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THE BINARY MORPHOLOGICAL SAMPLING THEOREM

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ABSTRACT

There are potential industrial applications for any methodology which inherently reduces processing time and cost and yet produces results sufficiently close to the result of full processing. It is for this reason that a morphological sampling theorem is important.

The morphological sampling theorem described in this paper states: (1) how a digital image must be morphologically filtered before sampling in order to preserve the relevant information after sampling; (2) to what precision an appropriately morphologically filtered image can be reconstructed after sampling; and (3) the relationship between morphologically operating before sampling and the more computationally efficient scheme of morphologically operating on the sampled image with a sampled structuring element.

The digital sampling theorem is developed first for the case of binary morphology and then it is extended to gray scale morphology through the use of the umbra homomorphism theorems.

1. Introduction

Morphological operations on images have relevance to conditioning, labeling, grouping, extracting and matching image processing operations. Thus from low level to intermediate to high level vision, morphological techniques are important. Indeed, many successful machine vision algorithms employed in industry on the factory floor, processing thousands of images per day in each application, are based on morphological techniques. Among the recent research papers on morphology are Crimmons and Brown (1985), Lee, Haralick, and Shapiro (1987), Haralick, Sternberg, and Zhuang (1987), and Maragos and Schafer (1987). Serra (1982) constitutes a comprehensive reference.

Many well-known relationships worked out in the classical context of the convolution operation have morphological analogs. In this paper, we introduce the digital morphological sampling theorem, which relates to morphology as the standard sampling theorem relates to signal processing and communications. The sampling theorem permits the development of a precise multiresolution approach to morphological processing.

Multiresolution techniques (Ahuja and Swamy, 1984; Klinger, 1984; Meresereau and Speake, Tanimoto, 1982; Uhr, 1983) have been useful for at least two fundamental reasons: (1) the representation they provide naturally permits a computational mechanism to focus on objects or features likely to be at least a given specified size (Crowley, 1984; Miller and Stout; Rosenfeld, 1983; Witkin, 1984), and (2) the computational mechanism can operate on only those resolution levels which just suffice for the detection and localization of objects or features of specified size while significantly reducing the number of operations performed (Burt, 1984; Dyer, 1982; Lougheed and McCubbrey, 1980).

The usual resolution hierarchy, called a pyramid, is produced by low pass filtering and then sampling to generate the next lower resolution level of the hierarchy. The basis for a morphological pyramid requires a morphological sampling theorem which explains how an appropriately morphologically filtered and sampled image relates to the unsampled image. It must explain what kinds of shapes are preserved and what kinds are suppressed or eliminated. It must explain the relationship between performing a less costly morphological filtering operation on the sampled image and performing the more costly equivalent morphological filtering operations on the original image. It is just these issues which we address in this paper.

We analyze the constraints on sampling and on image objects in order to speed up morphological operations without sacrificing accurate shape analysis. The following results are shown to be true under reasonable morphological sampling conditions. Before sets are sampled, they must be morphologically simplified by an opening or a closing. Such sampled sets can be reconstructed in two ways, by either a closing or dilation. In both reconstructions, the sampled reconstructed sets are equal to the sampled sets. A set contains its reconstruction by closing and is contained in its reconstruction by dilation; indeed, these are extremal bounding sets. That is, the largest set which downsamples to a given set is its reconstruction by dilation; the smallest is its reconstruction by closing. Furthermore, the distance from the maximal reconstruction to the minimal reconstruction is no more than the diameter of the reconstruction structuring element. Morphological sampling thus provides reconstructions positioned only to within some spatial tolerance which depends on the sampling interval. This spatial limitation contrasts with the sampling reconstruction process in signal processing from which only those frequencies below the Nyquist frequency can be reconstructed.

A number of relationships which follow from the morphological sampling theorem. These relationships govern the commutivity between sampling and then performing morphological operations in the sampled domain versus first performing the morphological operations and then sampling. We find that sampling a minimal reconstruction which has been dilated is identical to dilating the sample set with a sampled structuring element. Sampling a maximal reconstruction which has been eroded is identical to eroding the sampled set with a sampled structuring element. These results establish bounds which can be used to determine the difference between morphological operations in the sampled domains and operations in the original domain followed by sampling.

All set morphological relationships are immediately generalizable to gray scale morphology via the umbra homomorphism theorems. For grayscale images, the bounds which the reconstruction establishes are bounds in a spatial sense.

In Section 2 we review the basic definitions and properties for binary morphology operations. In Section 3, we develop the morphological sampling theorem for binary morphology. In Section 4, we derive the relationship between morphologically operating in the original domain and operating in the sampled domain. The homomorphism theorem between binary and grayscale morphology implies that each result in binary morphology has a corresponding result in grayscale morphology. Section 5 develops these grayscale generalizations. Section 6 discusses the computational advantages of operating on morphologically sampled images and shows how successively sampled images can be operated on in a resolution hierarchy called a pyramid. The final section summarizes the key points and contains conclusions.

2. Preliminaries

Let E denote the set of numbers used to index a row or column position on a binary image. We asume that the addition and subtraction operations are defined on E. The binary image itself can then be thought of as a subset of $E \times E$. Pixels are in this subset if and only if they have the binary value one on the image. This correspondence permits us to work with sets rather than with image functions, indeed, with sets in E^N. The first two operations of mathematical morphology are the dual operations of dilation and erosion. The *dilation* of a set $A \subseteq E^N$ with a set $B \subseteq E^N$ is defined by

$$A \oplus B = \{x \mid \text{for some } a \in A \text{ and } b \in B,\ x = a + b\}.$$

The *erosion* of A by B is defined by

$$A \ominus B = \{x \mid \text{for every } b \in B,\ x + b \in A\}.$$

The careful reader should beware that the symbol $\ominus$ used by Serra (1982) does not designate erosion. Rather, it designations Minkowski subtraction.

