Predicting Expected Grey Level Statistics of Opened Signals

Wendy Swan Costa * and Robert M. Haralick
Department of Electrical Engineering, FT-10
University of Washington
Seattle WA 98195, U.S.A

Abstract

The opening of a model signal with a convex, zero-height structuring element is studied empirically. Experiments are performed in which the input signal model parameters and the opening length are varied over an acceptable range and the corresponding grey level distributions in the opened signal are fit to Pearson distributions. Next, regressions are used to relate the Pearson distribution parameters to the input parameters, resulting in equations that may be used to predict the effect of an opening. Finally, characterization experiments show that the maximum absolute errors between actual and predicted cumulative distributions using these regression equations have a mean of 0.036 and a standard deviation of 0.011 (for a range of zero to one); the worst-case maximum absolute error encountered between the cumulative distributions is 0.066.

1 Introduction

Morphological opening operations are useful in discriminating between lengths of sequences which are well above background in a signal. To analyze algorithms for this detection task, it is necessary to know how a stochastic signal changes when it is opened. Because this operation is nonlinear, expressing it as a transformation of random variables becomes intractable even for quite simple random input signal models [5]. As an alternative to the analytical solution, we study the opening of a model signal by a convex, zero-height structuring element empirically, using the Pearson system to parameterize the distributions in the opened signal.

We model the input signal as a sequence of mutually independent Normal random variables well above the zero-valued background. Next, we show that the input variables affecting the grey level distributions in the opened signal are: the input signal mean and variance; the length of the opening structuring element; and the translation class[1] of a particular pixel in the opened signal. The empirical determination of the opened signal's distribution consists of varying these four input parameters over an acceptable range and running Monte Carlo experiments to determine the resulting distributions. These empirical distributions. which correspond to each processed set of experimental input variables, are fit to Pearson distributions [2], which are also described by four parameters. Thus, the experiments provide several data points indicating the relationships between sets of experimental input variables and the four Pearson parameters. Regression analysis is then performed to describe these relationships mathematically. Given the regression equations to predict the Pearson parameters of the opened signal, we are interested in the error between the actual and predicted distributions; thus a separate set of Monte Carlo trials is run to characterize these errors. As we are interested in the expected cumulative grey level distribution function for our particular application, the root mean square error and maximum absolute error between the actual and simulated cumulative grey level distributions are calculated for each input parameter set.

In summary, two Monte Carlo studies are made to determine an empirical relationship estimating the change in the average grey level from the transformation of random variables associated with opening a random grey level signal with a specific model by a simple zero-height structuring element. The following sections elaborate upon the input random signal model, opening process, and assumptions made by this estimation; also, the experiments are described in detail.

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[1] We define the translation class of a pixel to be the number of translations that the structuring element can make in the non-zero support of the sequence and contain that pixel. It is related to both the length of the sequence and the length of the structuring element.

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2 Random Signal Model

The random signal model describes the signal to be opened by a convex zero-height structuring element. Consider a discrete one-dimensional signal $f(x)$ which is a sequence of λ consecutive positive, non-zero random variables amid a zero-valued background. Without loss of generality, it may be assumed that the support F of non-zero $f(x)$ is $\{0, 1, \ldots, \lambda - 1\}$. Now allow the grey level $f(x)$ at each $x \in F$ to be a Normal random variable with mean b and variance σ^2. Furthermore, let the random variables $f(1), \ldots, f(\lambda)$ be mutually independent. If several such sequences were present in the signal with lengths $\lambda_1, \lambda_2, \ldots,$ and if each sequence were separated from the other by the zero-valued background, it may be shown that the the sequences are opened independently. Therefore, the input random signal may be parameterized by: its mean b, its variance σ^2, and the distribution of lengths of its non-zero sequences. The mean and variance of a signal will be used as input variables for the Monte Carlo studies; an analysis of the morphological opening operation will yield a method to transform a distribution of sequence lengths to a third parameter used as an input variable for the experiments.

