Application of Mathematical Morphology to Machine Vision

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Abstract

This paper gives an analysis of a variety of morphological vision procedures. The analysis is designed to illustrate the power and flexibility of mathematical morphology for the extraction of shape information from gray tone and binary images.

1 Introduction

Usually, when a vision expert is given a task, he or she first analyzes the task, makes a plan, produces a procedure, tries it, evaluates the test results, and refines or updates the procedure. In doing so, the expert uses the knowledge of the given problem domain and his or her knowledge of the vision algorithms and is able to reason through using the knowledge to determine a reasonable vision procedure.

It is natural for us to ask how a human vision expert can solve the vision tasks and to investigate the mechanism involved in the process of developing such solutions. In particular, the problem we hope to solve is to reduce any machine vision task to a sequence of given morphological operations. We must use them at the right moment, and with the right parameters to accomplish what the vision procedure is designed for. Research on this topic can be found in [3, 7, 12, 14, 16, 20].

Considering that a vision procedure can be decomposed into a sequence of primitive operators, the following constitutes a first set of questions that can be raised toward the solution of the automatic compilation of vision procedures: What is a set of basic primitive operators that do not require finer grained decomposition relative to the task at hand and what does each primitive operator do when applied to the image ? How do primitive operators aggregate together to encode knowledge which we express using the terms of coarse grain system ? Are there any properties of an image that suggest to a human expert which operator to use on the image to solve a particular vision problem ? These are questions regarding the knowledge about the primitive operators and the discrimination information contained in a distorted noisy image they work on. Even though there exists a number of known image descriptions available, an image in general contains a huge amount of information. However, it appears that there exists a set of image descriptions relevant to the selection of each primitive operator. An application of an operator results in an output image with a different set of descriptions that we can possibly predict. These operators can be described in terms of the relationship between the class of images on which the operator works and the parameters of the operator. We need to discover and explicitly describe this knowledge in order to utilize it in the automatic compilation of machine vision procedure.

The problem of the automatic compilation of vision procedures involves more subproblems to be solved than the ones listed above. For example, here we have not asked questions about the reasoning mechanism, the evaluation method, and the representational method. In this paper, as part of the solution to the automatic compilation problem, we only discuss the analysis performed on a set of morphological vision procedures developed by vision experts.

In section 2 we review the basic definitions for both binary and grayscale morphological operators. Haralick *et al* [6] has a more complete discussion. A set of example morphological algorithms developed by vision experts is described in section 3. A summary and some comments relative to the solution of automatic compilation of vision procedures are given in section 4.

2 Definitions

Let E denote the set of integers. A pixel in an image is a tuple $(r, c) \in E \times E$ where r and c represents the row and column coordinates of the pixel in an image. A binary image can be thought of as subset of $E \times E$ whose elements are the pixels of binary value one. The following are the definitions for the binary morphological operators used in this paper. For any set $A \subseteq E^N$ and $x \in E^N$, let A_x denote the translation of A by x which is defined by

$$A_x = \{y \mid \text{for some } a \in A, y = a + x\}$$

For any set $A \in E^N$, let $\check{A}$ denote the reflection of A about the origin which is defined by

$$\check{A} = \{x \mid \text{ for some } a \in A, x = -a\}$$

The dilation of a set $A \subseteq E^N$ by a set $B \subseteq E^N$ is defined by

$$\begin{aligned} A \oplus B &= \{x \mid \text{for some } a \in A \text{ and } b \in B, x = a + b\} \\ &= \bigcup_{b \in B} A_b \end{aligned}$$

The erosion of A by B is defined by

$$\begin{aligned} A \ominus B &= \{x \mid \text{for every } b \in B, x + b \in A\} \\ &= \bigcap_{b \in B} A_{-b} \end{aligned}$$

The opening of A by B is defined by

$$
\begin{aligned}
A \circ B &= (A \ominus B) \oplus B \\
&= \{x \mid \text{for some } y, x \in B_y \subseteq A\}
\end{aligned}
$$