For any set $A \subseteq E^N$ and $x \in E^N$, let A_x denote the *translation* of A by x;

$$A_x = \{y \mid \text{for some } a \in A,\ y = a + x\}.$$

For any set $A \subseteq E^N$, let $\check{A}$ denote the *reflection* of A about the origin;

$$\check{A} = \{x \mid \text{ for some } a \in A,\ x = -a\}.$$

Relationships satisfied by dilation and erosion include the following:

$$A \oplus B = B \oplus A$$

$$(A \oplus B) \oplus C = A \oplus (B \oplus C) \qquad (A \ominus B) \ominus C = A \ominus (B \oplus C)$$

$$(A \cup B) \oplus C = (A \oplus C) \cup (B \oplus C) \qquad (A \cap B) \ominus C = (A \ominus C) \cap (B \ominus C)$$

$$A \oplus B = \bigcup_{b \in B} A_b \qquad A \ominus B = \bigcap_{b \in B} A_{-b}$$

$$A \subseteq B \Rightarrow A \oplus C \subseteq B \oplus C \qquad A \subseteq B \Rightarrow A \ominus C \subseteq B \ominus C$$

$$(A \cap B) \oplus C \subseteq (A \oplus C) \cap (B \oplus C) \qquad (A \cup B) \ominus C \supseteq (A \ominus C) \cup (B \ominus C)$$

$$(A \oplus B)^C = A^C \ominus \check{B} \qquad A \ominus (B \cup C) = (A \ominus B) \cap (A \ominus C)$$

In practice, dilations and erosions are usually employed in pairs, either dilation of an image followed by the erosion of the dilated result, or image erosion followed by dilation. In either case, the result of iteratively applied dilations and erosions is an elimination of specific image detail smaller than the structuring element without the global geometric distortion of unsuppressed features. For example, opening an image with a disk structuring element smooths the contour, breaks narrow isthmuses, and eliminates small islands and sharp peaks or capes. Closing an image with a disk structuring element smooths the contours, fuses narrow breaks and long thin gulfs, eliminates small holes, and fills gaps on the contours.

Of particular significance is the fact that image transformations employing iteratively applied dilations and erosions are idempotent, that is, their reapplication effects no further changes to the previously transformed result. The practical importance of idempotent transformations is that they comprise complete and closed stages of image analysis algorithms because shapes can be naturally described in terms of under what structuring elements they can be opened or can be closed and yet remain the same. Their functionality corresponds closely to the specification of a signal by its bandwidth. Morphologically filtering an image by an opening or closing operation corresponds to the ideal non-realizable bandpass filters of conventional linear filtering. Once an image is ideal bandpassed filtered, further ideal bandpass filtering does not alter the result.

These properties motivate the importance of opening and closing, concepts first studied by Matheron (1967, 1975) who was interested in axiomatizing the concept of size. Both Matheron's (1975) definitions and Serra's (1982) definitions for opening and closing are identical to the ones given here, but their formulas appear different because they use the symbol $\ominus$ to mean Minkowski subtraction rather than erosion.

The morphological filtering operations of opening and closing are made up of dilation and erosion performed in different orders. The *opening* of A by B is defined by

$$A \circ B = (A \ominus B) \oplus B.$$

The *closing* of A by B is defined by

$$A \bullet B = (A \oplus B) \ominus B.$$

Opening and closing satisfy the following basic relationships:

$$(A \circ B) \circ B = A \circ B \qquad (A \bullet B) \bullet B = A \bullet B$$
$$A \circ B \subseteq A \qquad A \subseteq A \bullet B$$
$$A \subseteq B \Rightarrow A \circ C \subseteq B \circ C \qquad A \subseteq B \Rightarrow A \bullet C \subseteq B \bullet C$$
$$(A \circ B)^C = A^C \bullet \check{B} \qquad (A \bullet B)^C = A^C \circ \check{B}$$

The reason that openings and closings deal directly with shape properties is apparent from the following representation theorem for openings.

$$A \circ B = \{x \mid \text{for some } y,\ x \in B_y \subseteq A\}.$$

A opened by B contains only those points of A which can be covered by some translation B_y which is, in turn, entirely contained inside A. Thus x is a member of the opening if it lies in some area inside A which entirely contains a translated copy of the shape B. In this sense, A opened by B is the set of all points of A which can participate in areas of A which match B. If B is a disk of diameter d, for example, then $A \circ B$ would be that part of A which in no place is narrower than d.

The duality relationship $(A \circ B)^C = A^C \bullet \check{B}$ between opening and closing implies a corresponding representation theorem for closing

$$A \bullet B = \{x \mid x \in \check{B}_y \text{ implies } \check{B}_y \cap A \neq \emptyset\}.$$

A closed by B consists of all those points x for which x being covered by some translation $\check{B}_y$ implies that $\check{B}_y$ "hits" or intersects some part of A. A more extensive discussion of these relationships can be found in Haralick, Sternberg, and Zhuang (1987).

3. The Binary Digital Morphological Sampling Theorem

The preliminary part of this section sets the stage, discussing the appropriate morphological simplifying and filtering to be done before sampling. Certain relationships must be satisfied between the sampling set and the structuring element used for reconstruction. The main body of the section discusses two kinds of reconstructions of the sampled images: a maximal reconstruction accomplished by dilation and a minimal reconstruction accomplished by closing. Fundamental set bounding relationships are proved which show that the closing reconstruction of a set must be contained in the set itself which, in turn, must be contained in its dilation reconstruction. The closing reconstruction differs from the dilation reconstruction by just a dilation by the reconstruction structuring element, so the set bound relationships translate to geometric distance relationships. The section concludes by defining a suitable set distance function which measures the distance between the sampled set and the morphologically filtered set. The distance between the minimal reconstruction and the maximal reconstruction, and the distance between the morphologically filtered set and either of its reconstructions, are all less than the sampling distance.

