Binary Morphology: Working in the Sampled Domain

Robert M. Haralick, Xinhua Zhuang, Charlotte Lin and James Lee**

Department of Electrical Engineering
University of Washington
Seattle, WA 98195

*Boeing High Technology Center
PO Box 24969 / MS 7J-24
Seattle, WA 98124-6269

ABSTRACT

This paper describes the relationship between morphologically filtering and then sampling versus sampling and then morphologically filtering in the sampled domain. It also describes the relationships between morphologically filtering versus sampling, morphologically filtering in the sampled domain and then reconstructing. Unlike the standard communication sampling theory where for appropriately low pass filtered images there is commutivity between sampling and filtering, this is not the case for appropriately morphologically simplified images. The relationship which does exist shows that the commutivity holds to within one sampling interval distance in the unsampled domain and to within two sampling intervals in the sampled domain.

1. Introduction

Definitions for all the morphology concepts in this paper can be found in Haralick et al. (1987a). The reference list provides some pointers into the recent literature about mathematical morphology. Sampling a set is accomplished by taking its intersection with the sampling set. Reconstructing a sampled set can be done by dilating the set in the sampled domain or closing the set in the sampled domain. These provide maximal and minimal reconstructions. The relationship between the sampling set S and the reconstruction structuring element K is given by the following five conditions:

(1) $S = S \oplus S$

(2) $S = \check{S}$

(3) $K \cap S = \{0\}$

(4) $K = \check{K}$

(5) $a \in K_b \rightarrow K_a \cap K_b \cap S \neq \emptyset$

When K and S are so related, the binary morphological sampling theorem states that for any set F

(1) $F \cap S = [(F \cap S) \bullet K] \cap S$

(2) $F \cap S = [(F \cap S) \oplus K] \cap S$

(3) $(F \cap S) \bullet K \subseteq F \bullet K$

(4) $F \circ K \subseteq (F \cap S) \oplus K$

(5) If $F = F \circ K = F \bullet K$, then $(F \cap S) \bullet K \subseteq F \subseteq (F \cap S) \oplus K$

(6) If $A = A \bullet K$ and $A \cap S = F \cap S$, then $A \subseteq (F \cap S) \bullet K$ implies $A = (F \cap S) \bullet K$

(7) If $A = A \circ K$ and $A \cap S = F \cap S$, then $A \supseteq (F \cap S) \oplus K$ implies $A = (F \cap S) \oplus K$

These bounding relationships are presented in Haralick et al. (1987b). Figures 1 through 5 illustrate the set bounding constraints expressed by the binary morphological samling theorem. In this paper, we refine these bounding relationships by the use of the Hausdorf set metric.

2. The Distance Relationships

The reconstruction $(F \cap S) \oplus K$ is maximal with respect to the property of being open and downsampling to $F \cap S$, and the reconstruction $(F \cap S \bullet K$ is minimal with respect to the property of being closed and downsampling to $F \cap S$. But how far is the bound $F \ominus K$ from $F \bullet K$, how far is $F \circ K$ from the bound $F \oplus K$, and how far is the minimal reconstruction $(F \cap S) \bullet K$ from $(F \cap S) \oplus K$? This is important to know since $F \ominus K \subseteq (F \cap S) \bullet K \subseteq F$ when $F = F \bullet K$, and $F \subseteq (F \cap S) \oplus K \subseteq F \oplus K$ when $F = F \circ K$, and $(F \cap S) \bullet K \subseteq F \subseteq (F \cap S) \oplus K$ when $F = F \circ K = F \bullet K$. Notice that in all three cases the difference between the lower and the upper set bound is just a dilation by K. This motivates us to define a distance function to measure the distance between two sets and to work out the relation between the

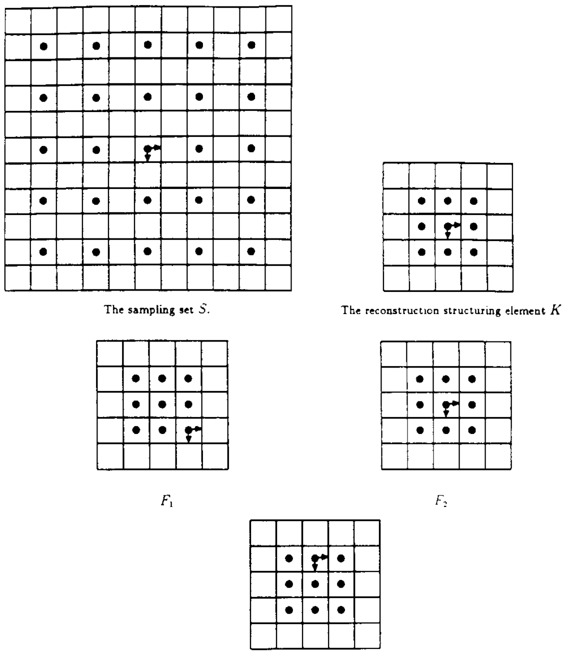


Figure 1 *illustrates a sampling set S, a reconstruction structuring element K, and three sets, F_1, F_2, and F_3, each of which is open under K.*

distance between a set and its dilation by K with the size of the set K. In this section we show that with a suitable definition of distance, all these distances are less than the radius of K. Since K is related to the sampling distance, all the above-mentioned distances are less than the sampling interval.

For the size of a set B, denoted by $r(B)$, we use the radius of its circumscribing disk. Thus, $r(B) = \min_{x \in B} \max_{y \in B} \|x - y\|$. The more mathematically correct forms of inf for min and sup for max may be substituted when the space E is the real line. In this case the proofs in this section require similar modifications. For a set A which contains a set B, a natural pseudo-distance from A to B is defined by $\rho\ (A,B) = \max_{x \in A} \min_{y \in B} \|x - y\|$. Proposition 1 proves that (1) $\rho(A,B) \geq 0$, (2) $\rho(A,B) = 0$ implies $A \subseteq B$, and (3)$\rho(A,C) \leq \rho(A,B) + \rho(B,C) + r(B)$. The asymmetric relation (2) is weaker than the corresponding metric requirement that $\rho\ (A,B) = 0$ if and only if $A = B$, and relation (3) is weaker than the metric triangle inequality.

Proposition 1

(1) $\rho\ (A,B) \geq 0$

(2) $\rho\ (A,B) = 0$ if and only if $A \subseteq B$

(3) $\rho\ (A,C) \leq \rho\ (A,B) + \rho\ (B,C) + r(B)$

Proof

(1) $\rho(A,B) \geq 0$ since $\rho(A,B) = \max_{x \in A} \min_{y \in B} \|x - y\|$ and $\|x - y\| \geq 0$.

(2) Suppose $\rho(A,B) = 0$. Then $\max_{a \in A} \min_{b \in B} \|a - b\| = 0$. Since $\|a - b\| \geq 0, \max_{a \in A} \min_{b \in B} \|a - b\| = 0$ implies for every $a \in A$, $\min_{b \in B} \|a - b\| = 0$. But $\|a - b\| = 0$ if and only if $a = b$. Hence, for every $a \in A$, there exists a $b \in B$ satisfying $a = b$. i.e., $A \subseteq B$. Suppose $A \subseteq B$. Then for each $a \in A$, $\min_{b \in B} \|a - b\| = 0$. Hence, $\max_{a \in A} \min_{b \in B} \|a - b\| = 0$.