3 Morphological Opening Operation

Given the random signal described above, we would like to describe the grey level distribution of its opening with a one-dimensional, convex, zero-height structuring element. Let the structuring element k be of length T_λ. Since the structuring element is convex, its support K may be given by $K = \{0, 1, \ldots, T_\lambda - 1\}$. By definition [3], an opening is an erosion followed by a dilation:

$$f \circ k = (f \ominus k) \oplus k, \tag{1}$$

where the erosion is given by $(f \ominus k)(x) = min_{z \in K}[f(x + z) - k(z)]$ and the dilation by $(f \oplus k)(x) = max_{\substack{z \in K \\ x - z \in F}} [f(x - z) + k(z)]$. Substituting $k(z) = 0$ for all z and letting $f'(x) = (f \ominus k)(x)$, definition 1 becomes

$$(f \circ k)(x) = max[f'(x - T_\lambda + 1), \ldots, f'(x)]. \tag{2}$$

Note, however, that any member of $\{f(x), \ldots, f(T_\lambda - 1 + x)\}$ which is zero-valued (i.e., background) will cause $f'(x)$ to be zero, also. Careful inspection of equation 2 at different values of x reveals that these degenerate zero-valued arguments give rise to different distributions for $(f \circ k)(x)$. The most obvious consequence of the opening is that $(f \circ k)(x)$ is zero for all locations x that are members of a sequence of length $\lambda < T_\lambda$. This implies that the opening length T_λ acts as a *length threshold* for the sequences in the input signal. Next, note that $f(x)$ is positively correlated to $f'(x+1), f'(x+2), \ldots, f'(x+T_\lambda-1)$, which hinders the theoretical analysis of the transformation of variables introduced by the dilation. However, because the random variables $f(x)$, $x = 0, 1, \ldots, \lambda - 1$, are independently and identically distributed, $f'(x) \stackrel{d}{=} f'(x - \Delta x)$ for $x \in F'$ and $x - \Delta x \in F'$. From this, it follows that

$$max[f'(x), \ldots, f'(x + c_t - 1)] \stackrel{d}{=} max[f'(x - \Delta x), \ldots, f'(x - \Delta x + c_t - 1)] \tag{3}$$

for all Δx such that $x - \Delta x, \ldots, x - \Delta x + c_t - 1 \in F'$. Therefore, the number of non-zero elements from which the maximum is chosen (c_t in equation 3) distinguishes the distributions $(f \circ k)(x)$ at different values of x. This number of elements corresponds to the number of translations that the structuring element may make in the non-zero support F of the input model and still contain the point x (see Figure 1), so we will refer to c_t as the *translation class* of the pixel at point x.

Now, using the concept of the translation class, equation 2 may be rewritten with its degenerate terms removed by introducing another random variable $H(c_t)$ such that $H(c_t) = max[f'(0), \ldots, f'(c_t - 1)]$, where c_t is the translation class of the distribution at x. In terms of this new variable, the grey level distribution of $(f \circ k)(x)$ may be broken down into several cases, which are given below. If $\lambda < T_\lambda$,

$$(f \circ k)(x) \stackrel{d}{=} 0. \tag{4}$$

If $T_\lambda \leq \lambda \leq 2T_\lambda - 1$,

$$(f \circ k)(x) \stackrel{d}{=} \begin{cases} H(x + 1) \text{ for } 0 \leq x < \lambda - T_\lambda \\ H(\lambda - T_\lambda + 1) \text{ for } \lambda - T_\lambda \leq x < T_\lambda \\ H(\lambda - x) \text{ for } T_\lambda \leq x < \lambda \end{cases} \tag{5}$$

If $\lambda > 2T_\lambda - 1$,

$$(f \circ k)(x) \stackrel{d}{=} \begin{cases} H(x + 1) \text{ for } 0 \leq x < T_\lambda \\ H(T_\lambda) \text{ for } T_\lambda \leq x < \lambda - T_\lambda \\ H(T_\lambda - x) \text{ for } \lambda - T_\lambda \leq x < \lambda \end{cases} \tag{6}$$

These relationships describe the transformation of any distribution of sequence lengths, $\lambda_1, \lambda_2, \ldots$, to a weighted combination of distributions $H(c_t)$, where $c_t = 1, 2, \ldots, T_\lambda$. This ability to transform a length distribution to a function of the translation class implies that the third parameter for the Monte Carlo studies should be the translation class, whose range depends upon the length of T_λ of the opening structuring element. Therefore, the fourth input variable to the Monte Carlo studies must necessarily be the length T_λ of the opening structuring element.