The first conceptual issue which arises in developing a morphological sampling theorem is how to remove small objects, object protrusions, object intrusions and holes before sampling. It is exactly the presence of this kind of small detail before sampling which causes the sampled result to be unrepresentative of the original, just as in signal processing, the presence of frequencies higher than the Nyquist frequency causes the sampled signal to be unrepresentative of the original signal. This "aliasing" means that signals must be low pass filtered before sampling. Likewise in morphology, the sets must be morphologically filtered and simplified before sampling. Small objects and object protrusions can be eliminated by a suitable opening operation. Small object intrusions and holes can be eliminated by a suitable closing. Since opening and closing are duals, we develop our motivation by just considering the opening operation.

Opening a set F by a structuring element K in order to eliminate small details of F raises, in turn, the issue of how K should relate to the sampling set S. If the sample points of S are too finely spaced, little will be accomplished by the reduction in resolution. On the other hand, if S is too coarse relative to K, objects preserved in the opening may be missed by the sampling. S and K can be coordinated by demanding that there be a way to reconstruct the opened image from the sampled opened image. Of course, details smaller than K, are removed by the opening and cannot be reconstructed.

One natural way to reconstruct a sampled opening is by dilation. If S and K were coordinated to make the reconstructed image (first opened, then sampled, and then dilated) the same as the opened image, we would have a morphological sampling theorem nearly identical to the standard sampling theorem of signal processing. However, morphology cannot provide a perfect reconstruction, as is illustrated by the following one-dimensional continuous domain example.

Let the image F be the union of three topologically open intervals

$$F = (3.1,\ 7.4) \cup (11.5,\ 11.6) \cup (18.9,\ 19.8),$$

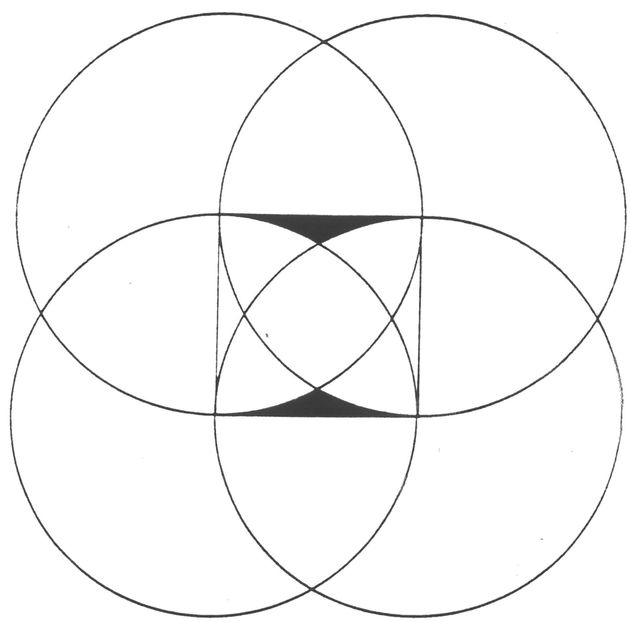


Figure 1 *illustrates how two points can be chosen no further apart than the sample distance yet there is no sample point which is simultaneously less distant than the sample distance to each of them. Take the two points to be opposite each other each interior to one of the shaded regions. Consider the sample points to be the corners of the square.*

where (x, y) denotes the topologically open interval between x and y. We can remove all details of less than length 2 by opening with the structuring element $K = (-1,\ 1)$ consisting of the toplogically open interval from -1 to 1. Then the opened image $F \circ K = (3.1,\ 7.4)$. What should the corresponding sample set be? Consider a sampling set $S = \{x \mid x \text{ an integer}\}$, with a sample spacing of unity; other spacings such as .2, .5 or .7 could illustrate the same sampling concept as well. The sampled opened image $(F \circ K) \cap S = \{4, 5, 6, 7\}$. Dilating by K to reconstruct the image produces $[(F \circ K) \cap S] \oplus K = (3, 8)$, an interval which properly contains $F \circ K$. The dilation fills in between the sample points, but cannot "know" to expand on the left end by a length of .9 and yet expand by .4 on the right end. However, the reconstruction is the largest one for which the sampled reconstruction $\{[(F \circ K) \cap S] \oplus K\} \cap S$ produces the sampled opening $(F \circ K) \cap S = \{4, 5, 6, 7\}$. This is easily seen in the example because substituting the closed interval $[3, 8]$ for the open interval $(3,\ 8)$ produces the sampled closed interval $[3, 8] \cap S = \{3, 4, 5, 6, 7, 8, \}$ which properly contains $(F \circ K) \cap S = \{4, 5, 6, 7\}$.

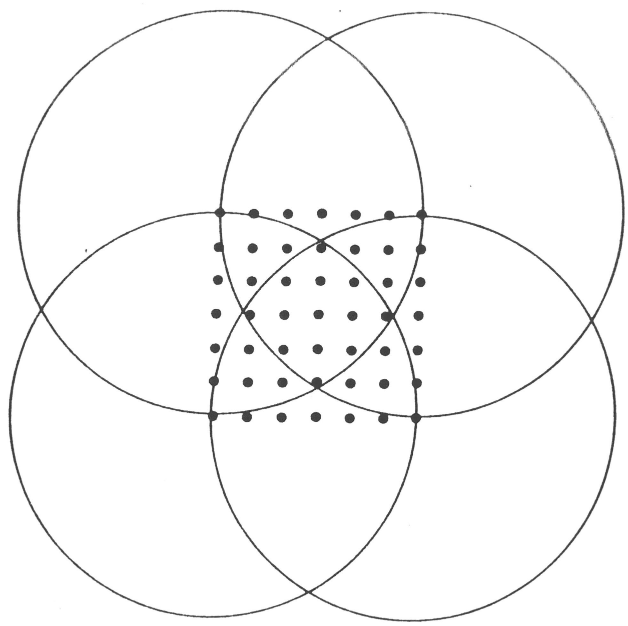


Figure 2 *illustrates how the condition $x \in K_y \Rightarrow K_x \cap K_y \cap S \neq \emptyset$ can be satisfied in the digital case where K is a circular disc. Notice that for any pair of digital points in the regions corresponding to the shaded region of Figure 1, the distance between them is not less than the radius of the open disc K. Hence the difficulty illustrated in Figure 1 cannot arise.*

