THE CONSISTENT LABELING PROBLEM

Robert M. Haralick
Department of Electrical Engineering
Department of Computer Science
University of Kansas
Lawrence, Kansas 66045

Linda G. Shapiro
Department of Computer Science
Kansas State University
Manhattan, Kansas 66506

Submitted to the IEEE Computer Society Workshop
on Pattern Recognition and Artificial Intelligence

December 1977

THE CONSISTENT LABELING PROBLEM

SUMMARY

In this paper we formulate a general network constraint analysis problem which we call the labeling problem. The labeling problem is a generalization of specific problems from each of several different specialty areas. Some of these specific problems include the subgraph isomorphism problem (Ullman, 1976), the graph homomorphism problem (Harary, 1969), the automata homomorphism problem (Ginzberg, 1968), the graph coloring problem (Harary, 1969), the relational homomorphism problem (Haralick and Kartus, 1976), the packing problem (Deutsch, 1966), the scene labeling problem (Barrow and Tenenbaum, 1976), the shape matching problem (Davis, 1976), and the Latin square puzzle (Whitehead, 1972). The generalized problem involves a set of units which usually represent a set of objects to be given names, a set of labels which are the possible names for the units, and a compatibility model containing ordered groups of units which mutually constrain one another and ordered groups of unit-label pairs which are compatible. The compatibility model is sometimes called a world model. The problem is to find a label for each unit such that the resulting set of unit-label pairs is consistent with the constraints of the world model.

Before we can fully state the labeling problem we need some additional concepts and definitions. Let $U = \{1,\dots,M\}$ be a set of M units and let L be a set of labels. If $u_1,\dots,u_N \in U$ and $\ell_1,\dots,\ell_N \in L$, then we call the N-tuple $(\ell_1,\dots,\ell_N)$ a labeling of units $(u_1,\dots,u_N)$. The labeling problem is to use the world model to find a particular kind of labeling called a consistent labeling for all M units in U.

The problem of labeling is that not all of the labelings in L^M are consistent because some of the units are apriori known to mutually constrain

one another. If an N-tuple of units $(u_1,\ldots,u_N)$ are known to mutually constrain one another, then not all labelings are permitted or legal for units $(u_1,\ldots,u_N)$. The compatibility model tells us which units mutually constrain one another N at a time and which labelings are permitted or legal for those units which do constrain one another. One way of representing this compatibility model is by a quadruple (U,L,T,R) where $T \subseteq U^N$ is the set of all N-tuples of units which mutually constrain one another and the <u>constraint relation</u> $R \subseteq (U \times L)^N$ is the set of all 2N-tuples $(u_1,\ell_1,\ldots,u_N,\ell_N)$ where $(\ell_1,\ldots,\ell_N)$ is a permitted or legal labeling of units $(u_1,\ldots,u_N)$.

A labeling $(\ell_1,\ldots,\ell_P)$ is a <u>consistent labeling</u> of units $(u_1,\ldots,u_P)$ with respect to the compatibility model (U,L,T,R) if and only if $\{i_1,\ldots,i_N\} \subseteq \{1,\ldots,P\}$ and $(u_{i_1},\ldots,u_{i_N}) \in T$ imply the 2N-tuple $(u_{i_1},\ell_{i_1},\ldots,u_{i_N},\ell_{i_N}) \in R$; that is, the labeling $(\ell_{i_1},\ldots,\ell_{i_N})$ is a permitted or legal labeling of units $(u_{i_1},\ldots,u_{i_N})$. When U and L are understood, such a labeling $(\ell_1,\ldots,\ell_P)$ is called a <u>(T,R)-consistent labeling</u> of $(u_1,\ldots,u_P)$. The <u>consistent labeling problem</u> is to find all consistent labelings of units (1,...,M) with respect to the compatibility model.

In this paper we discuss the consistent labeling problem and define two generalized operators that can be used to aid in finding solutions to a given labeling problem. In Section II we describe how a variety of combinatorial problems are special cases of the consistent labeling problem. In Section III we give a brief perspective of a procedure for solving the consistent labeling problem and discuss how researchers have used "relaxation" and "look-ahead" operators to accelerate the search time. Section III also gives a short summary of a recent paper (Haralick et. al., 1977) which addressed the labeling problem.

In Section IV we show the relationship between (T,R)-consistency as defined in this paper and global consistency with respect to R, as defined in the Haralick et. al. paper, and we define the look-ahead operator ϕ_{KP} which generalizes the ϕ_P operator in the earlier paper. In Section V we develop theorems characterizing the operation of the look-ahead operator ϕ_{KP}. In Section VI we introduce another look-ahead operator Ψ_{KP} and show the relationship between ϕ_{KP} and Ψ_{KP}. In Section VII, we discuss implementation issues and define a recursive procedure which implements a tree search incorporating the ϕ_{KP} operator. In Section VIII, we discuss the complexity of any label finding procedure.