The closing of A by B is defined by

$$
\begin{aligned}
A \bullet B &= (A \oplus B) \ominus B \\
&= \{x | x \in \check{B}_y \text{ implies } \check{B}_y \cap A \neq \phi\}
\end{aligned}
$$

The hit–or–miss transformation of a set A by a structuring element $B = \{B^0, B^1\}$ is defined as:

$$A \circledast B = (A \ominus B^0) - (A \oplus \check{B}^1)$$

where the minus sign means a set subtraction. A point x belongs to $A \circledast B$ if and only if $B_x^0 \subset A$ and $B_x^1 \subset A^c$.

Let $M = \{1, \cdots, m\}$ and $N = \{1, \cdots, n\}$ be the sets of indices. Let $I^k, k \in M$ and $J^k, k \in N$ be the connected components in binary images I and J respectively. the conditional dilation of the image I with respect to the image J by a structuring element B is defined as:

$$I \oplus |_J B = \bigcup_i \{J^i | (J^i \oplus B) \cap I^k \neq \phi, \text{ for some } k \in M\}$$

This operation extracts only the connected components of J, when dilated by B intersect with I. The conditional dilation can be implemented as the following iterative procedure.

$$
\begin{aligned}
I_0 &= I \\
I_n &= (I_{n-1} \oplus B) \cap J
\end{aligned}
$$

and

$$I \oplus |_J B = I_n$$

where n satisfies $I_n = I_m$, for all $m > n$.

The morphological operators can be defined for a real valued function $f : E^N \to R$, where R is a set of real numbers. For any function $f : F \to R$, define the reflection of f by $\check{f} : \check{F} \to R$ where

$$\check{f}(x) = f(-x)$$

Now, before we define the grayscale morphological operators, let us define the umbra and top. A set $A \subseteq E^N \times R$ is an umbra if and only if $(x, y) \in A$ implies that $(x, z) \in A$ for every $z \leq y$. The top of A is a function $T[A]$ mapping the spatial domain of $A, \{x \in E^N | \text{for some } y \in E, (x, y) \in A\}$, to the top surface of A,

$$T[A](x) = \max\{y \in R \mid (x, y) \in A\}$$

If $F \subseteq E^N$ and $f : F \to R$, the umbra of f is the set

$$U[f] = \{(x, y) \in (F \times R) \mid y \leq f(x)\}$$

Now, the definitions for the morphological operators follow: For $F, K \subseteq E^N, f : F \to R$, and $k : K \to R$, the dilation of f by k is the mapping $(f \oplus k) : (F \oplus K) \to R$ defined by

$$\begin{aligned}(f \oplus k)(x) &= T[U[f] \oplus U[k]](x) \\ &= \max_{z \in K, (x-z) \in F} \{f(x - z) + k(z)\}\end{aligned}$$

The erosion of f by k is the mapping $(f \ominus k) : (F \ominus K) \to R$ defined by

$$\begin{aligned}(f \ominus k)(x) &= T[U[f] \ominus U[k]](x) \\ &= \min_{z \in K} \{f(x + z) - k(z)\}\end{aligned}$$

The opening $(f \circ k)$ and closing $(f \bullet k)$ for functions are similarly defined as:

$$\begin{aligned}f \circ k &= (f \ominus k) \oplus k \\ f \bullet k &= (f \oplus k) \ominus k\end{aligned}$$

We use the following special type of binary structuring elements in our discussion in this chapter.

- (c, r): A point at column c and row r.
- box(w, h): A box of width w and height h.
- diamond(r): A diamond whose diagonal is r pixels long.
- disk(r): A disk of radius r such that disk$(r) = \{x \mid \|x\| \leq r\}$.
- line$((c_1, r_1), (c_2, r_2))$: A straight line segment connecting two points (c_1, r_1) and (c_2, r_2).
- ring(r): An annulus defined by disk$(r + 1)$ - disk(r).