In the remainder of this section we give a complete derivation of the results illustrated in the example. First, note that to use a structuring element K as a "reconstruction kernel," K must be large enough to ensure that the dilation of the sampling set S by K covers the entire space E^N. For technical reasons apparent in the derivations, we also require that K be symmetric, $K = \check{K}$. In the standard sampling theorem, the period of the highest frequency present must be sampled at least twice in order to properly reconstruct the signal from its sampled form. In mathematical morphology, there is an analogous requirement. The sample spacing must be small enough that the diameter of K is just smaller than these two sample intervals. Hence, the diameter of K is large enough that it can contain two sample points but not three sample points. We express this relationship by requiring that

$$x \in K_y \Rightarrow K_x \cap K_y \cap S \neq \emptyset$$
$$\text{and } K \cap S = \{0\}.$$

The first condition implies that the dilation of sample points fills the whole space; that is, $S \oplus K = E^N$ when K is not empty. If the points in the sampling set S are spaced no further than d apart, then the corresponding reconstructing kernel K could be the topologically open ball of radius $2d$. In this case, $x \in K_y \Rightarrow K_x \cap K_y \cap S \neq \emptyset$. Notice that two points which are d apart can lie on the diameter of d. But since the ball is topologically open, the diameter cannot contain 3 points spaced d apart. Hence, the radius of K is just smaller than the sampling inteval. Also notice that if a sample point falls in the center of K, K will not contain another sample point.

Why does the morphological sampling theorem we develop here pertain mainly to the digital domain. Consider the two-dimensional continuous case in which there is a regular square grid sampling, with the sample interval in each direction being of length L. To guarantee that $K \cap S = \{0\}$, the biggest possible disc K is the open disc having radius L.

The difficulty occurs with the condition $x \in K_y \Rightarrow K_x \cap K_y \cap S \neq \emptyset$. Figure 1 shows a square whose length L side is the sampling interval. It also shows several translates of K, and a disc of the radius L. Select two points which are no further from each other than distance L in the following way. Take one point x to be in the interior of one shaded region of Figure 1. Take the other point y to be opposite it interior to the other shaded region. With this selection the distance between the two points is guaranteed to be less than L. Yet it is apparent from the geometry that since none of the four open discs of diameter L can contain the two points which are distance L apart, the condition $x \in K_y \Rightarrow K_x \cap K_y \cap S \neq \emptyset$ cannot be satisfied. This is because each open disc represents exactly the set of points each having the property that, if an open disc were centered at the point, the open disc would contain a sample point. Hence, with the disc K being defined by the L_2 norm, there can be no morphological sampling theorem in the continuous case. In fact, the only norm by which K can be defined which yields a morphological sampling theorem in the continuous case is the L_∞ norm.

Because the shaded region in Figure 2 is so narrow, this difficulty does not arise in the digital case. Suppose that the original domain is discrete with nearest points at distance one from each other. Then the condition $x \in K_y \Rightarrow K_x \cap K_y \cap S \neq \emptyset$ is easily satisfied for any $L \in \{2,3,4,5,6,7\}$ since in this case $L(1 - \frac{\sqrt{3}}{2}) < 1$. Figure 2 illustrates the case where the sample interval L is six. Notice that the distance between any pair of digital points, one from a region corresponding to one of the shaded regions of Figure 1 and the other from the region opposite it, must be greater than L. Hence, for any two such points x and y, it is not the case that $x \in K_y$. So the difficulty with $x \in K_y$ and $K_x \cap K_y \cap S = \emptyset$ cannot arise.

We now prove some propositions which lead to the binary morphological sampling theorem. In what follows, the set $F \subseteq E^N$, the reconstruction structuring element will be denoted by $K \subseteq E^N$, and the sampling set will be denoted by $S \subseteq E^N$. Although not necessary for every proposition, we assume that S and K obey the following five conditions:

$$
\begin{aligned}
&(1) S = S \oplus S,\\
&(2) S = \check{S},\\
&(3) K \cap S = \{0\},\\
&(4) K = \check{K},\\
&(5) a \in K_b \Rightarrow K_a \cap K_b \cap S \neq \emptyset.
\end{aligned}
$$

Figure 3 illustrates the S associated with a 3 to 1 downsampling. Figure 4 illustrates a structuring element K satisfying (3), (4) and (5). Since the dilation operation is commutative and associative, conditions (1) through (3) imply that the sampling set S with the dilation operation comprises an abelian group with the origin being its unit element. Thus, if $x \in S$, then $S_x = S$, and also since $K \cap S = \{0\}$, $x \in S$ implies $K_x \cap S = \{x\}$. Both these facts are utilized in a number of the proofs to follow.

3.1 The Set Bounding Relationships

It is obvious that since $0 \in K$, the reconstruction of a sampled set $F \cap S$ by dilation with K produces a superset of the sampled set $F \cap S$. That is, $F \cap S \subseteq (F \cap S) \oplus K$. The reconstruction by dilation is open so that $[(F \cap S) \oplus K] \circ K = (F \cap S) \oplus K$. Moreover, as stated in the next proposition, the erosion and dilation of the original image F by K bound the reconstructed sampled image.