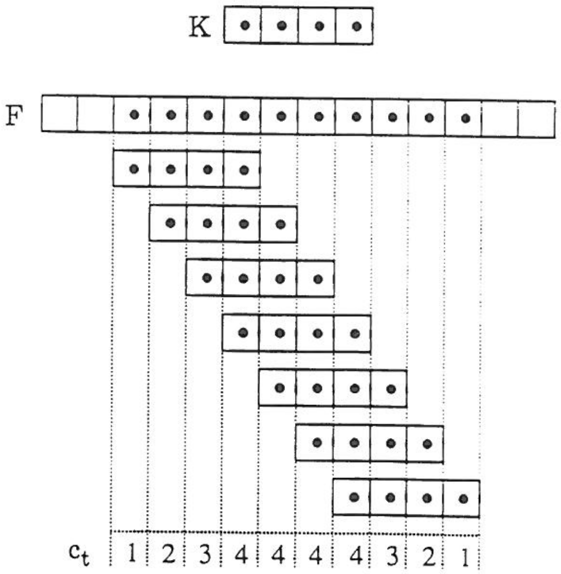

Figure 1: Illustration of the translation classes c_t of impulses within a signal, whose non-zero support is F, opened by a convex structuring element K.

4 Pearson Parameters

Now that the input parameters (the signal mean b, the signal variance σ^2, the opening length T_λ, and the translation class c_t) causing variation in the opened signal pixels' grey level distributions have been identified, a protocol can be established to observe these grey level distributions as the input parameters are varied over a specified range. We chose Pearson distributions to model the opened signal pixels' grey level distributions because they can flexibly describe unimodal distributions. Pearson distributions have four parameters which may be computed given the first four moments of a distribution. These four parameters are estimated as the input is varied. Each step of this experimental protocol is discussed in more detail below.

4.1 Random Signal Generation

Several images are generated so that each image provides a sample of grey levels for a given combination of the input variables $\{b, \sigma^2, \text{and } T_\lambda\}$ and for all of the translation classes c_t such that $c_t \in \{1, 2, \dots, T_\lambda\}$. First, an image containing 5,000 independent line segments of length $2T_\lambda$ on a zero-valued background is created such that the grey levels of the pixels within the line segments are independent Normal random variables with mean b and standard deviation σ. Next, the image is opened with a zero-height line-shaped structuring element of length T_λ. Finally, the mask image which indicates the translation classes of the pixels in the model image is generated. Since the length of the line segments in each image is twice the length of the structuring element used to open that image, each experimental image contains two pixels per line segment that represent each translation class, yielding a sample of 10,000 pixels per translation class.

The input parameters are sampled as follows to create the opened images: mean (b) of the pre-opened normal distribution = 80, 99, 118, 137, 156; standard deviation (σ) of the pre-opened normal distribution = 5, 10, 15, 20, 25; opening length (T_λ) = 2, 4, 8, 16, 32; and translation class (c_t) = 1, 2, ..., T_λ. All possible combinations of the b, σ, and T_λ values listed above are generated. These experimental images, then, provide the statistics which are used to fit the grey level distributions resulting from each set of input parameters to a Pearson model by estimation of the Pearson parameters.

4.2 Estimating the Pearson Parameters

The method of moments [2] is used to estimate the Pearson parameters of the grey level distributions in the opened images described above. A Pearson distribution is parameterized by its mean $\mu_1\prime$, its variance μ_2, the root of its coefficient of skewness $\sqrt{\beta_1}$, and its coefficient of kurtosis β_2. In this method, the sample mean ($\mu_1\prime$) and its second, third, and fourth central moments (μ_2, μ_3, and μ_4, respectively) are used directly to estimate the Pearson family parameters for the sample as follows [2]: $\hat{\mu}_1\prime = \mu_1\prime$; $\hat{\mu}_2 = \mu_2$; $\widehat{\sqrt{\beta_1}} = \mu_3/{\mu_2}^{3/2}$; and $\hat{\beta}_2 = \mu_4/{\mu_2}^2$. The sample moments are gathered for all translation classes in each experimental image. Sheppard's corrections [2] are then applied to the moments to correct for the integer binning of the grey levels, and the Pearson parameters are calculated from the corrected moments using the relations given above. Scatterplots of the resulting estimated Pearson parameters versus the experimental input parameters (b, σ, T_λ, c_t) are shown in Figure 2.