(3)

$$
\begin{aligned}
\rho(A,C) &= \max_{a \in A} \min_{c \in C} \|a - c\| \\
&\leq \max_{a \in A} \min_{c \in C} \|a - b\| + \|b - c\| \\
&\quad \text{for every } b \in B \\
&\leq \max_{a \in A} \{\|a - b\| + \min_{c \in C} \|b - c\|\} \\
&\quad \text{for every } b \in B \\
&\leq \max_{a \in A} \{\|a - b\| + \max_{b' \in B} \\
&\quad \min_{c \in C} \|b' - c\|\} \\
&\quad \text{for every } b \in B \\
&\leq \rho(B,C) + \max_{a \in A} \|a - b\| \\
&\quad \text{for every } b \in B
\end{aligned}
$$

$$
\begin{aligned}
\text{But } \max_{a \in A} \|a - b\| &= \max_{a \in A} \|a - b' + b' - b\| \\
&\quad \text{for every } b,\ b' \in B \\
&\leq \max_{a \in A} \|a - b'\| + \|b' - b\| \\
&\quad \text{for every } b,\ b' \in B \\
\text{Finally } \max_{a \in A} \|a - b\| &\leq \|b' - b\| + \max_{a \in A} \|a - b'\| \\
&\quad \text{for every } b,\ b' \in B \\
\text{Thus } \max_{a \in A} \|a - b\| &\leq \max_{b' \in B} \|b' - b\| + \min_{b' \in B} \\
&\quad \max_{a \in A} \|a - b'\| \text{ for every } b \in B \\
&\leq \max_{b' \in B} \|b' - b\| + \rho\ (A,B) \\
\text{Finally } \rho\ (A,C) &\leq \rho\ (B,C) + \max_{b' \in B} \|b' - b\| \\
&\quad \text{for every } b \in B \\
&\leq \rho\ (A,B) + \rho\ (B,C) + \\
&\quad \min_{b \in B} \max_{b' \in B} \|b' - b\| \\
&\leq \rho\ (A,B) + \rho\ (B,C) + r(B).
\end{aligned}
$$

The pseudo distance ρ has a very direct interpretation. $\rho(A,B)$ is the radius of the smallest disk which when used as a structuring element to dilate B produces a result which contains A.

Proposition 2

Let $disk\ (r) = \{x|\ \|x\| \leq r\}$ and $A, B \subseteq F^N$. Then $\max_{a \in A} \min_{b \in B} \|a - b\| = inf\{r | A \subseteq B \oplus disk(r)\}$.

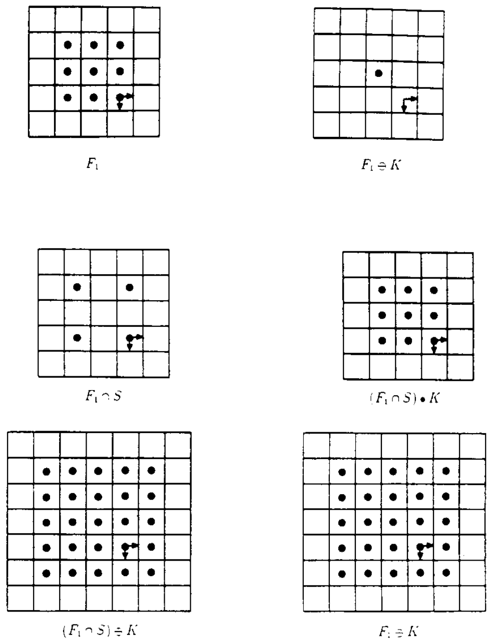


Figure 2 *shows how the erosion and dilation of F_1 bound the minimal reconstruction $(F_1 \cap S) \bullet K$ and the maximal reconstruction $(F_1 \cap S) \oplus K$, respectively, which in turn bound F_1. because F_1 is both open and closed under K.*

Proof

Let $\rho_0 = \max_{a \in A} \min_{b \in B} \|a - b\|$ and $r_0 = inf\ \{r | A \subseteq B \oplus disk(r)\}$. Let $a \in A$ be given. Let $b_0 \in B$ satisfy $\|a - b_0\| = \min_{b \in B} \|a - b\|$. Now, $\rho_0 = \max_{x \in A} \min_{y \in B} \|x - y\| \geq \min_{b \in B} \|a - b\|$. Hence, $\rho_0 \geq \|a - b_0\|$ so that $a - b_0 \in disk(\rho_0)$. Now, $b_0 \in B$ and $a - b_0 \in disk(\rho_0)$ implies $a = b_0 + (a - b_0) \in B \oplus disk(\rho_0)$. Hence $A \subseteq B \oplus disk(\rho_0)$. Since $r_0 = inf\ \{r | A \subseteq B \oplus disk(r)\}$, $r_0 \leq \rho_0$. Suppose $A \subseteq B \oplus disk(r_0)$. Then $\max_{a \in A} \min_{b \in B \oplus disk(r_0)} \|a - b\| = 0$. Hence, $\max_{a \in A} \min_{b \in B} \min_{y \in disk(r_0)} \|a - b - y\| = 0$. But $\|(a - b) - y\| \geq \|a - b\| - \|y\|$. Therefore,

$$\begin{aligned} 0 &\geq \max_{a \in A} \min_{b \in B} \min_{y \in disk(r_0)} \|a - b\| - \|y\| \\ &\geq \max_{a \in A} \min_{b \in B} \|a - b\| + \min_{y \in disk(r_0)} -\|y\| \\ &\geq \max_{a \in A} \min_{b \in B} \|a - b\| - \max_{y \in disk(r_0)} \|y\| \end{aligned}$$

Now $\rho_0 = \max_{a \in A} \min_{b \in B} \|a - b\|$ and $r_0 = \max_{y \in disk(r_0)} \|y\|$ implies $0 \geq \rho_0 - r_0$ so that $r_0 \geq \rho_0$. Finally, $r_0 \leq \rho_0$ and $r_0 \geq \rho_0$ implies $r_0 = \rho_0$. ∎

The pseudo distance ρ can be used as the basis for a true set metric by making it symmetric. We define the set metric $\rho_M(A, B) = \max\{\rho(A, B), \rho(B, A)\}$, also called the Hausdorf metric. Before we actually prove that ρ_M is indeed a metric, we note that $\rho_M(A, B) = inf\ \{r | A \subseteq B \oplus disk(r)$ and $B \subseteq A \oplus disk(r)\}$. This happens since

$$\begin{aligned} \rho_m(A, B) &= \max\{\rho(A, B), \rho(B, A)\} \\ &= \max\Big\{inf\{r | A \subseteq B \oplus disk(r)\} \\ &\quad inf\ \{r | B \subseteq A \oplus disk(r)\}\Big\} \\ &= inf\{r | A \subseteq B \oplus disk(r) \text{ and } B \subseteq A \oplus disk(r)\} \end{aligned}$$

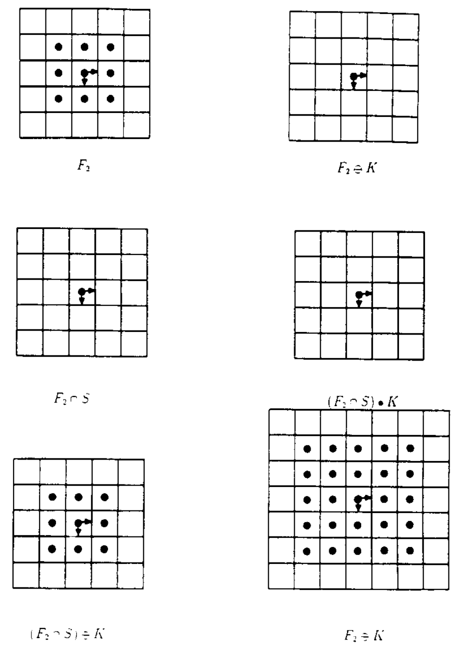


Figure 3 *shows a second example of how the erosion and dilation of F_2 bound the minimal reconstruction $(F_2 \cap S) \bullet K$ and the maximal reconstruction $(F_2 \cap S) \oplus K$, respectively, which in turn bound F_2.*

Proposition 3

(1) $\rho_M(A, B) \geq 0$

(2) $\rho_M(A, B) = 0$ if and only if $A = B$

(3) $\rho_M(A, B) = \rho_M(B, A)$

(4) $\rho_M(A, C) \leq \rho_M(A, B) + \rho_M(B, C)$

Proof (1) $\rho_M(A, B) \geq 0$ since $\rho(A, B) \geq 0$ and $\rho(B, A) \geq 0$.

(2) Suppose $\rho_M(A, B) = 0$. Then $\max\{\rho(A, B), \rho(B, A)\} = 0$. Since $\rho(A, B) \geq 0$ and $\rho(B, A) \geq 0$, we must have $\rho(A, B) = 0$ and $\rho(B, A) = 0$. But $\rho(A, B) = 0$ implies $A \subseteq B$ and $\rho(B, A) = 0$ implies $B \subseteq A$. Now $A \subseteq B$ and $B \subseteq A$ implies $A = B$. Also $\rho_M(A, A) = \max\{\rho(A, A), \rho(A, A)\} = \rho(A, A)$. Since $A \subseteq A, \rho(A, A) = 0$

(3) Immediate from symmetry of max.