Proposition 1

Let $F,\ K,\ S \subseteq E^N$. Suppose $S \oplus K = E^N$ and $K = \check{K}$. Then $F \ominus K \subseteq (F \cap S) \oplus K \subseteq F \oplus K$.

Proposition 1 shows that the reconstruction by dilation cannot be too far away from F since the reconstruction is constrained to lie between F eroded by K and F dilated by K. Proposition 2 strengthens the closeness between F and the dilation reconstruction $(F \cap S) \oplus K$. Sampling F and sampling the dilation reconstruction of F produce identical results.

Proposition 2

$F \cap S = [(F \cap S) \oplus K] \cap S$

Proof

$$
\begin{aligned}
[(F \cap S) \oplus K] \cap S &= [\bigcup_{x \in F \cap S} K_x] \cap S \\
&= \bigcup_{x \in F \cap S} K_x \cap S \\
&= \bigcup_{x \in F \cap S} K_x \cap S_x \\
&= \bigcup_{x \in F \cap S} (K \cap S)_x \\
&= \bigcup_{x \in F \cap S} \{0\}_x \\
&= F \cap S
\end{aligned}
$$

■

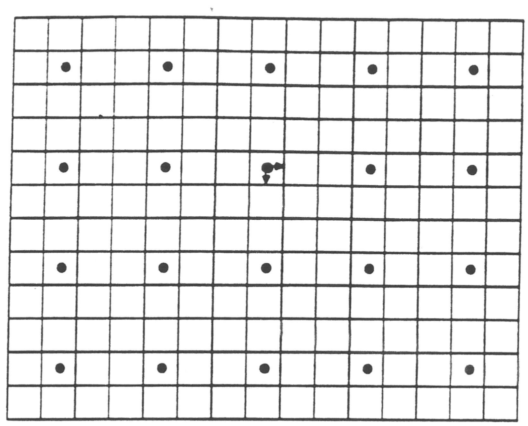


Figure 3 *illustrates sampling every third pixel by row and by column. The sampling set S is represented by all points which are shown as "•".*

From this result it rapidly follows that sampling followed by a dilation reconstruction is an idempotent operation. That is, $\left([(F \cap S) \oplus K] \cap S\right) \oplus K = (F \cap S) \oplus K$.

Considering sampling followed by reconstruction as an operation we discover that it is an increasing operation, distributes over union but not over intersection. That is:

$$(1) F_1 \subseteq F_2 \text{ implies } (F_1 \cap S) \oplus K \subseteq (F_2 \cap S) \oplus K$$
$$(2) \left((F_1 \cup F_2) \cap S\right) \oplus K = \left[(F_1 \cap S) \oplus K\right] \cup \left[(F_2 \cap S) \oplus K\right]$$
$$(3) \left((F_1 \cap F_2) \cap S\right) \oplus K \subseteq \left[(F_1 \cap S) \oplus K\right] \cap \left[(F_2 \cap S) \oplus K\right]$$

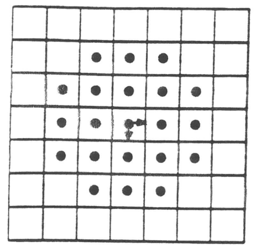


Figure 4 *illustrates a symmetric structuring element K which is a digital disc of radius $\sqrt{5}$. For the sampling set S of Figure 3, $K \cap S = \{0\}$ and $x \in K_y$ implies $K_x \cap K_y \cap S \neq \emptyset$.*

Proposition 3 states that the dilation reconstruction of a sampled F is always a superset of F opened by the reconstruction structuring element K. Hence, if F is open under K, then F is contained in its dilation reconstruction.

Proposition 3

$$F \circ K \subseteq (F \cap S) \oplus K$$

Corollary:

$$F \circ K \subseteq \left[(F \circ K) \cap S\right] \oplus K.$$

Thus the reconstruction of the opened sampled image F is bounded by $F \circ K$ on the low side and $F \circ K$ dilated by K on the high side.

$$F \circ K \subseteq \left[(F \circ K) \cap S\right] \oplus K \subseteq (F \circ K) \oplus K$$

If F is morphologically simplified and filtered so that $F = F \circ K$, then the previous bounds reduce to

$$F \subseteq (F \cap S) \oplus K \subseteq F \oplus K$$

By reconsidering our example $F = (3.1,\ 7.4) \cup (11.5,\ 11.6) \cup (18.9,\ 19.8)$ which is not open under $K = (-1, 1)$, we can see that such an F is not necessarily a lower bound for the reconstruction. In this case $F \cap S = \{4, 5, 6, 7, 19\}$ and the reconstruction $(F \cap S) \oplus K = (3, 8) \cup (18, 20)$, which does not contain F. This suggests that the condition that F be open under K is essential in order to have $F \subseteq (F \cap S) \oplus K$.

We now show one last relation between the reconstruction $(F \cap S) \oplus K$ and F. The reconstruction $(F \cap S) \oplus K$ is the largest open set which when sampled produces $F \cap S$.

Proposition 4

Let $A \subseteq E^N$ satisfy $A \cap S = F \cap S$ and $A = A \circ K$. Then $A \supseteq (F \cap S) \oplus K$ implies $A = (F \cap S) \oplus K$.

Thus, we have established the maximality of the reconstruction $(F \cap S) \oplus K$ with respect to the two properties of being open and downsampling to $F \cap S$. What about a minimal reconstruction? Certainly we would expect a minimal reconstruction to be contained in the maximal reconstruction and contain the sampled image. Since closing is extensive, we immediately have $F \cap S \subseteq (F \cap S) \bullet K$. Since $0 \in K$, erosion is an anti-extensive operation. Hence, $(F \cap S) \bullet K = [(F \cap S) \oplus K] \ominus K \subseteq (F \cap S) \oplus K$. These relations suggest the possibility of a reconstruction by closing. The next proposition shows that a closing reconstruction has set bounds similar to the dilation reconstruction.