The following structuring function are used for the grayscale operations. The function values are all zero while the domain of the structuring functions are different.

- rod(r): The domain of function is a disk of radius r.
- brick(w, h): The domain of function is a box of width w and height h.

3 Compendium of Morphological Image Processing Algorithms

This section presents a collection of morphological image processing algorithms developed by vision experts. It is hoped that by analyzing the algorithms in this collection we can find the knowledge and the reasoning mechanism employed by vision experts in their process of developing those algorithms. Each of the following subsections provides an analysis for each algorithm. The analysis is divided into four parts - input image description, goal, morphological procedure, and reasoning and knowledge. The description of the input image that the algorithm is expecting and the goal image that the algorithm is supposed to produce are given in plain English. The morphological procedure that accomplishes the task is presented. The procedure is written in two different fashions. It is first given in step by step fashion, where each step corresponds to a single meaningful primitive morphological operation. A short description of each step is also given. In each step i, G_i or B_i represents the grayscale or the binary image produced at the step i, where G_0 or B_0 designates the original input image. Next, the procedure is expressed in as compact an equation as possible using the smallest number of intermediate images. Finally, we present the reasoning and knowledge constituting the morphologic basis for the algorithm.

3.1 Identify leads coming from a SMD component

input image: [1]

- The image consists of one very dark component mounted on a medium bright background.
- The background can have both dark and bright stripes.
- The component is rectangular shaped and located in the middle of the image. It is the largest object in the image. The leads are horizontally oriented in a vertical column and connected to the two vertical boundaries of the component. The body of the rectangle may contain medium bright or bright objects which are smaller or thinner than the leads.
- The leads are very bright and small rectangular shaped objects. The distance between a pair of leads next to each other is longer than the height of one lead. The leads themselves may contain some small dark areas.

goal:

- Detect only the bright almost rectangular shaped blobs with known size, the leads, which are located inside the very dark large rectangular shaped component in the middle of the image. The small dark areas that may exist inside the lead should also be detected as being a part of the lead. Thus, each detected lead should be a convex shape with no holes in it.

morphological procedure:

1. $B_1 = G_0 < 200$:
   Binarization to extract dark region.
2. $B_2 = B_1 \oplus \text{box}(5,5)$:
   Improve connectivity. It fills holes caused by the medium bright objects inside the component, but not the leads.

3. $B_3 = B_2 \ominus \mathrm{box}(80, 100)$:
   The component is the only blob which is larger than a box which is 80 pixels in column and 100 pixels in row. An erosion with a large box removes all blobs smaller than the size of the box.

4. $G_4 = B_3 \oplus \mathrm{box}(160, 90)$:
   Generate a mask that covers the component and its leads. The width of the box is bigger than the box used in the previous erosion.

5. $G_5 = G_0 \wedge G_4$:
   Extract the part of the image which contains the area of interest, the component with leads.

6. $G_6 = G_5 \circ \mathrm{rod}(4)$:
   Removes small noisy relatively bright areas within the dark extracted area.

7. $B_7 = G_6 > 174$:
   Binarization to extract the bright leads.

8. $B_8 = B_7 \bullet \mathrm{box}(8, 2)$:
   Improve connectivity of the detected leads.

9. $B_9 = B_8 \circ \mathrm{box}(20, 3)$:
   Removes noisy blobs smaller than the leads.

$$B_a = ((G_0 < 200) \oplus \mathrm{box}(80, 100) \ominus \mathrm{box}(80, 100) \oplus \mathrm{box}(160, 90)) \wedge G_0$$

$$B_b = ((B_a \circ \mathrm{rod}(4)) > 174) \bullet \mathrm{box}(8, 2) \circ \mathrm{box}(20, 3)$$

:easoning and knowledge:

- We need to distinguish the bright blobs in the component from the ones in the background. Using the fact that the leads are ***inside*** the dark component, try to find the dark component including the leads. Use the detected component as a mask to select only the leads (steps 1 to 5).
- To distinguish the darker component from the leads, use a thresholding for absolute darkness (step 1). We can predict that the thresholding will produce an image consisting of the component with many holes because of the leads and the smaller bright objects inside the component. It could also have dark blobs in the background.
- Remove holes inside the component by a dilation operation (step 2).
- Remove the dark objects in the background by an opening operation (step 3 and 4). These two operations should be applied in the order mentioned above.
- We need to find only the connected convex blobs (leads) with certain size and shape (steps 6 and 9).

