z_0 at (X_0, Y_0) is known, then the absolute depth z at any image point (X,Y) is given by

$$z = \frac{\alpha}{\alpha_0} z_0 \qquad (69)$$

where α is the solution of $k_{03}^{*}\begin{bmatrix}X\\Y\end{bmatrix}+\alpha\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}k_{01}^{*}\\k_{02}^{*}\end{bmatrix}$.

Example 2. $k = 0$. Then, there is a unique Ω such that for the optic flow image

$$\begin{vmatrix}\omega_1 & \omega_2\\ X & Y\end{vmatrix}\begin{bmatrix}X\\Y\end{bmatrix}+\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}\begin{vmatrix}\omega_2 & \omega_3\\ Y & 1\end{vmatrix}\\ \begin{vmatrix}\omega_3 & \omega_1\\ 1 & X\end{vmatrix}\end{bmatrix} \tag{70}$$

and hence (see (50))

$$\frac{\dot{z}}{z}=\begin{vmatrix}\omega_1 & \omega_2\\ X & Y\end{vmatrix} \tag{71}$$

We can demonstrate that, in the case of $k = 0$, it is impossible to determine the relative depth from the optic flow image without any other information.

We have proved (see Reference) that the rigid body motion is uniquely determined by the optic flow image plus depth information of four uncoplanar spatial points. Due to Theorem 5, the conditions can be reduced to the optic flow image plus depth information of one spatial point.

The reason is simple since depth information of one spatial point plus the optic flow image determines uniquely the motion of the spatial point.

V. Algorithm. Simulation Results.

Step 1. Compute h: $Wh=0$ $(\|h\|=1)$

Step 2. Let $k^* = [h_7, h_8, h_9]'$

Step 3. If $|k_1^*| \geq |k_2^*|, |k_3^*|$, then

$$\left\{\omega_1=\frac{h_1-h_2-h_3}{2k_1^*},\quad \omega_2=\frac{h_4-k_2^*\omega_1}{k_1^*},\quad \omega_3=\frac{h_5-k_3^*\omega_1}{k_1^*};\ \text{go to step 6}\right\}$$

Step 4. If $|k_2^*| \geq |k_3^*|$, then

$$\left\{\omega_2=\frac{h_2-h_1-h_3}{2k_2^*},\quad \omega_1=\frac{h_4-k_3^*\omega_2}{k_2^*},\quad \omega_3=\frac{h_6-k_3^*\omega_2}{k_2^*};\ \text{go to step 6}\right\}$$

Step 5. Let

$$\left\{\omega_3=\frac{h_3-h_1-h_2}{2k_3^*},\quad \omega_1=\frac{h_5-k_1^*\omega_3}{k_3^*},\quad \omega_2=\frac{h_6-k_2^*\omega_3}{k_3^*}\right\}$$

Step 6. Output $\Omega = [\omega_1, \omega_2, \omega_3]'$

Step 7. If there are two optic flow image points such that the corresponding a's are zero, then
{ output $k=0$, stop }

Step 8. If there is an image point such that

$$\|a\|b\| \pm b\|a\|\| < \|a\|b\| \pm b\|a\|\|$$

then

$$\left\{\text{output } \frac{k}{\|k\|}=\pm\frac{k^*}{\|k^*\|},\ \text{stop}\right\}$$

Simulation Results

To verify that the theory works, we generate optic flow images of an ellipsoid in motion, and apply the algorithm to the optic flow image obtained to recover the motion parameters and relative depth of the ellipsoid.

First, we compute velocity at surface points of the ellipsoid moving with $\kappa = [1, 1, 1]'$ and $\Omega = [0, 0, .5]'$ in the half space $z < 0$ and project the velocity on the image plane at $z=1$ to obtain the ideal optic flow image as in Fig. 1. Applying the algorithm on the optic flow image obtained, we recover the motion parameters as

$$\kappa^* = [0.5773502, 0.5773503, 0.5773503]'$$

$$\Omega = [-6.7 * 10^{-8}, -3.4 * 10^{-8}, 0.5000000]'$$

and obtain the relative depth image in Fig. 2 whose ideal depth image is in Fig. 3.

VI. Summary

Given an optic flow image coming from a part on the rigid body which can not be contained in a quadratic surface passing through the origin, then the mode of motion, the rotation velocity, the translation orientation and the relative depth all can be uniquely determined by solving a smallest eigen value-vector problem of 9x9 nonnegative matrix. When k=0, at least six image points are needed and when $k \neq 0$, at least eight image points are needed to recover the motion. With the aid of one spatial point motion, the rigid body motion in uniquely determined.

In the forthcoming paper, the problem of rigid body motion and the optic flow under a small perturbation is considered and solved.