Proposition 5

Let $F, K, S \subseteq E^N$. If $K = \check{K}$ and $x \in K_y$ implies $K_x \cap K_y \cap S \neq \emptyset$ and $0 \in K$ then $F \ominus K \subseteq (F \cap S) \bullet K \subseteq (F \cap S) \oplus K \subseteq F \oplus K$.

For true reconstruction, the sampled reconstruction should be identical to the sampled image. Indeed, this is the case.

Proposition 6

$[(F \cap S) \bullet K] \cap S = F \cap S$.

Consider our example $F = (3.1,\ 7.4) \cup (11.5,\ 11.6) \cup (18.9,\ 19.8)$, which is closed under $K = (-1,\ 1)$. If the sampling set S is the integers then $F \cap S = \{4, 5, 6, 7, 19\}$. Closing $F \cap S$ with K can be visualized via the opening/closing duality $(F \cap S) \bullet K = \left((F \cap S)^C \circ \check{K}\right)^C$. Opening the set $(F \cap S)^C$ with $\check{K} = K$ produces $(F \cap S)^C \circ K = \{\, x \neq 19 \mid x < 4 \text{ or } > 7\}$. Hence $(F \cap S) \bullet K = \left((F \cap S)^C \circ K\right)^C = \{x \mid x = 19 \text{ or } 4 \leq x \leq 7\}$, and sampling produces $\left[(F \cap S) \bullet K\right] \cap S = \{4, 5, 6, 7, 19\} = F \cap S$.

From the previous proposition it rapidly follows that sampling followed by a reconstruction by closing is an idempotent operation. That is, $[(F \cap S) \bullet K] \cap S \bullet K = (F \cap S) \bullet K$

A reconstruction by closing is obviously closed under K. Moreover, it can be quickly determined that

$$F_1 \subseteq F_2 \text{implies } (F_1 \cap S) \bullet K \subseteq (F_2 \cap S) \bullet K$$

$$[(F_1 \cup F_2) \cap S] \bullet K \supseteq [(F_1 \cap S) \bullet K] \cup [(F_2 \cap S) \bullet K]$$

$$[(F_1 \cap F_2) \cap S] \bullet K \subseteq [(F_1 \cap S) \bullet K] \cap [(F_2 \cap S) \bullet K]$$

Furthermore, the closing reconstruction of a sampled F is always a subset of F closed by the reconstruction structuring element K. That is, $(F \cap S) \bullet K \subseteq F \bullet K$, so that $((F \bullet K) \cap S) \bullet K \subseteq F \bullet K$. Hence a closing reconstruction of an image which is closed before sampling will be a subset of the closed image.

By considering a simple example $F = \{0, 1\}$, which is not closed under $K = (-1, 1)$, we can see that F is not necessarily an upper bound for the reconstruction. In this case, $F \cap S = \{0, 1\} = F$ and the reconstruction $(F \cap S) \bullet K = F \bullet K = [0, 1]$ which properly contains F. This suggests that the condition that F be closed under K is essential in order to have $(F \cap S) \bullet K \subseteq F$.

We now show one last relation between the reconstruction $(F \cap S) \bullet K$ and F. The reconstruction $(F \cap S) \bullet K$ is the smallest closed set which when sampled produces $F \cap S$.

Proposition 7

Let $A \subseteq E^N$ satisfy $A \cap S = F \cap S$ and $A = A \bullet K$. Then $A \subseteq (F \cap S) \bullet K$ implies $A = (F \cap S) \bullet K$.

Proof

Suppose $A \subseteq (F \cap S) \bullet K$. Now $A \cap S = F \cap S$ implies $(A \cap S) \bullet K = (F \cap S) \bullet K$. Since $(A \cap S) \bullet K \subseteq A \bullet K$ and $A \bullet K = A$, we obtain $(F \cap S) \bullet K \subseteq A$. But $A \subseteq (F \cap S) \bullet K$ and $A \supseteq (F \cap S) \bullet K$ imply $A = (F \cap S) \bullet K$.

■

3.2 Examples

To better illustrate the bounding relationships developed in the previous section between a set and its sample reconstructions, we show three simple examples. The domain of these examples is defined as $E \times E$ where E is the set of integers. The sample set S is chosen as the set of even numbers in both row and column directions. Thus,

$$S = \{(r, c) | r \in E \text{ and is even}; \quad c \in E \text{ and is even}\}.$$

K is chosen as a box of size 3×3 whose center is defined as the origin. The sets S, K, and the three example sets F_1, F_2, and F_3 are shown in Figure 5. The sets F_1, F_2, and F_3 are 3×3 boxes having different origins and the condition $F = F \circ K$ holds for all these example sets.

The results of $F \ominus K, F \cap S, (F \cap S) \bullet K, (F \cap S) \oplus K$, and $F \oplus K$ for sets F_1, F_2 and F_3 are shown in Figures 6,7, and 8 respectively.