(4) Let $\rho_{12} = \rho_M(A,B)$ and $\rho_{23} = \rho_M(B,C)$. Then $A \subseteq B \oplus disk(\rho_{12}), B \subseteq C \oplus disk(\rho_{23}), B \subseteq A \oplus disk(\rho_{12}), C \subseteq B \oplus disk(\rho_{23})$. Hence, $A \subseteq C \oplus disk(\rho_{23}) \oplus disk(\rho_{12}) \subseteq C \oplus disk(\rho_{12} + \rho_{23})$ and $C \subseteq A \oplus disk(\rho_{12}) \oplus disk(\rho_{23}) \subseteq A \oplus disk(\rho_{12}+\rho_{23})$. Now $A \subseteq C \oplus disk(\rho_{12}+\rho_{23})$ and $C \subseteq A \oplus disk(\rho_{12}+\rho_{23})$ imply $\rho_{12}+\rho_{23} \geq inf\ \{r | A \subseteq C \oplus disk(r),\ C \subseteq A \oplus disk(r)\}$. Therefore, $\rho_M(A,B) + \rho_M(B,C) \geq \rho_M(A,C)$.

A strong relationship between the set distance and the dilation of sets must be developed to translate set bounding relationships to distance bounding relationships. We show that $\rho(A \oplus B, C \oplus D) \leq \rho(A,C) + \rho(B,D)$ and then quickly extend the result to $\rho_M(A \oplus B, C \oplus D) \leq \rho_M(A,C) + \rho_M(B,D)$.

Proposition 4

(1) $\rho(A \oplus B, C \oplus D) \leq \rho(A,C) + \rho(B,D)$

(2) $\rho_M(A \oplus B, C \oplus D) \leq \rho_M(A,C) + \rho_M(B,D)$

Proof

$$
\begin{aligned}
(1)\quad \rho(A \oplus B, C \oplus D) &= \max_{x \in A \oplus B} \min_{y \in C \oplus D} \|x - y\| \\
&= \max_{a \in A} \max_{b \in B} \min_{c \in C} \min_{d \in D} \|a + b - c - d\| \\
&\leq \max_{a \in A} \max_{b \in B} \min_{d \in D} \min_{c \in C} [\|a - c\| + \|b - d\|] \\
&\leq \max_{a \in A} \max_{b \in B} \min_{d \in D} [(\min_{c \in C} \|a - c\|) + \|b - d\|] \\
&\leq \max_{a \in A} \min_{c \in C} \|a - c\| + \max_{b \in B} \min_{d \in D} \|b - d\| \\
&\leq \rho(A,C) + \rho(B,D)
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad \rho_M(A \oplus B, C \oplus D) &= \max\{\rho(A \oplus B, C \oplus D), \rho(C \oplus D, A \oplus B)\} \\
&\leq \max\{\rho(A,C) + \rho(B,D), \rho(C,A) + \rho(D,B)\} \\
&\leq \max\{\rho(A,C), \rho(C,A)\} + \max\{\rho(B,D), \rho(D,B)\} \\
&\leq \rho_M(A,C) + \rho_M(B,D)
\end{aligned}
$$

■

From this last result, it is apparent that dilating two sets with the same structuring element cannot increase the distance between the sets. Dilation tends to suppress differences between sets, making them more similar. More precisely, if $B = D = K$, then $\rho_M(A \oplus K, C \oplus K) \leq \rho_M(A,C)$. It is also apparent that $\rho_M(A, A \oplus K) = \rho_M(A \oplus \{0\}, A \oplus K) \leq \rho_M(A,A) + \rho_M(\{0\},K) = \rho_M(\{0\},K) \leq \max_{k \in K} \|k\|$. Indeed, since the reconstruction structuring element $K = \check{K}$ and $\emptyset \in K$, the radius of the circumscribing disk is precisely $\max_{k \in K} \|k\|$. Hence, the distance between A and $A \oplus K$ is no more than the radius of the circumscribing disc of K.

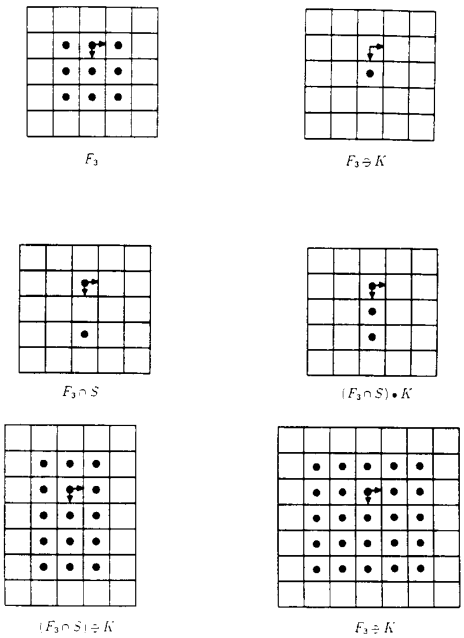


Figure 4 *shows a third example of how erosion and dilation of F_2 bound (in this case properly) the minimal reconstruction $(F_3 \cap S) \bullet K$ and the maximal reconstruction $(F_3 \cap S) \oplus K$, respectively, which in turn bound (in this case properly) F_3.*

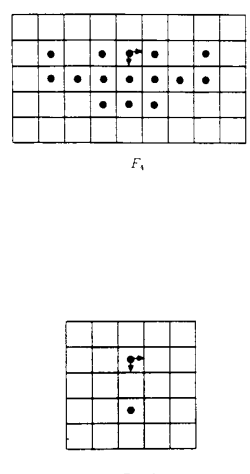


Figure 5 *shows a set F_4 which is not open under K. Its sampling $F_4 \cap S$ is identical to the sampling of F_3 yet the maximal reconstruction $(F_4 \cap S) \oplus K$ does not constitute an upper bound for F_4 as in the previous examples.*

Proposition 5

If $K = \check{K}$ and $0 \in K$, then $r(K) = \max_{k \in K} \|k\|$.

Proof

$$r(K) = \min_{x \in K} \max_{y \in K} \|x - y\| \leq \max_{y \in K} \|0 - y\| = \max_{y \in K} \|y\|$$

$$\begin{aligned}
\text{and } \max_{y \in K} \|y\| &= \frac{1}{2} \max_{y \in K} \|y - x + x + y\| \\
&\quad \text{for } x \in K \\
&\leq \frac{1}{2}\{\max_{y \in K} \|y - x\| + \max_{y \in K} \|x + y\|\} \\
&\quad \text{for } x \in K \\
&\leq \frac{1}{2}\{\max_{y \in K} \|x - y\| + \max_{y \in K} \|x - y\|\} \\
&\quad \text{for } x \in K \\
&\leq \max_{y \in K} \|x - y\| \text{for } x \in K \\
&\leq \min_{x \in K} \max_{y \in K} \|x - y\| = r(K)
\end{aligned}$$

■

Since $\rho_M(A, A \oplus K) \leq \max_{k \in K} \|k\|$ and $\max_{k \in K} \|k\| = r(K)$, we have $\rho_M(A, A \oplus K) \leq r(K)$. Also, since $A \bullet K \supseteq A$, $\rho_M(A \bullet K, A) = \rho(A \bullet K, A)$. Since $0 \in K$, $A \bullet K \subseteq A \oplus K$. Hence, $\rho_M(A \bullet K, A) = \rho(A \bullet K, A) \leq \rho\big((A \bullet K) \oplus K, A\big) = \rho(A \oplus K, A) \leq r(K)$.

It immediately follows that the distance between the minimal and maximal reconstructions, which differ only by a dilation by K, is no greater than the size of the reconstruction structuring element; that is, $\rho_M((F \cap S) \bullet K, (F \cap S) \oplus K) \leq r(K)$. When $F = F \circ K = F \bullet K, (F \cap S) \bullet K \subseteq F \subseteq (F \cap S) \oplus K$. Since the distance between the minimal and maximal reconstruction is no greater than $r(K)$ it is unsurprising that the distance between F and either of the reconstructions is no greater than $r(K)$.

Proposition 6

If $A \subseteq B \subseteq C$, then (1) $\rho_M(A, B) \leq \rho_M(A, C)$ and (2) $\rho_M(B, C) \leq \rho_M(A, C)$.