5 Regression of Pearson Parameters

The sets of estimated parameters corresponding to different input signal, opening, and translation class parameters are useful only if they can be mathematically related to the Pearson model parameters. To accomplish this, a multivariable regression is performed for each of the Pearson model parameters on functions of the four input variables. For each regression model used, the bases are calculated from the original data and then supplied to the regression. The final relationships between each of the Pearson parameters and the input variables and summaries of the regressions are given below.

5.1 Mean of Opened Sample

The empirical expression obtained for the mean is:

$$\hat{\mu_1'} = b + \sigma(-0.4512 - 0.4715 \ln T_\lambda + 0.4297 \frac{c_t}{T_\lambda}). \quad (7)$$

This regression produces a coefficient of determination of $r^2 = 1.0000$ and an F-statistic of 35538134 on 4 and 1546 degrees of freedom, yielding a p-value of 0.

5.2 Variance of Opened Sample

The empirical expression obtained for the variance is:

$$\hat{\mu_2} = -0.0642\sigma^{1.9950} T_\lambda^{-0.4390} e^{[0.3417 \frac{c_t}{T_\lambda} - 0.6601(\frac{c_t}{T_\lambda})^2]}. \quad (8)$$

This relationship is obtained by regressing the natural logarithm of the variance estimates on ($\ln\sigma$, $\ln T_\lambda$, c_t, and c_t^2). The regression has a coefficient of determination of $r^2 = 0.9981$ and an F-statistic of 207425.6 on 4 and 1545 degrees of freedom, yielding a p-value of 0.

5.3 Skewness of Opened Sample

The empirical expression obtained for the root of the coefficient of skewness is:

$$\widehat{\sqrt{\beta_1}} = 0.0672 - 0.0637T_\lambda + 0.0035T_\lambda^2 - 0.0001T_\lambda^3 + 0.1665\frac{c_t}{T_\lambda} - 1.5254(\frac{c_t}{T_\lambda})^2 + 1.2164(\frac{c_t}{T_\lambda})^3. \quad (9)$$

The coefficient of determination for this regression is $r^2 = 0.8703$ and its F-statistic is 1726.218 on 6 and 1543 degrees of freedom, yielding a p-value of 0.

5.4 Kurtosis of Opened Sample

The empirical expression obtained for the coefficient of kurtosis is:

$$\hat{\beta_2} = 2.7470 + 0.0215T_\lambda - 0.0006T_\lambda^2 - 0.9779\frac{c_t}{T_\lambda} + 3.9935(\frac{c_t}{T_\lambda})^2 - 2.4765(\frac{c_t}{T_\lambda})^3, \quad (10)$$

This regression produces a coefficient of determination of $r^2 = 0.8885$ and an F-statistic of 2460.367 on 5 and 1543 degrees of freedom, yielding a p-value of 0.

5.5 Analysis of Regression Results

The r^2 values of the regressions performed indicate that suitable regression models have been used for the mean and the variance of the opened signal samples, whereas we have not been able to fit the coefficients of skew and kurtosis quite as well. Since the mean and variance have a more profound effect upon the output distributions, we are willing to accept more error in the other parameters. The manner in which this error affects the predicted distributions is the topic of the next section.

6 Error Characterization

To compare the predicted output distributions given the input variables, a separate error characterization experiment is performed. In this experiment, the input variables are sampled differently than in the experiment used to gather the estimated grey level moments. Opened model images are then generated according to this new sampling scheme and histogrammed. The sampled values of the input variables are also used to generate predicted grey level distributions for each opened model image using the regression results of Section 5 as the parameters of the Pearson distributions. The actual and predicted grey level cumulative mass functions are compared in terms of their mean square errors and their maximum absolute errors.