3.2.1 Example 1

All the pixels contained in the vertical boundaries of F_1 have even column coordinates and those in the horizontal boundaries of F_1 have even row coordinates. Since the sample set S consists of pairs of even numbers and F_1 is a 3×3 box, the set $F_1 \cap S$ consists of the four corner points of F_1 and is contained in the boundary set of F_1. Hence the closing reconstruction of $F_1 \cap S$ recovers F_1 and the dilation reconstruction of $F_1 \cap S$ is equivalent to $F \oplus K$. In fact, the following two equalities hold only when (1) the sampling is every other row and column, (2) a set's vertical boundaries have even column coordinates, and (3) its horizontal boundaries have even row coordinates

$$(F \cap S) \bullet K = F \text{ and}$$
$$(F \cap S) \oplus K = F \oplus K.$$

The bounding relationships for F_1, illustrated in Figure 6, are

$$F_1 \ominus K \subseteq (F_1 \cap S) \bullet K = F_1 \subseteq (F_1 \cap S) \oplus K = F_1 \oplus K.$$

3.2.2 Example 2

Since all pixels contained in the vertical boundaries of F_2 have odd column coordinates and those in the horizontal boundaries of F_2 have odd row coordinates and F_2 is a small 3×3 box, the set $F_2 \cap S$ does not contain any part of the boundary of F_2. Thus the closing reconstruction of $F_2 \cap S$ equals $F_2 \ominus K$ and the dilation reconstruction of $F_2 \cap S$ is equivalent to F_2. Similar to the example 1, the following equalities hold only when the sampling is every other row and column and has its odd column coordinates in its vertical boundaries and its odd row coordinates in its horizontal boundaries.

$$F \ominus K = (F \cap S) \bullet K \text{ and}$$
$$F = (F \cap S) \oplus K.$$

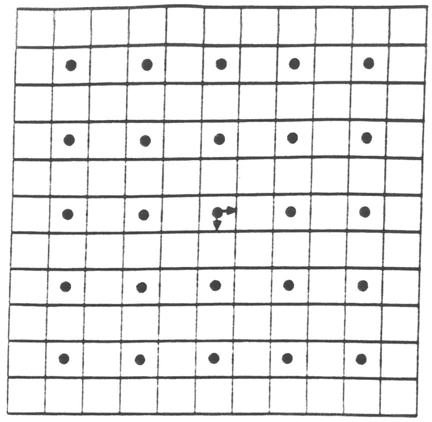
The bounding relationships for F_2, illustrated in Figure 7, are

$$F_2 \ominus K = (F_2 \cap S) \bullet K \subseteq F_2 = (F_2 \cap S) \oplus K \subseteq F_2 \oplus K.$$

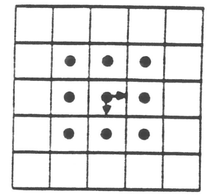
3.2.3 Example 3

The pixels contained in the vertical boundaries of F_3 have odd column coordinates and the pixels in the horizontal boundaries of F_3 have even row coordinates. Hence, no equalities should exist in the bounding relationship. This is illustrated in Figure 8. The bounding relationships for F_3 are

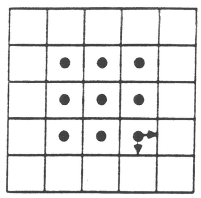
$$F_3 \ominus K \subseteq (F_3 \cap S) \bullet K \subseteq F_3 \subseteq (F_3 \cap S) \oplus K \subseteq F_3 \oplus K.$$



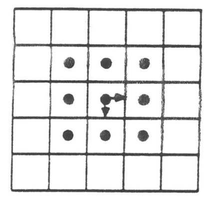
The sampling set S.



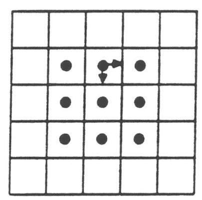
The reconstruction structuring element K



F_1



F_2



F_3.

Figure 5 *illustrates a sampling set* S*, a reconstruction structuring element* K*, and three sets,* $F_1, F_2,$ *and* F_3*, each of which is open under* K.

To show why the opening condition $F = F \circ K$ is needed for the bounding relationships involving F, we show an example set F_4 which deviates from the set F_3 by adding six extra points to it (see Figure 9). The sample and reconstruction results of $F_4, F_4 \cap S, (F_4 \cap S) \bullet K,$ and $(F_4 \cap S) \oplus K$ are exactly the same as the results for F_3. However, no bounding relationships between F_4 and its sample reconstructions are applicable. If we open F_4 by K, the bounding relationships exist because $F_4 \circ K = F_3$.

4. Conclusion

We have shown that before a binary image can be sampled, it must be morphologically simplified by an opening or a closing with the reconstruction structuring element. A sampled image has a minimal and maximal reconstruction. The minimal reconstruction is generated by closing the sampled image with the reconstruction structuring element and is a valid reconstruction when the morphological simplification done before sampling is a closing. The maximal reconstruction is generated by dilating the sampled image with the reconstructed structuring element and is a valid reconstruction when the morphological simplification done before sampling is an opening. The spatial meaning of the minimal and maximal reconstruction is direct. The minimal and maximal reconstruction delineate the spatial bounds within which the image event on the unsampled morphologically simplified image actually occurs. That is, the uncertainty due to sampling is precisely specified by the bounds given by the minimal and maximal reconstruction.

The extension of all that has occurred for the binary image morphological sampling theorem has been done for grayscale images, but due to lack of space, they were not reported on here.