Proof

(1) Since $A \subseteq B, \rho_M(A, B) = \rho(B, A)$, then

$$\begin{aligned}
\rho(B, A) &= \max_{b \in B} \min_{a \in A} \|b - a\| \\
&\leq \max_{c \in C} \min_{a \in A} \|c - a\| \text{ since } B \subseteq C \\
&\leq \rho(C, A) = \rho_M(A, C) \text{ since } A \subseteq C.
\end{aligned}$$

(2) The proof of (2) is similar to (1) with B taking the role of A and C taking the role of B. ■

Now it immediately follows that if $F = F \circ K = F \bullet K$, $\rho_M\big(F, (F \cap S) \oplus K\big) \leq r(K)$ and $\rho_M\big(F, (F \cap S) \bullet K\big) \leq r(K)$.

These distance bounds can actually be shown under slightly less restrictive conditions. Suppose that $F = F \circ K$. Then it follows that $F \subseteq (F \cap S) \oplus K$. Since $F \cap S \subseteq F, (F \cap S) \oplus K \subseteq F \oplus K$. Hence $F \subseteq (F \cap S) \oplus K \subseteq F \oplus K$. But $\rho_M(F, F \oplus K) \leq r(K)$. Hence $\rho_M\big(F, (F \cap S) \oplus K\big) \leq r(K)$ and $\rho_M\big((F \cap S) \oplus K, F \oplus K\big) \leq r(K)$. It goes similarly with the closing reconstruction.

When the image F is open under K, the distance between F and its sampling $F \cap S$ can be no greater than $r(K)$. Why? It is certainly the case that $F \cap S \subseteq F \subseteq (F \cap S) \oplus K$. Hence $\rho_M(F, F \cap S) \leq \rho_M(F \cap S, (F \cap S) \oplus K) \leq r(K)$.

If two sets are both open under the reconstruction structuring element K, then the distance between the sets must be no greater than the distance between their samplings plus the size of K.

Proposition 7

If $A = A \circ K$ and $B = B \circ K$, then $\rho_M(A, B) \leq \rho_M(A \cap S, B \cap S) + r(K)$

Proof

Consider $\rho(A, B)$. $\rho(A, B) \leq \rho(A, B \cap S)$. Since $A = A \circ K$, $A \subseteq (A \cap S) \oplus K$. Hence $\rho(A, B) \leq \rho(A, B \cap S) \leq \rho\big((A \cap S) \oplus K, B \cap S\big) \leq \rho(A \cap S, B \cap S) + r(K)$. Similarly, since $B = B \circ K, \rho(B, A) \leq \rho(B \cap S, A \cap S) + r(K)$.

$$\begin{aligned}
\text{Now } \rho_M(A, B) &= \max\{\rho(A, B), \rho(B, A)\} \\
&\leq \max\{\rho(A \cap S, B \cap S) + r(K), \\
&\quad \rho(B \cap S, A \cap S) + r(K)\} \\
&= r(K) + \max\{\rho(A \cap S, B \cap S), \\
&\quad \rho(B \cap S, A \cap S)\} \\
&= r(K) + \rho_M(A \cap S, B \cap S).
\end{aligned}$$

From this last result it is easy to see that if F is closed under K, then the distance between F and its minimal reconstruction $(F \cap S) \bullet K$ is no greater than $r(K)$. Consider,

$$\begin{aligned}
\rho_M(F, (F \cap S) \bullet K) &\leq \rho_M\big(F \cap S, ((F \cap S) \bullet K) \cap S\big) \\
&\quad + r(K) \\
&= \rho_M(F \cap S, F \cap S) + r(K) = r(K)
\end{aligned}$$

These distance relationships mean that just as the standard sampling theorem cannot produce a reconstruction with frequencies higher than the Nyquist frequency, the morphological sampling theorem cannot produce a reconstruction whose positional accuracy is better than the radius of the circumscribing disk of the reconstruction structuring element K. Since the diameter of this disk is just short of being large enough to contain two sample intervals, the morphological sampling theorem cannot produce a reconstruction whose positional accuracy is better than the sampling interval.

2.1 Examples

We use the example sets F_1, F_2, F_3, and F_4 in computing the distance between the original images and the sample reconstruction images. The values $\max_{y \in K} \|x - y\|$ for each $x \in K$ are shown in Figure 6. The minimum value among them, $\sqrt{2}$, is the radius $r(K)$ since $r(K) = \min_{x \in K} \max_{y \in K} \|x - y\|$.

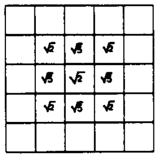


Figure 6 *The* $\max_{y \in K} \|x - y\|$ *values for all* $x \in K$*, where* K *is the digital disc having radius* $\sqrt{2}$.

We now measure the distance between two sample reconstructions for all the example sets. To compute $\rho_M((F_1 \cap S) \bullet K,\ (F_1 \cap S) \oplus K)$ we first compute $\rho((F_1 \cap S) \oplus K,\ (F_1 \cap S) \bullet K)$ and $\rho((F_1 \cap S) \bullet K,\ (F_1 \cap S) \oplus K)$. The values $\min_{y \in (F_1 \cap S) \bullet K} \|x - y\|$ for all $x \in (F_1 \cap S) \oplus K$ are shown in Figure 11. The maximum value among them, $\sqrt{2}$, is the distance $\rho((F_1 \cap S) \oplus K, (F_1 \cap S) \bullet K)$. Similarly, we can compute $\rho((F_1 \cap S) \bullet K,\ (F_1 \cap S) \oplus K)$ which equals 0. Thus, $\rho_M((F_1 \cap S) \bullet K,\ (F_1 \cap S) \oplus K)$ equals $\sqrt{2}$ which is exactly the radius $r(K)$. Similarly, the distance between two reconstructions for sets F_2, F_3, and F_4 can be measured and they are all equal to $r(K)$.

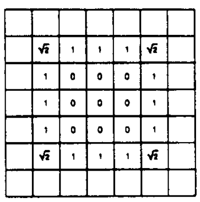


Figure 7 *The* $\min_{y \in (F_1 \cap S) \bullet K} \|x - y\|$ *for all* $x \in (F_1 \cap S) \oplus K$.

What is the distance $\rho_M(F, (F \cap S) \oplus K)$ for the example sets? Since $F_1 = (F_1 \cap S) \bullet K$, $\rho_M(F_1, (F_1 \cap S) \oplus K) = \rho_M((F_1 \cap S) \bullet K,\ (F_1 \cap S) \oplus K) = r(K)$. It is easy to see $\rho_M((F_2, (F_2 \cap S) \oplus K) = 0$ because $F_2 = (F_2 \cap S) \oplus K$. Figure 12 shows the values $\min_{y \in F_3} \|x - y\|$ for all $x \in (F_3 \cap S) \oplus K$, their maximum value being $\rho((F_3 \cap S) \oplus K,\ F_3) = 1$ Since $F_3 \subseteq (F_3 \cap S) \oplus K, \rho(F_3, (F_3 \cap S) \oplus K)$ equals 0. Hence, $\rho_M((F_3 \cap S) \oplus K, F_3) = 1 < r(K)$.

The distance $\rho(F_4, (F_4 \cap S) \oplus K)$ is interesting since $F_4 \neq F_4 \circ K$. The $\min_{y \in (F_4 \cap S) \oplus K} \|x - y\|$ values for all $x \in F_4$ are shown in Figure 9a, their maximum value being $\rho(F_4, (F_4 \cap S) \oplus K) = 2$. The $\min_{y \in F_4} \|x - y\|$ values for all $x \in (F_4 \cap S) \oplus K$ are shown in Figure 9(b), the maximum value is $\rho((F_4 \cap S) \oplus K, F_4) = 1$ Thus, the distance $\rho_M(F_4, (F_4 \cap S) \oplus K)$ is equal to 2 which is greater than $r(K)$. This shows why the condition $F = F \circ K$ is required to bound the difference between F and its maximum reconstruction $(F \cap S) \oplus K$. Similarly, we find

$$\rho_M((F_1 \cap S) \oplus K, F_1 \oplus K) = 0 < r(K)$$
$$\rho_M((F_2 \cap S) \oplus K, F_2 \oplus K) = \sqrt{2} = r(K)$$
$$\rho_M((F_3 \cap S) \oplus K, F_3 \oplus K) = 1 < r(K)$$
$$\rho_M((F_4 \cap S) \oplus K, F_4 \oplus K) = 2 > r(K)$$

Note that since $F_4 \neq F_4 \circ K$, $\rho_M((F_4 \cap S) \oplus K, F_4 \oplus K) \not\leq r(K)$. Using the minimum reconstruction, the positional accuracy for the example sets are

$$\rho_M(F_1, (F_1 \cap S) \bullet K) = 0 < r(K)$$
$$\rho_M(F_2, (F_2 \cap S) \bullet K) = \sqrt{2} = r(K)$$
$$\rho_M(F_3, (F_3 \cap S) \bullet K) = 1 < r(K)$$
$$\rho_M(F_4, (F_4 \cap S) \bullet K) = 3 > r(K)$$

Also, since $F_4 \neq F_4 \bullet K$, $\rho_M(F_4, (F_4 \cap S) \bullet K) \not\leq r(K)$.