- A grayscale opening operation can be used to remove small noisy peaks (step 6).
- To detect bright leads, use a thresholding for absolute brightness (step 7).
- Fill in the holes inside the detected leads by a closing operation (step 8). It also connects small but clustered pixels. Step 6 helped to preclude this condition.
- Select only the leads by an opening operation by a proper structuring element that can be fit inside the leads but not inside other noisy blobs (step 9).

3.2 Hot spot detection in IR images

input image: [4]

- The input image consists of bright blobs with different sizes, shapes, and brightness in a generally relatively dark background.
- The background is quite arbitrary except the fact that it is relatively dark.

goal:

- To detect relatively bright blobs with certain size and shape.
- Each bright blob contains some points that are brighter than a given brightness value.

morphological procedure:

1. $B_1 = G_0 > 112$:
   Extract bright blobs with absolute brightness value greater than 112.
2. $G_2 = G_0 \bullet \mathrm{rod}(2)$:
   Removes small relatively dark areas.
3. $G_3 = G_2 - (G_2 \circ \mathrm{rod}(8))$:
   Detects relatively bright blobs whose size is smaller than a disk of radius 8.
4. $G_4 = G_3 \circ \mathrm{rod}(1)$:
   Removes small relatively bright peaks whose support is is less than a disk of radius 1.
5. $B_5 = G_4 > 13$:
   Binarization to detect bright areas.
6. $B_6 = B_1 \oplus |_{B_5} \mathrm{disk}(1)$:
   Combine information in B_1 and B_5. Detect relatively bright blobs which contain at least one pixel whose grayscale value is greater than 112.

$$G_a = G_0 \bullet \mathrm{rod}(2)$$
$$B_b = ((G_a - G_a \circ \mathrm{rod}(8)) \circ \mathrm{rod}(1)) > 13$$
$$B_c = (G_0 > 112) \oplus |_{B_b} \mathrm{disk}(1)$$

reasoning and knowledge:

- By thresholding the image at the specified level it is guaranteed that we get at least some part of the blobs to be detected (step 1).
- Relatively bright areas with certain sizes can be detected by a thresholding of an opening residue operation (step 2 to 5).
- The detected objects that are larger than the specified size can be filtered out by an opening residue operation (step 3).
- The detected objects which are smaller than the specified size can be removed by an opening operation (step 4).
- When one operator detects some part of the objects to be detected and the other operator detects all the objects to be detected together with some more unwanted objects, conditionally dilate the result of the first operator with respect to the result of the second operator (step 6).

3.3 Inspection of watch gears for missing or broken teeth

input image: [19]

- The input image is a binary image with the watch gears as foreground pixels. The background has no other objects. There can be at most two watch gears in the image.
- Each watch gear is a disk with teeth around the boundary of the disk. Some of its teeth may be broken or missing causing different shape than expected. The inside of the disk contains four holes with a known size, shape, and arrangement.
- If there are two watch gears in the image, they can touch each other at their boundaries. They never occlude each other.

goal

- Find the location of any missing or broken teeth. The broken tooth is the one whose height is shorter than the height of the normal tooth.

morphological procedure:

1. $B_1 = B_0 \ominus \text{ring}(30, 16)$:
   Find the center of the gear holes inside the body.
2. $B_2 = B_1 \oplus \text{octagon}(35)$:
   Construct an octagonal hole mask centered at the centers of the gear holes found in step 1.
3. $B_3 = B_0 \vee B_2$:
   Fill in the holes by combining the original image with the mask found in step 2.
4. $B_4 = B_3 \circ \text{disk}(75)$:
   Select only the gear body without the teeth.