6.1 Generating Actual Distributions

The input variables are sampled and used to generate images as in Section 4.1. The samples are taken as follows: mean (b) of the pre-opened normal distribution = 85, 100, 115, 130; standard deviation (σ) of the pre-opened normal distribution = 6, 12, 18, 24; opening length (T_λ) = 5, 10, 15, 20; and translation class (c_t) = 1, 2, ..., T_λ. Each opened model image is then histogrammed to obtain its probability mass function and its cumulative mass function is computed.

6.2 Predicting Grey Level Distributions

For each sample described by a unique (b, σ, T_λ, c_t) combination, the parameters of the output grey level (Pearson) distribution are predicted using the regression results in equations 7 through 10. Since we would like to compare the predicted distributions to the actual distributions for these cases, the predicted Pearson parameters are used to compute the predicted binned distributions to be used for comparison. The Pearson distributions are computed using a method which estimates eleven percentage points given the four Pearson parameters [1]. A four-point polynomial interpolation (using Neville's algorithm [4]) is then performed at integer range values to give the binned predicted output cumulative distributions. The limits of the grey level ranges are taken from the ranges of the corresponding actual distributions (thus, the total probabilities of the two mass functions are not necessarily equivalent).

6.3 Prediction Errors

Given the actual and predicted distributions (i.e., the probability and cumulative mass functions for each case), the errors between these distributions corresponding to each (b, σ, T_λ, c_t) combination are summarized by a mean square error and a maximum absolute error describing the discrepancies between the cumulative mass functions.

The average and worst case root mean square errors encountered between the actual and predicted cumulative mass functions are 0.015 and 0.023, respectively. One should note that the mean square error does not indicate whether there is a bias or other structure in the error (e. g., it may be heaviest in the steepest tail), so it must be used with caution. Since we are interested in using the cumulative distributions in future algorithms, this statistic would be useful if the predicted cumulative probability at any grey level could be described by some zero-mean random variable given a particular set { b, σ, T_λ, c_t } of input variables. Considering our data, this assumption does not seem appropriate. Instead, we consider the maximum absolute errors that we make in predicting the cumulative grey level distributions of opened signals.

The maximum absolute error between the predicted and actual grey level cumulative distribution functions indicates the largest prediction error that one is likely to make at any point in the distribution. As with the mean square error, the maximum absolute errors found in the characterization experiments do not exhibit much structure as functions of the input parameters. Therefore, we consider the collection of maximum absolute errors for each prediction in the characterization as a single sample. The largest of these errors encountered in the characterization is 0.066, their mean is 0.036, and their standard deviation is 0.011 (where the range of the cumulative probabilities is between zero and one). These statistics may be used to predict an approximate upper bound on the difference between the actual and predicted grey level cumulative distributions of a pixel in an opened signal, regardless of the values of the input parameters (provided that they're within the range of the characterization).

7 Conclusion

In conclusion, a model has been established to allow the investigation of the grey level distributions of classes of pixels associated with the transformation of random variables induced by a morphological opening operation. A Monte Carlo simulation of the operation was performed to obtain the first four grey level moments of several populations corresponding to different input signal and opening process parameters. Next, four multivariable regressions were performed to mathematically describe the four Pearson distribution parameters (obtainable by a transformation of the moments) as functions of the input signal and opening process parameters. These relationships may now be used to describe the expected grey level distributions of this particular signal model following an opening operation. Finally, a second experiment was performed to characterize the error in the predicted moments given a particular method of computing the Pearson distributions.

The value of this work lies in: (1) the use of an explicit model under which the opening operation may be analyzed; (2) the development of prediction equations for the Pearson parameters, which may be used in characterizing processing algorithms; and (3) the characterization scheme, which provides a means to approximate the error associated with the predicted grey level cumulative distribution of a class of pixels in an opened signal, as well as a means to check the errors if the regression models and/or the grey level (Pearson) distribution computation method change. Knowing the average grey level distribution in a signal comprised of these model sequences is useful if one wants to follow an opening operation by a point operator which depends only upon the grey level value of the pixel (e.g., thresholding). Thus, this study is a first step toward the empirical analysis of the opening operation and is sufficient to study our simple detection algorithms.