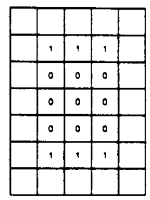
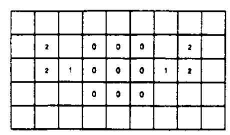
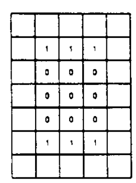


Figure 8 *shows* $\min_{y \in F_3} \|x - y\|$ *for each* $x \in (F_3 \cap S) \oplus K$.



(a)



(b)

Figure 9(a) *shows values for* $\min_{y \in (F_4 \cap S) \oplus K} \|x - y\|$ *for all* $x \in F_4$. **Figure 9(b)** *shows values for* $\min_{y \in F_4} \|x - y\|$ *for all* $x \in (F_4 \cap S) \oplus K$. *The maximum among all these values is 2. Hence* $\rho_M(F_4 \cap S) \oplus k) = 2 > r(K)$.

2.2 The Reconstruction Distance Bounds Theorem

This section summarizes the results developed thus far which constitute the reconstruction distance bounds theorem.

Reconstruction Distance Bounds Theorem

Let $F, K, S \subseteq E^N$. Suppose K and S satisfy the sampling conditions

(1) $S \oplus S = S$
(2) $S = \check{S}$
(3) $K \cap S = \{0\}$
(4) $K = \check{K}$
(5) $x \in K_y$ implies $K_x \cap K_y \cap S \neq \emptyset$

Then
(1) If $F = F \bullet K$, then $\rho_M\big(F, (F \cap S) \bullet K\big) \leq r(K)$
(2) If $F = F \circ K$, then $\rho_M\big((F \cap S) \oplus K, F\big) \leq r(K)$

3. Operating In the Sampled Domain

The previous section established the relationship between the information contained in the sampled set and the information contained in the unsampled set. It shows that a minimal and maximal reconstruction can be computed from the sampled set. When the set is smooth enough with respect to the sampling S (that is, when the set is both open and closed under the reconstruction structuring element), then the minimal and maximal reconstructions bound the unsampled set, differing from it by no more than the sampling interval length.

Not addressed is the relationship between the computationally more efficient procedure of morphologically operating in the sampled domain versus the less computationally efficient procedure of morphologically operating in the unsampled domain. In this section we quantify just exactly how close a morphological operation in the sampled domain can come to the corresponding morphological operation in the original domain. Thus we answer the question of how to compute the largest length of sampling interval which can produce an answer close enough to the desired answer when morphologically operating in the sampled domain.

The first proposition shows that a sampled dilation contains the dilation of the sampled sets and a sampled erosion is contained in the erosion of the sampled sets.

Proposition 8

Let $B \subseteq E^N$ be the structuring element employed in the dilation or erosion. Then

(1) $(F \cap S) \oplus (B \cap S) \subseteq (F \oplus B) \cap S$

(2) $(F \cap S) \ominus (B \cap S) \supseteq (F \ominus B) \cap S$

Proof

(1) $F \cap S \subseteq F$ and $B \cap S \subseteq B$. Hence $(F \cap S) \oplus (B \cap S) \subseteq F \oplus B$. Also $F \cap S \subseteq S$ and $B \cap S \subseteq S$. Hence $(F \cap S) \oplus (B \cap S) \subseteq S \oplus S$. But $S \oplus S = S$. Then, $(F \cap S) \oplus (B \cap S) \subseteq (F \oplus B) \cap S$.
(2) By (1) $[(F \ominus B) \cap S] \oplus (B \cap S) \subseteq [(F \ominus B) \oplus B] \cap S = (F \circ B) \cap S \subseteq F \cap S$. But $[(F \ominus B) \cap S] \oplus (B \cap S) \subseteq (F \cap S)$ if and only if $(F \cap S) \ominus (B \cap S) \supseteq (F \ominus B) \cap S$. ∎

Unfortunately, the containment relations cannot, in general, be strengthened to equalities. But we can determine the conditions under which the equality occurs and the distance between sets such as $(F \cap S) \oplus (B \cap S)$ and $(F \oplus B) \cap S$. In the sampled domain, we compare the scheme of sampling and then performing the dilation in the sampled domain to dilating first and then sampling. We also inquire about how different things could be in the unsampled domain by comparing performing the dilation in the sampled space and then reconstructing versus performing the dilation in the unsampled domain. The next proposition shows that this difference in the sampled domain cannot be more than $2r(K)$.

Proposition 9

If $F = F \circ K$ and $B = B \circ K$, then $\rho_M\big((F \oplus B) \cap S, (F \cap S) \oplus (B \cap S)\big) \leq 2r(K)$

Proof

First consider $\rho\big((F \oplus B) \cap S, (F \cap S) \oplus (B \cap S)\big) \leq \rho\big(F \oplus B, (F \cap S) \oplus (B \cap S)\big)$ Since $F = F \circ K$ and $B = B \circ K$, $F \subseteq (F \cap S) \oplus K$ and $B \subseteq (B \cap S) \oplus K$. Hence,

$$\begin{aligned}\rho\big(F \oplus B, (F \cap S) \oplus (B \cap S)\big) &\leq \rho\big((F \cap S) \oplus K \oplus \\ &\quad (B \cap S) \oplus K, (F \cap S) \oplus (B \cap S)\big) \\ &\leq \rho\big([(F \cap S) \oplus (B \cap S)] \oplus K \oplus K, \\ &\quad (F \cap S) \oplus (B \cap S)\big) \\ &\leq r(K \oplus K) \leq 2r(K)\end{aligned}$$

Next note that $\rho\big((F \cap S) \oplus (B \cap S), (F \oplus B) \cap S\big) = 0$. Since $(F \cap S) \oplus (B \cap S) \subseteq (F \oplus B) \cap S$. Now $\rho_M\big((F \oplus B) \cap S, (F \cap S) \oplus (B \cap S)\big) = \max\big\{\rho\big((F \oplus B) \cap S, (F \cap S) \oplus (B \cap S)\big), \rho\big((F \cap S) \oplus (B \cap S), (F \oplus B) \cap S\big)\big\} \leq \max\{2r(K), r0\} = 2r(K)$ ∎

Whereas dilation tends to suppress differences, erosion tends to accentuate differences. Consider the following example. Let F be a disk of radius 12 and B be a disk of radius 10. Then $F \ominus B$ is a disk of radius 2. Now define F' to be a disk of radius 12 with its center point deleted. Notice that the pseudo set distance between F and F' is zero. But although F' close to F, $F' \ominus B = \emptyset$. The difference of one point makes all the difference.

More formally, consider the difference between the erosion of F and the erosion of $F \oplus K$.

$$\rho_M\Big((F \oplus K) \ominus B, F \ominus B\Big) = \rho\Big((F \oplus K) \ominus B, F \ominus B\Big)$$
$$\geq \rho\Big((F \ominus B) \oplus K, F \ominus B\Big)$$

since $(F \oplus K) \ominus B \subseteq (F \ominus B) \oplus K$ where $\rho((F \ominus B) \oplus K, F \ominus B)$ is no greater than and could be as close to $r(K)$ as possible.

Thus we cannot expect that the difference between performing an erosion in the sampled domain versus performing a sampling of the erosion in the unsampled domain is no greater than the size of K. However, we do obtain set bounding relationships for dilation and erosion using the following relationships:

Dilating (eroding) a sampled set by a sampled structuring element is equivalent to sampling the dilation (erosion) of the unsampled set by the sampled structuring element.