5. $B_5 = B_4 \oplus \text{disk}(3.5)$:
Extend the gear body a little bit so that it covers the broken teeth completely but not the normal teeth.

6. $B_6 = (B_5 \oplus \text{disk}(7.5)) - B_5$:
Construct a ring around the extended gear body that covers only the normal teeth.

7. $B_7 = B_0 \wedge B_6$:
Mask out only the normal teeth.

8. $B_8 = B_7 \oplus \text{disk}(6.5)$:
Connect the teeth by dilating them by a disk whose diameter is equal to the tip_to_tip gear tooth spacing.

9. $B_9 = B_6 > B_8$:
Determine the place where the teeth can not be connected by the dilation operation in step 9 by taking the difference between the connected teeth and the teeth mask constructed in step 6.

$$
\begin{aligned}
B_a &= (((B_0 \ominus \text{ring}(30,16)) \oplus \text{octagon}(35)) \vee B_0) \\
&= \quad \circ\ \text{disk}(75) \oplus \text{disk}(3.5) \\
B_b &= B_a < (B_a \oplus \text{disk}(7.5)) \\
B_c &= B_b > ((B_b \wedge B_0) \oplus \text{disk}(6.5))
\end{aligned}
$$

reasoning and knowledge

- Find the difference between the ones we want to detect and the others. Does the missing or broken teeth have unique characteristics ? If they are missing, the spacing between the teeth in the missing part of the teeth train is longer than the known regular spacing. If they are broken, the height of the broken tooth is shorter than the standard tooth.
- Isolate the normal teeth from the main gear body. Main body can be extracted by opening operation (step 1 to 4).
- If no proper structuring element can be found for the opening, fill in the holes in the body (step 1 to 3) and then open it with a proper disk (step 4).
- The normal tooth is taller than the broken tooth. To generate a mask that covers only the normal teeth, extend the gear body by the maximum height of broken tooth (step 5), and then get the annulus surrounding the extended gear body by a dilation residue operation (step 6).
- Since the spacing between the teeth where a tooth is missing is longer than normal tip–to–tip gear tooth spacing, if we dilate the teeth only image by a disk whose diameter is equal to the tip–to–tip gear tooth spacing, all teeth will be connected except at the location where a tooth is missing (step 8).

- Disconnected portion of the teeth can be detected by subtracting the result obtained in step 8 from the normal teeth mask (step 9).

3.4 Road detection in radar images

input image: [9]

- There are two images. One shows only a part of the desired roads, while the other image shows all the roads with many other noisy objects.
- The image contains dark valleys with all different width and length. Some are clustered and some are isolated. They are relatively dark within a relatively bright region of the image.
- The image consists of vertical stripes of large dark and bright regions with similar size. The dark regions contains small bright blobs.

goal

- Detect only the dark valleys with certain width and certain length within the bright region of the image. Detected valleys must be connected.

morphological procedure:

1. $G_1 = G_0 \bullet \text{rod}(1)$:
   Removes small dark noisy spots. G_0 is the image that shows only parts of all the roads to be detected with small amount of noisy valleys.

2. $B_2 = ((G_1 \bullet \text{rod}(5)) - G_1) > thr_1$:
   Detects relatively dark valleys whose width is smaller than 10 pixels.

3. $B_3 = \text{sieve\_filter}(B_2, \text{box\_frame}, \text{box}, N)$:
   Removes small isolated objects and retains only the connected longer valleys.
   The sieve_filter(I, box_frame, box, N), which sifts the image through increasingly coarse sieves, is defined as follows:

   $I^1 = I$
   for each $i = 1, \cdots, N$
   $\quad A = I^i \oplus \text{box-frame}_i$
   $\quad B = A \ominus \text{box}_i$
   $\quad I^{i+1} = I^i \wedge B$

   where the sizes of box_frame$_i$ and box$_i$ increase as i increases.