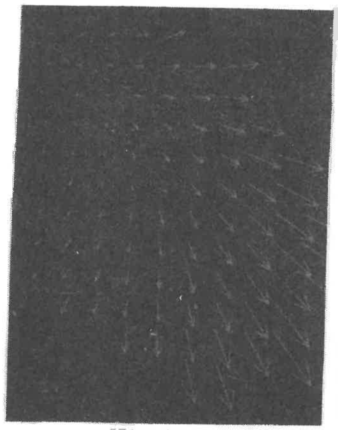

Fig. 1

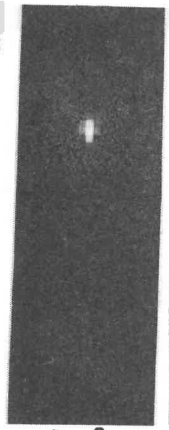

Fig. 2

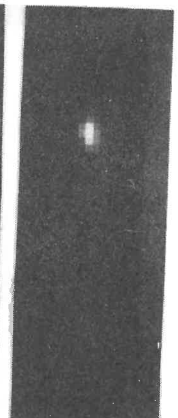

Fig. 3

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Gradient-based Estimation of Optical Flow with Global Optimization

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ABSTRACT

Gradient-based methods offer a promising approach for the estimation of optical flow. A major problem with gradient-based methods is that large errors can be made where the image is highly textured and at discontinuities in optical flow. Methods which operate globally can propagate these errors across the image, totally disrupting the estimation process. We examine how the introduction confidence measurements -- judgements of the accuracy of flow estimates -- can be used to locate poor estimates are prevent their propagation.

1. Introduction.

The velocity field that represents the motion of object points across an image is called the optical flow field. Optical flow results from relative motion between a camera and objects in the scene. One class of techniques for the estimation of optical flow utilizes a relationship between the motion of surfaces and the derivatives of image brightness [2, 3, 4, 7, 8, 10, 12, 13, 14, 15]. The major difficulty with gradient-based methods is their sensitivity to conditions commonly encountered in real imagery. Highly textured regions, motion boundaries, and depth discontinuities can all be troublesome for gradient-based methods. Fortunately, the areas characterized by these difficult conditions are usually small and localized.

The estimation errors caused by these conditions are especially problematic for methods that operate globally [4, 14]. Even though the error prone regions are sparsely distributed, the global method can propagate estimation errors made in the areas throughout the image. In this paper we examine how estimates of the accuracy of optical flow computations can be used to prevent error propagation.

2. The Gradient Constraint Equation.

The gradient constraint equation relates velocity on the image (u, v) and the image brightness function $I(x, y, t)$. The common assumption of gradient-based techniques is that the observed brightness (intensity on the image plane) of any object point is constant over time. Consequently, any change in intensity at a point on the image must be due to motion. Relative motion between an object and a camera will cause the position of a point on the image located at (x, y) at time t to change position on the image over a time interval δt. By the constant brightness assumption, the intensity of the object point will be the same in images sampled at times t and $t + \delta t$. The constant brightness assumption can be formally stated as

$$I(x, y, t) = I(x + \delta x, y + \delta y, t + \delta t). \tag{1}$$

Expanding the image brightness function in a Taylor's series around the point (x, y, t) we obtain

$$I(x + \delta x, y + \delta y, t + \delta t) = \tag{2}$$
$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t + h.o.t.$$

A series of simple operations leads to the gradient constraint equation:

$$0 = I_x u + I_y v + I_t \tag{3}$$

where

$$I_x = \frac{\partial I}{\partial x}, \quad I_y = \frac{\partial I}{\partial y}, \quad I_t = \frac{\partial I}{\partial t}.$$

A detailed derivation is given in [4].

3. Gradient-Based Algorithms.

The gradient constraint equation does not by itself provide a means for calculating optical flow. The equation only constrains the values of u and v to lie on a line when plotted in flow coordinates.

The gradient constraint is usually coupled with an assumption that nearby points move in a like manner to arrive at algorithms which solve for optical flow. Groups of constraint equations are used to collectively constrain the optical flow at a pixel. Horn and Schunck developed a method which globally minimizes an error function based upon the gradient constraint and the local variation of optical flow [4]. Another approach that has been widely investigated operates locally by solving a set of constraint lines from a small neighborhood as a system of linear equations [5, 6, 8, 10, 12, 13, 15].

The local and global methods rely on a similar assumption of smoothness in the optical flow field. Both methods require that flow vary slowly across the image. The locally constructed system of constraint equations is solved as if optical flow is constant over the neighborhood from which the constraint lines are collected. When optical flow is not constant the local method can provide a good approximation where flow varies slowly over small neighborhoods. The global method seeks a solution which minimizes local variation in flow. The important difference between methods of local and global optimization is not the constraint that they place on the scene but the computational method that they use to apply the constraint. There are contrasting aspects in the performance of the two approaches that are directly related to the difference in the scope of interactions across the image.

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