Lemma:

(1) $(F \cap S) \oplus (B \cap S) = [F \oplus (B \cap S)] \cap S$

(2) $(F \cap S) \ominus (B \cap S) = [F \ominus (B \cap S)] \cap S$

Proof

(1)

$$[F \oplus (B \cap S)] \cap S = \Big(\bigcup_{x \in B \cap S} F_x\Big) \cap S$$
$$= \bigcup_{x \in B \cap S} (F_x \cap S)$$

But $x \in S$ implies $S = S_x$. Hence,

$$[F \oplus (B \cap S)] \cap S = \bigcup_{x \in B \cap S} F_x \cap S_x$$
$$= \bigcup_{x \in B \cap S} (F \cap S)_x$$
$$= (F \cap S) \oplus (B \cap S)$$

(2)

$$[F \ominus (B \cap S)] \cap S = \Big(\bigcap_{x \in B \cap S} F_{-x}\Big) \cap S$$
$$= \Big(\bigcap_{x \in \check{B} \cap \check{S}} F_x\Big) \cap S$$
$$= \bigcap_{x \in \check{B} \cap \check{S}} (F_x \cap S)$$
$$= \bigcap_{x \in \check{B} \cap \check{S}} (F_x \cap S_x) \text{ since } x \in S \text{ implies } S_x = S$$
$$= \bigcap_{x \in \check{B} \cap \check{S}} (F \cap S)_x$$
$$= (F \cap S) \ominus (B \cap S)$$

■

Moreover, the dilation of the minimal reconstruction by a structuring element B open under K is contained in the dilation of the maximal reconstruction by the sampled structuring element $B \cap S$.

Lemma:

Let $B = B \circ K$. Then $[(F \cap S) \bullet K] \oplus B \subseteq [(F \cap S) \oplus K] \oplus (B \cap S)$

Proof

Let $x \in [(F \cap S) \bullet K] \oplus B$. Then there exists an $f \in (F \cap S) \bullet K$ and $b \in B$ such that $x = f + b$. Since $B = B \circ K$, $b \in B$ implies there exists a y such that $b \in K_y \subseteq B$. But because of the sampling constraint between K and S, $b \in K_y$ implies $K_b \cap K_y \cap S \neq \emptyset$. Therefore, there exists a $z \in K_b \cap K_y \cap S$. Now $z \in K_b$ implies that $z = k + b$ for some $k \in K$. Since it is also the case that $z \in K_y$, it must be that $z \in B$ because $K_y \subseteq B$. Recall that $x = f + b = f + z - k = (f - k) + z$. Since $f \in (F \cap S) \bullet K = [(F \cap S) \oplus K] \ominus K$ and since $-k \in \check{K} = K$, $f - k \in \Big([(F \cap S) \oplus K] \ominus K\Big) \oplus K = (F \cap S) \oplus K$. Since $z \in B$ and $z \in S$, $z \in B \cap S$. Finally, $f - k \in (F \cap S) \oplus K$ and $z \in B \cap S$ imply $x = (f - k) + z \in [(F \cap S) \oplus K] \oplus (B \cap S)$.

■

Now we see that dilation in the sampled domain and dilation in the unsampled domain are equivalent exactly when the structuring element B of the dilation is open under K, and when the image F is its minimal reconstruction.

Sample Dilation Theorem

Let $B = B \circ K$. Then $(F \cap S) \oplus (B \cap S) = \{[(F \cap S) \bullet K] \oplus B\} \cap S$

Proof

$(F \cap S) \oplus (B \cap S) = \Big((F \cap S) \oplus B\Big) \cap S$ is always true. Since $F \cap S \subseteq (F \cap S) \bullet K$, $\Big((F \cap S) \oplus B\Big) \cap S \subseteq \{[(F \cap S) \bullet K] \oplus B\} \cap S$. But $[(F \cap S) \bullet K] \oplus B \subseteq [(F \cap S) \oplus K] \oplus (B \cap S)$ when $B = B \circ K$. Hence $(F \cap S) \oplus (B \cap S) \subseteq \{[(F \cap S) \bullet K] \oplus B\} \cap S \subseteq \{[(F \cap S) \oplus K] \oplus (B \cap S)\} \cap S$. Now $\{[(F \cap S) \oplus K] \oplus (B \cap S)\} \cap S = \{[(F \cap S) \oplus K] \cap S\} \oplus (B \cap S)$. Since $[(F \cap S) \oplus K] \cap S = F \cap S$ always holds under the sampling conditions, there results $(F \cap S) \oplus (B \cap S) \subseteq [(F \cap S) \bullet K] \oplus (B \cap S) \subseteq (F \cap S) \oplus (B \cap S)$ so that $(F \cap S) \oplus (B \cap S) = [(F \cap S) \bullet K] \oplus (B \cap S)$. ■

The equality relationship established in the theorem immediately leads to a set bounding relationship for dilation.

$$[(F \ominus K) \oplus B] \cap S \subseteq \{[(F \cap S) \bullet K] \oplus B\} \cap S =$$
$$(F \cap S) \oplus (B \cap S) \subseteq (F \oplus B) \cap S$$

Also from the theorem, it becomes apparent that the difference between the maximally reconstructed dilation and the dilation of the minimal reconstruction can be no more than the size of K when B is open under K. This happens because

$$\rho_M([(F \cap S) \oplus (B \cap S)] \oplus K, [(F \cap S) \bullet K] \oplus B) \leq$$
$$\rho_M(\{[(F \cap S) \oplus (B \cap S)] \oplus K\} \cap S,$$
$$\{[(F \cap S) \bullet K] \oplus B\} \cap S) + r(K)$$
$$\leq \rho_M((F \cap S) \oplus (B \cap S),$$
$$(F \cap S) \oplus (B \cap S)) + r(K) = r(K)$$

Similarly, eroding a sampled image by a sampled structuring element is equivalent to eroding the maximal reconstruction by the structuring element and then sampling when the structuring element is open under K.

Sample Erosion Theorem

Let $B = B \circ K$. Then $(F \cap S) \ominus (B \cap S) = \{[(F \cap S) \oplus K] \ominus B\} \cap S$

Proof

The sampling conditions imply $[(F \cap S) \oplus K] \cap S = F \cap S$. Hence,

$$\begin{aligned}(F \cap S) \ominus (B \cap S) &= \{[(F \cap S) \oplus K] \cap S\} \ominus (B \cap S) \\ &= \{[(F \cap S) \oplus K] \ominus (B \cap S)\} \cap S \\ &\supseteq \{[(F \cap S) \oplus K] \ominus B\} \cap S\end{aligned}$$

■

Under the sampling conditions, $(F \cap S) \ominus (B \cap S) \subseteq S$. So to complete the equality, we need to show that $(F \cap S) \ominus (B \cap S) \subseteq [(F \cap S) \oplus K] \ominus B$. Let $x \in (F \cap S) \ominus (B \cap S)$. Then $(B \cap S)_x \subseteq F \cap S$. Since $B = B \circ K$, $B \subseteq (B \cap S) \oplus K$. Hence $B_x \subseteq (B \cap S)_x \oplus K$. But $(B \cap S)_x \subseteq F \cap S$ so that $B_x \subseteq (F \cap S) \oplus K$. Now by definition of erosion, if $B_x \subseteq (F \cap S) \oplus K$, then $x \in [(F \cap S) \oplus K] \ominus B$

This theorem immediately leads to some set bounding relationships for erosion

$$(F \ominus B) \cap S \subseteq (F \cap S) \ominus (B \cap S) = \{[(F \cap S) \oplus K] \ominus B\} \cap S \subseteq [(F \oplus K) \ominus B] \cap S$$

Theorem 3 also makes it apparent that the difference between the maximally reconstructed erosion and the erosion of the maximal reconstruction can be no more than the size of K when both B and the erosion of the maximal reconstruction are open under K. This happens because

$$\begin{aligned}&\rho_M\Big([(F \cap S) \ominus (B \cap S)] \oplus K, [(F \cap S) \oplus K] \ominus B\Big) \\ &\leq \rho_M\Big(\{[(F \cap S) \ominus (B \cap S)] \oplus K\} \cap S, \\ &\quad \{[(F \cap S) \oplus K] \ominus B\} \cap S\Big) + r(K) \\ &\leq \rho_M\Big((F \cap S) \ominus (B \cap S), (F \cap S) \ominus (B \cap S)\Big) + r(K) = r(K)\end{aligned}$$

Just as it was the case that dilating (eroding) a sampled set by a sampled structuring element is equivalent to sampling the dilation (erosion) of the unsampled set by the sampled structuring element, so it is also the case that opening (closing) a sampled set by a sampled structuring element is equivalent to sampling the opening (closing) of the unsampled set by the sampled structuring element. These relationships are useful in establishing when the opening and closing operation are equivalent in the sampled and unsampled domain.