4. $B_4 = B_3 \bullet \text{disk}(20)$:
   Connects a set of isolated blobs close to each other and makes a single cluster which is larger than the valleys we want to detect. This will also connect the disconnected valleys.

5. $B_5 = (B_4 \circ \text{disk}(5) \oplus \text{disk}(8)$:
   Generates a mask that covers the larger cluster of noisy textured blobs.

6. $B_6 = B_4 - B_5$:
   Removes the noisy textured blobs from the valley image.

7. $B_7 = B_6 \oplus |_{B_0} \text{disk}(1)$:
   B_0 can be obtained using the same procedure (step 1 to 6) from the second image that shows all the valleys clearly including a lot of noisy valleys. Since B_6 contains pixels of valleys with certain amount of confidence, conditionally dilate B_6 with respect to B_0 to recover all the valleys that we want to detect.

reasoning and knowledge

- The small noisy peaks and pits can be removed by an opening and closing operations respectively (step 1). A closing operation will help remove smaller noisy valleys.
- The valleys can be detected by thresholding a closing residue operation (step 2).
- The narrow valleys can be smoothed out by a closing operation (step 1).
- The shorter valleys can be removed by the sieve filter (step 3).
- If a group of small isolated objects make a cluster, they cannot be removed easily by the sieve filter only. If the size of the cluster is bigger than the valleys and quite far apart from the valleys, they can be made to form a single larger set by a closing operation (step 4). Then, it can be removed by an opening residue operation (step 5 and 6).
- If we can mark part of the objects we want to extract, they can be extended to recover the original whole object by a conditional dilation operation (step 7).

3.5 Airplane detection

input image:

- The image contains a relatively bright airplane shaped objects with some horizontal and vertical bright stripes in a generally dark background. The object can be partially occluded by the bright stripes. The lengths of strips are longer than the airplane.

goal

- Detect only the airplane shaped object.

morphological procedure:

1. $G_1 = G_0 - (G_0 \circ \text{brick}(1, 49, 0))$:
   Removes vertical relatively bright strips that are longer than 49 pixels.

2. $G_2 = G_1 - (G_1 \circ \text{brick}(89, 1, 0))$:
   Removes horizontal relatively bright stripes that are longer than 89 pixels.

3. $G_3 = G_2 \bullet \text{brick}(5,5,0)$:
Removes small dark pits that are smaller than the size of 5 pixels by 5 pixels square. These dark areas are the noise inside the bright airplane.

4. $B_4 = G_3 > 14$:
Threshold to detect bright areas.

$$G_a = G_0 - G_0 \circ \text{brick}(1,49,0)$$
$$B_b = ((G_a - G_a \circ \text{brick}(89,1,0)) \bullet \text{brick}(5,5,1)) > 14$$

reasoning and knowledge

- The horizontal and vertical stripes can be removed by an opening residue operation by a proper structuring element.

3.6 IR ship recognition

input image:

- It contains a single relatively highly contrasted dark ship in the near middle of the image. The boundary of the ship is quite blurred. The ship shows up as a long horizontal shape with some vertical structures at the top of it. The bottom of the ship is almost straight horizontal parallel to the water level.

goal

- Detect significant features of the ship. This means finding the horizontal edges comming from the top boundary of the ship.

morphological procedure:

1. $G_1 = G_0 \bullet \text{rod}(3)$:
Removes small noisy dark pits or ridges.

2. $B_2 = (G_1 \bullet \text{brick}(1,70,0) - G_1) > 40$:
Finds relatively dark areas whose shape can not be covered by a long vertical stripe. In this example, ship is the only object that can be detected by this thresholding a closing residue operation.