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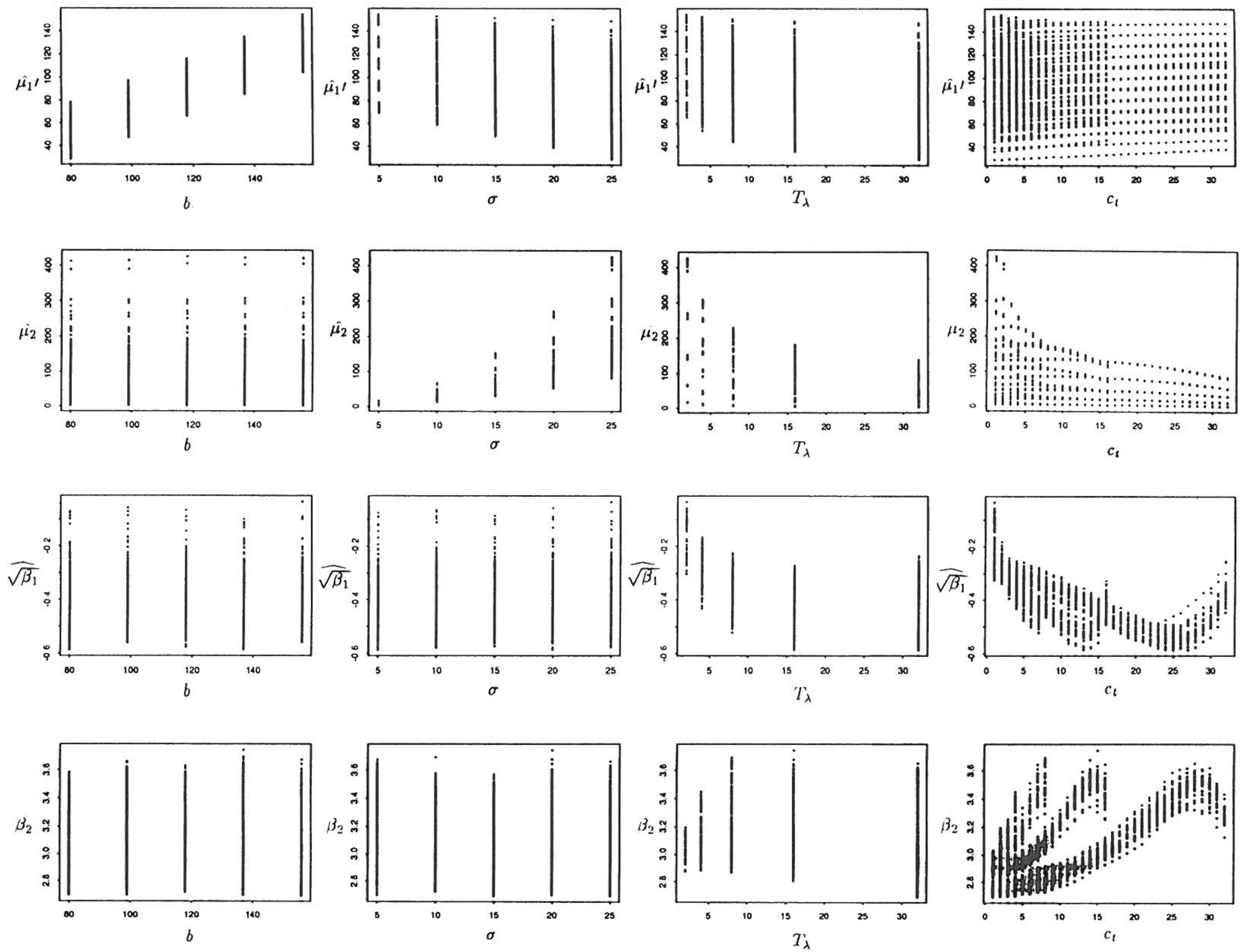

Figure 2: Scatterplot of the input parameters (the signal mean b, the signal standard deviation σ, the opening length T_λ, and the translation class c_t) versus the estimated Pearson parameters (the mean $\hat{\mu_1}\prime$, the variance $\hat{\mu_2}$, the root of the coefficient of skewness $\widehat{\sqrt{\beta_1}}$, and the coefficient of kurtosis $\hat{\beta_2}$) of the opened signal.