Proposition 10

$[F \circ (B \cap S)] \cap S = (F \cap S) \circ (B \cap S)$

Proof

Let $x \in [F \circ (B \cap S)] \cap S$. Then $x \in F \circ (B \cap S)$ and $x \in S$. But $x \in F \circ (B \cap S)$ if and only if for some $y \in F \ominus (B \cap S), x \in (B \cap S)_y \subseteq F$. Now $x \in (B \cap S)_y$ implies $x = b + y$ where $b \in B \cap S$. Then $y = x - b$. Since $b \in S$ and since $S - \check{S}$, $-b \in S$. Since $x \in S$ and $-b \in S$, $x - b \in S \oplus S$. But $S \oplus S = S$. Then $y \in S$. Now we show that $y \in S$ and $(B \cap S)_y \subseteq F$ imply $(B \cap S)_y \subseteq F \cap S$. Let $z \in (B \cap S)_y$ Since $(B \cap S)_y \subseteq F$, $z \in F$. Now $z \in (B \cap S)_y = B_y \cap S_y = B_y \cap S$ since $y \in S$. Hence $z \in S$. Hence $z \in F \cap S$. Finally, $x \in (B \cap S)_y \subseteq F \cap S$ implies $x \in (F \cap S) \circ (B \cap S)$. Thus $[F \circ (B \cap S)] \cap S \subseteq (F \cap S) \circ (B \cap S)$.

Now suppose $x \in (F \cap S) \circ (B \cap S)$. Then there exists a $y \in (F \cap S) \ominus (B \cap S)$ such that $x \in (B \cap S)_y \subseteq F \cap S$. But $F \cap S \subseteq F$. Then $x \in (B \cap S)_y \subseteq F$ and this implies that $x \in F \circ (B \cap S)$. Also, $x \in F \cap S$ implies $x \in S$. Then $x \in [F \circ (B \cap S)] \cap S$. This establishes that $(F \cap S) \circ (B \cap S) \subseteq [F \circ (B \cap S)] \cap S$ ■

Proposition 11

$[F \bullet (B \cap S)] \cap S = (F \cap S) \bullet (B \cap S)$

Proof

Let $x \in [F \bullet (B \cap S)] \cap S$. Then $x \in F \bullet (B \cap S)$ and $x \in S$. But $x \in F \bullet (B \cap S)$ if and only if $x \in (B \check{\cap} S)_y$ implies $x \in (B \check{\cap} S)_y \cap F \neq \emptyset$. Let y satisfy $x \in (B \check{\cap} S)_y$. Then $x = b + y$ where $b \in \check{B} \cap \check{S}$. Then $y = x - b$. Since $x \in S$ and $-b \in S$, $y \in S \oplus S = S$. Now if $y \in S$, then $(B \cap S)_y \cap F = \check{B}_y \cap \check{S}_y \cap F = \check{B}_y \cap \check{S}_y \cap S \cap F = (B \check{\cap} S)_y \cap (F \cap S)$. Now $x \in (B \check{\cap} S)_y$ implies $(B \check{\cap} S)_y \cap F \neq \emptyset$. Since $(B \check{\cap} S)_y \cap F = (B \check{\cap} S)_y \cap (F \cap S), (B \check{\cap} S)_y \cap (F \cap S) \neq \emptyset$. This implies that $x \in (F \cap S) \bullet (B \cap S)$.

Let $x \in (F \cap S) \bullet (B \cap S)$. Then $x \in (B \check{\cap} S)_y$ implies $(B \check{\cap} S)_y \cap (F \cap S) \neq \emptyset$. But $(B \check{\cap} S)_y \cap (F \cap S) \subseteq (B \check{\cap} S)_y \cap F$. Hence $(B \check{\cap} S)_y \cap F \neq \emptyset$ and this implies that $x \in F \bullet (B \cap S)$. Also,

$$\begin{aligned}(F \cap S) \bullet (B \cap S) &= [(F \cap S) \ominus (B \cap S)] \oplus (B \cap S) \\ &= \{[F \ominus (B \cap S)] \cap S\} \oplus (B \cap S) \\ &\subseteq S \oplus (B \cap S) \subseteq S \oplus S \subseteq S\end{aligned}$$

Hence $x \in S$. Finally, $x \in F \bullet (B \cap S)$ and $x \in S$ imply $x \in [F \bullet (B \cap S)] \cap S$. ■

The bounding relationships between the sampled and unsampled domains for the opening and closing operation now follow immediately.

Sample Opening and Closing Bounds Theorem

Suppose $B = B \circ K$, then

(1) $\{F \circ [(B \cap S) \oplus K]\} \cap S \subseteq (F \cap S) \circ (B \cap S) \subseteq \{[(F \cap S) \oplus K] \circ B\} \cap S$

(2) $\{[(F\cap S)\bullet K]\bullet B\}\cap S \subseteq (F\cap S)\bullet(B\cap S) \subseteq \{F\bullet[(B\cap S)\oplus K]\}\cap S$

Proof

(1) Notice that $[(B\cap S)\oplus K]\circ(B\cap S) = (B\cap S)\oplus K$. Under this condition, $\{F\circ[(B\cap S)\oplus K]\}\cap S \subseteq [F\circ(B\cap S)]\cap S$. But by a previous proposition $[F\circ(B\cap S)]\cap S = (F\cap S)\circ(B\cap S)$. Now suppose $x\in(F\cap S)\circ(B\cap S)$. Then there exists a y such that $x\in(B\cap S)_y\subseteq F\cap S$. But $(B\cap S)_y\subseteq(F\cap S)$ implies $(B\cap S)_y\oplus K\subseteq(F\cap S)\oplus K$ since dilation is an increasing operation. Hence, $[(B\cap S)\oplus K]_y\subseteq(F\cap S)\oplus K$. Since $B=B\circ K$, $B\subseteq(B\cap S)\oplus K$. Then, $B_y\subseteq(F\cap S)\oplus K$. Also, $x\in(B\cap S)_y$ implies $x\in B_y$. Finally, $x\in B_y\subseteq(F\cap S)\oplus K$ implies $x\in[(F\cap S)\oplus K]\circ B$.

(2) By a previous proposition $(F\cap S)\bullet(B\cap S) = [F\bullet(B\cap S)]\cap S$. Since $[(B\cap S)\oplus K]\circ(B\cap S) = (B\cap S)\oplus K$, $[F\bullet(B\cap S)]\cap S \subseteq \{F\bullet[(B\cap S)\oplus K]\}\cap S$. Let $R=(F\cap S)\bullet K$. Since $B=B\circ K$, $R\oplus B$ is open under K. Hence $R\oplus B\subseteq[(R\oplus B)\cap S]\oplus K$. Now

$$(R\bullet B)\cap S = [(R\oplus B)\ominus B]\cap S \subseteq \Big(\{[(R\oplus B)\cap S]\oplus K\}\ominus B\Big)\cap S$$

But the sampled erosion of a maximal reconstruction is the erosion of the sampled set by the sampled structuring element. Hence,

$$\Big(\{[(R\oplus B)\cap S]\oplus K\}\ominus B\Big)\cap S = \Big([(R\oplus B)\cap S]\ominus(B\cap S)$$

And the sampled dilation of a minimal reconstruction is the dilation of the sampled set by the sampled structuring element. Hence,

$$[(R\oplus B)\cap S]\ominus(B\cap S) = [(R\cap S)\oplus(B\cap S)]\ominus(B\cap S)$$

Finally, $R\cap S = [(F\cap S)\bullet K]\cap S = F\cap S$ so that $\Big([(F\cap S)\bullet K]\bullet B\Big)\cap S\subseteq(F\cap S)\bullet(B\cap S)$.

∎

The bounding relationships immediately imply the following equivalence for the opening and closing operations between the sampled and unsampled domains.

Sample Opening and Closing Theorem

Suppose $B=B\circ K$.