3. $B_3 = B_2 \ominus \text{disk}(3)$:
Removes any unwanted dark areas that can be detected by step 2. It also shrinks the area of the detected ship, thus making sure we only detect inside parts of the ship.

4. $B_4 = ((G_1 \oplus \text{brick}(12,2,0)) - G_1) > 20$:
If we let the origin of the brick to be at the center of the top row of its support, a thresholding a closing residue operation will find the top boundary of the ship.

5. $B_5 = B_4 \wedge (G_1 < 85)$:
   The detected boundary should be inside the dark area (ship) whose brightness value is less than 85. The grayscale value of the ship area should be absolutely less than 85.

6. $B_6 = B_3 \vee B_5$:
   Combines information gathered through two mutually compensating operations.

7. $B_7 = B_6 \wedge (B_2 \oplus \text{disk}(8))$:
   Make sure the areas detected in B_6 come from B_2.

8. $B_8 = B_7 \bullet \text{box}(2, 10)$:
   Fills in small zig_zag boundaries.

9. $B_9 = (B_8 \bullet \text{disk}(4) \ominus \text{disk}(4)) \vee B_8$:
   Fills in possible holes inside the ship.

10. $B_{10} = (B_9 \oplus \text{points}\{(0,0),(0,-2)\}) > B_9$:
    Detects only the top edges.

reasoning and knowledge

- The uneven structures on the top of the objects are the significant features. Try to determine the characteristics of these features.
- The relatively dark region of horizontally long object can be detected by a thresholding a closing residue operation especially by a vertical line segment longer than the height of the object (step 2).
- The top boundary can be detected by a thresholding a dilation residue operation by a brick structuring element whose origin is at the top row instead of the center row (step 4).
- To have consistent features, we need to smooth the boundaries This can be done by a closing operation. The structuring element used in this operation should be small enough not to destroy the details of the shape but to remove small noisy jaggedness of the boundaries (step 8).
- To fill in holes inside the detected ship but not to destroy the details of the boundary, close the image and shrink it by an erosion operation. Then or the result with the original image (step 9).
- If one operation finds the core of the ship while another operation can find the top portion of the ship more easily but not the core, combine the result of both operation for the better recognition of the ship (step 6). We can find an object if different part of the object can be detected by different operations.
- The top boundaries can be detected by a dilation residue operation (step 10). The structuring element consists of two points, one at the origin and the other at vertically shifted point.

3.7 Barcode detection

input image:

- It is a binary image with barcodes as foreground pixels. The barcode is shown as a group of small boxes with the same height but with different widths. They are highly regular. The size of the group of barcode is known. However, they can be arbitrarily oriented and positioned.
- There are many noisy blobs with similar sizes but not regularly grouped as the barcodes. The background can also contain large and small objects allowing it to be quite arbitrary.

goal

- Find the barcodes, clusters of bars with known size, shape, and arrangement. The bars are parallel to each other. They can almost be included in a single rectangular shaped box. The distance between each bar is almost same.

morphological procedure:

1. $B_1 = B_0 - (B_0 \circ \text{disk}(6))$:
   Removes objects larger than the size of barcode.
2. $B_2 = B_1 \circ \text{disk}(2)$:
   Removes objects smaller than the size of barcode.
3. $B_3 = B_2 \ominus \text{points}\{(0,3),(0,4),(0,-3),(0,-4),(3,0),(4,0),(-3,0),(-4,0)\}$:
   Detects regular patterns (barcodes) by an erosion operation with special set of points.
4. $B_4 = B_3 \oplus \text{box}(15,15)$:
   Generates the barcode mask by dilating the seeds detected in step 3.
5. $B_5 = B_2 \wedge B_4$:
   Masks out only the barcodes.
6. $B_6 = B_5 \ominus \text{disk}(1)$:
   Separates possibly connected bars in each barcode group.

$$B_a = (B_0 - B_0 \circ \text{disk}(6)) \circ \text{disk}(2)$$
$$B_b = (B_a \wedge (B_a \ominus \text{pointset} \oplus \text{box}(15,15))) \ominus \text{disk}(1)$$

reasoning and knowledge

- Remove larger blobs by an opening residue operation (step 1).
- Remove smaller blobs by an opening operation (step 2).
- The clusters of blobs arranged regularly can be detected by an erosion operation by a set of regularly arranged points (step 3). The set of points should be center symmetric so that it can detect arbitrarily positioned barcodes.