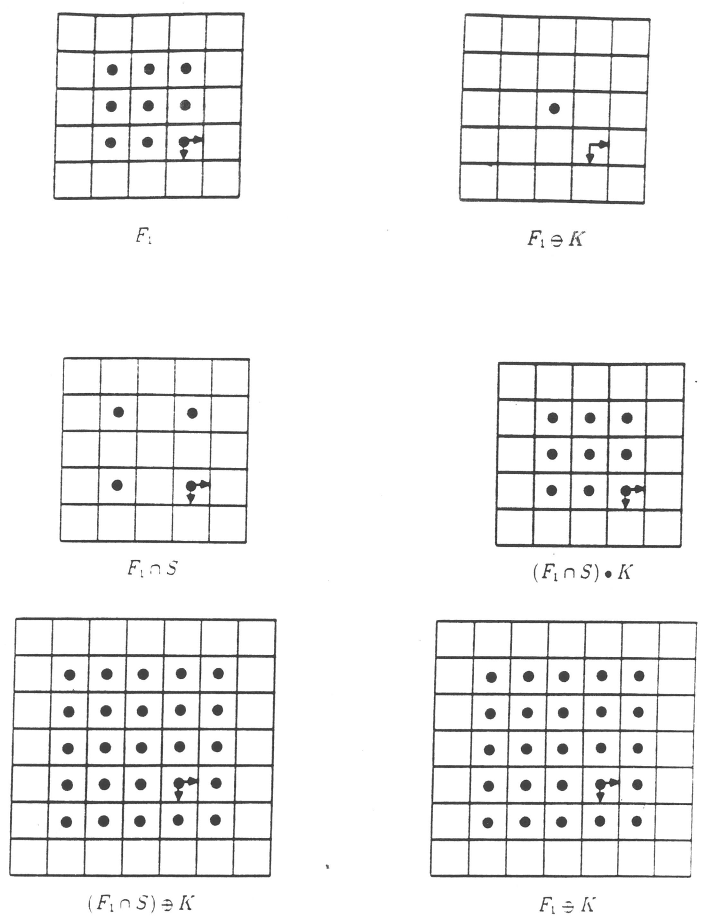


Figure 6 *shows how the erosion and dilation of* F_1 *bound the the minimal reconstruction* $(F_1 \cap S) \bullet K$ *and the maximal reconstruction* $(F_1 \cap S) \oplus K$*, respectively, which in turn bound* F_1. *because* F_1 *is both open and closed under* K.

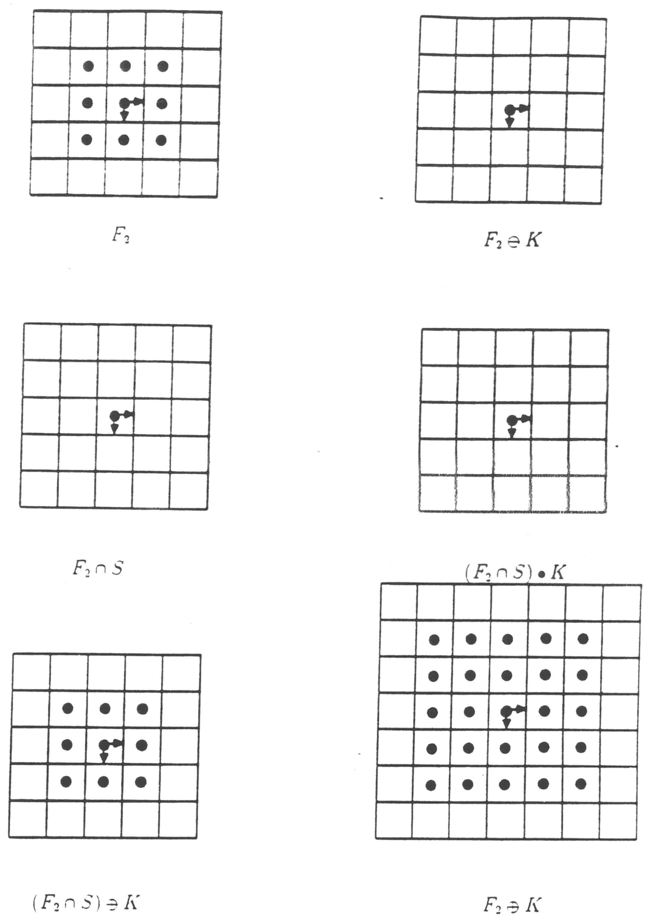


Figure 7 *shows a second example of how the erosion and dilation of* F_2 *bound the minimal reconstruction* $(F_2 \cap S) \bullet K$ *and the maximal reconstruction* $(F_2 \cap S) \oplus K$*, respectively, which in turn bound* F_2.

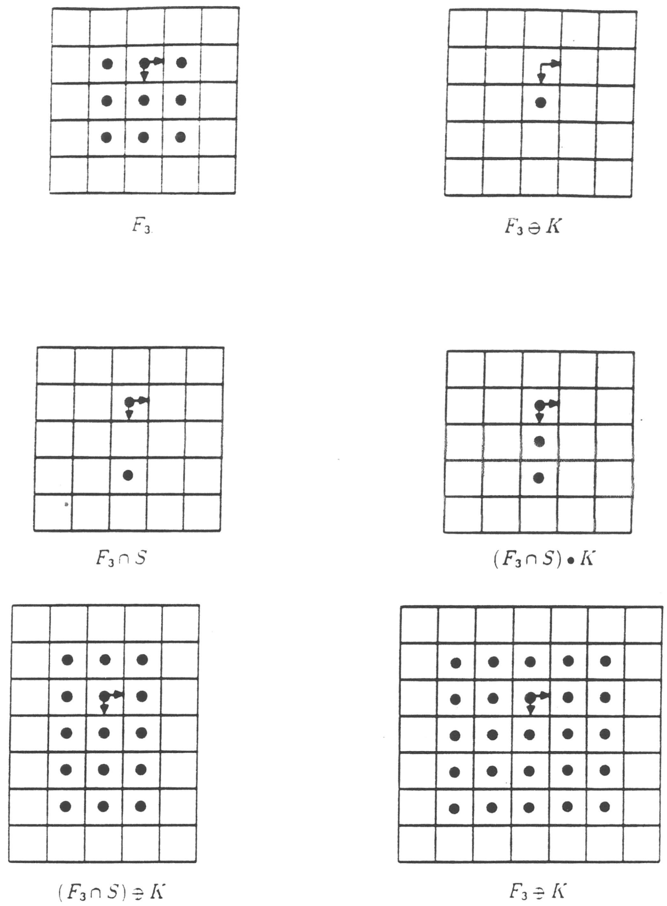


Figure 8 *shows a third example of how erosion and dilation of* F_2 *bound (in this case properly) the minimal reconstruction* $(F_3 \cap S) \bullet K$ *and the maximal reconstruction* $(F_3 \cap S) \oplus K$*, respectively, which in turn bound (in this case properly)* F_3.

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