(1) If $F=(F\cap S)\oplus K$ and $B=(B\cap S)\oplus K$, then $(F\cap S)\circ(B\cap S)=(F\circ B)\cap S$

(2) If $F=(F\cap S)\bullet K$ and $B=(B\cap S)\oplus K$, then $(F\cap S)\bullet(B\cap S)=(F\bullet B)\cap S$

Proof

(1) If $F=(F\cap S)\oplus K$ and $B=(B\cap S)\oplus K$, the bounding relationship for opening becomes

$$(F\circ B)\cap S\subseteq(F\cap S)\circ(B\cap S)\subseteq(F\circ B)\cap S$$

from which we immediately obtain $(F\circ B)\cap S=(F\cap S)\circ(B\cap S)$

(2) If $F=(F\cap S)\bullet K$ and $B=(B\cap S)\oplus K$, the bounding relationship for closing becomes

$$(F\bullet B)\cap S\subseteq(F\cap S)\bullet(B\cap S)\subseteq(F\bullet B)\cap S$$

from which we immediately obtain $(F\bullet B)\cap S=(F\cap S)\bullet(B\cap S)$. ∎

3.1 Examples

A simple example illustrates the bounding relationships of morphological operations operating in the pre- and post-sampled domain. The sample set S and the set K we used are those defined in the previous examples (see Figure 1). The sets F,B, and K are defined in Figure 10. It is clear that $B=B\circ K$. In Figure 11, we show the results of down-sampling every other row and every other column, $F\cap S$, $B\cap S$, and the sampled domain morphological operations, $(F\cap S)\oplus(B\cap S)$, $(F\cap S)\ominus(B\cap S)$. The results $[(F\cap S)\bullet K]\oplus B$, $[(F\cap S)\oplus K]\ominus B$, $\{[(F\cap S)\bullet K]\oplus B\}\cap S$,and $\{[(F\cap S)\oplus K]\ominus B\}\cap S$ are shown in Figure 12. Note that the following equalities hold:

$$(F\cap S)\oplus(B\cap S)=\{[(F\cap S)\bullet K]\oplus B\}\cap S$$

and

$$(F\cap S)\ominus(B\cap S)=\{[(F\cap S)\oplus K]\ominus B\}\cap S.$$

Figure 13 shows $(F\oplus B)\cap S$, $(F\ominus B)\cap S$, $(F\oplus(B\cap S))$, and $(F\ominus(B\cap S))$. Note that the following are true:

$$(F\cap S)\oplus(B\cap S)\subseteq(F\oplus B)\cap S$$

and

$$(F\cap S)\ominus(B\cap S)\supseteq(F\ominus B)\cap S.$$

It can be easily verified that

$$(F\cap S)\oplus(B\cap S)=[F\oplus(B\cap S)]\cap S$$

and

$$(F\cap S)\ominus(B\cap S)=[F\ominus(B\cap S)]\cap S.$$

In practical multiresolution image processing applications we would like to perform morphological operations in the sampled domain to reduce the computational expense. How well can a morphological operation be performed in the sampled domain rather than the original domain can be answered by the relationships and distances between $(F\cap S)\oplus(B\cap S)$ and $(F\oplus B)\cap S$ as well as $(F\cap S)\ominus(B\cap S)$ and $(F\ominus B)\cap S$. Unfortunately, the distance

$$\rho_M((F\cap S)\oplus(B\cap S),(F\oplus B)\cap S)<2r(K)$$

can be guaranteed only when $F=F\circ K$ and $B=B\circ K$. It can be very big when the set F is not open. The set F of

Figure 10 is an example having a large difference between the pre- and post- sampled dilations because the conditions $F = F \circ K$ and $B = B \circ K$ are not satisfied.

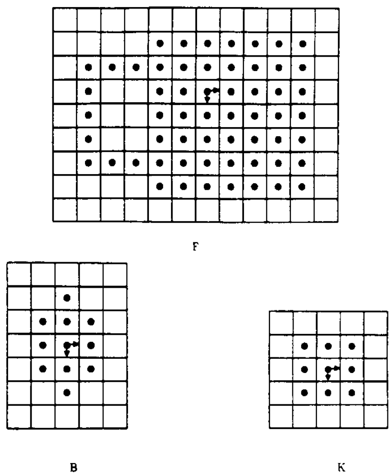


Figure 10 *illustrates the sets F, B, and K.*

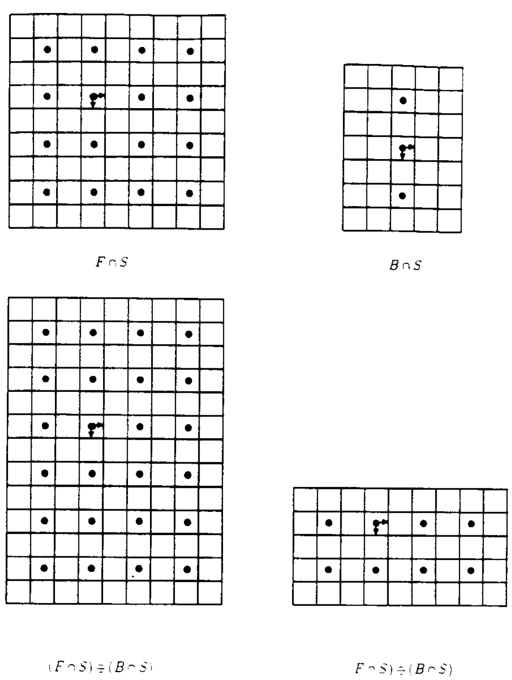


Figure 11 *shows the results of sampling the F and B of Figure 14 and performing the dilation and erosion of $F \cap S$ by $B \cap S$ in the sampled domain.*

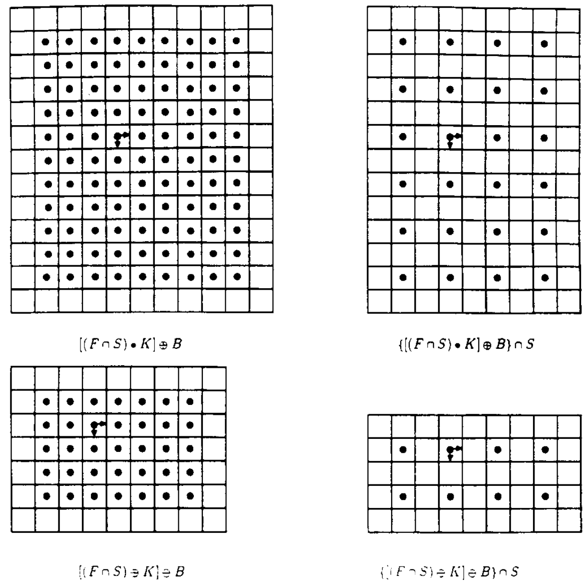


Figure 12 *shows the dilation and erosion of the minimal and maximal reconstruction of F by the structuring element B and also shows the sampling of this dilation and erosion.*

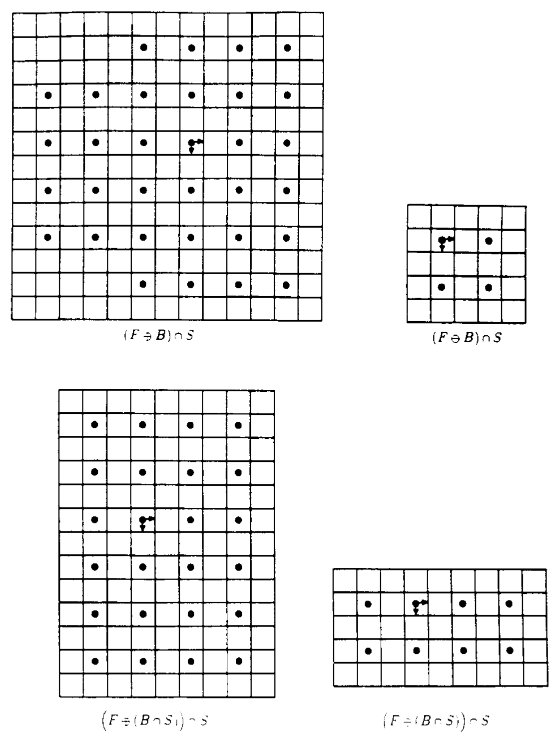


Figure 13 *shows some morphological operations in the original domain followed by sampling.*

4. Conclusion

We have refined the set bounding relationships of the binary morphological sampling theorem to distance bounding relationships. We have shown that on all comparisons between morphologically filtering versus sampling, morphologically filtering in the sampled domain, and reconstructing, the differences by the Hausdorf set metric will be no more than the sampling interval. For comparison in the sampled domain, that is, between morphologically filtering and sampling versus sampling and morphologically filtering in the sampled domain, the difference by the Hausdorf set metric will be no more than twice the sampling interval.

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SESSION 3.3 – NAVIGATION

Chair

H. Samet
University of Maryland