3.8 Recognition of broken rice grains

input image: [16]

- It is a grayscale image where the grains show up as bright highly contrasted objects with respect to the dark background.
- The grains can touch each other.

goal

- Find nonconvex bright blobs with certain size which do not touch the boundary of the input image. The detected blobs should be isolated.

morphological procedure:

1. $B_1 = (G_0 - (G_0 \circ \mathrm{rod}_1)) > \mathrm{thr}_1$:
   Detects relatively bright blobs whose size are smaller than the rod_1 used in the opening operation.
2. $B_2 = \mathrm{watershed}(B_1)$:
   Separates touching blobs by a watershed operation. The watershed operation is described in [15].
3. $B_3 = (G_0 \geq 0) - ((G_0 \geq 0) \ominus \mathrm{disk}(1))$:
   Finds the boundary of the input image.
4. $B_4 = B_3 \oplus |_{B_2} \mathrm{disk}(1)$:
   Extracts blobs touching the boundary of the image.
5. $B_5 = B_2 - (B_2 \circ \mathrm{disk}(2))$:
   Marks concave blobs which are not open by a disk.
6. $B_6 = B_5 \oplus |_{B_2} \mathrm{disk}(1)$:
   Recover the concave blobs from the seeds marked by step 5.
7. $B_7 = B_6 > B_4$:
   Remove blobs touching the boundary of the image from the detected concave blobs.

reasoning and knowledge

- The convex shaped objects are open to a convex shaped disk which is smaller than the size of the objects (step 5).
- If operators are are independent, we can use them in any order in the procedure.
- Some operations presuppose special operations. For example the operation which extracts convex grains presupposes the segmentation by watershed which separates touching grains.

3.9 Particle marking

input image: [15]

- In a dark background, there are some distinguishably bright objects, cells with nuclei and the cytoplasm.
- The medium bright cells may or may not contain very bright nuclei. The shape of nuclei is convex round or sometimes smeared making them to appear as concave blobs. The cytoplasm is medium bright and does not contain nuclei.

goal

- Find medium bright cells with a convex shaped nuclei inside them. The size of nuclei is approximately known.

morphological procedure:

1. $B_1 = G_0 > \mathrm{thr}_1$:
   Detects only the very bright nuclei.
2. $B_2 = G_0 > \mathrm{thr}_2$:
   Detects all bright blobs. $B_2 \supset B_1$ since $\mathrm{thr}_1 > \mathrm{thr}_2$.
3. $B_3 = B_1 \oplus |_{B_2} \mathrm{disk}(1)$:
   Detects cells with nuclei in them.
4. $B_4 = (B_1 \bullet \mathrm{disk}(5)) - B_1$:
   Marks concave blobs.
5. $B_5 = B_4 \oplus |_{B_3} \mathrm{disk}(1)$:
   Recovers concave blobs from the image of cells with nuclei.
6. $B_6 = B_3 - B_5$:
   Genarates an image of convex shaped cells with nuclei.
7. $B_7 = B_1 \ominus \mathrm{disk}(10)$:
   Marks nuclei larger than the disk of radius 10 pixels.
8. $B_8 = B_7 \oplus |_{B_6} \mathrm{disk}(1)$:
   Recovers convex cells with nuclei. The nuclei are larger than the disk of radius 10 pixels.

reasoning and knowledge

- Use two thresholding operations with different threshold values to detect areas with different grayscale intensity values (step 1 and 2).
- To remove the cytoplasm without nuclei, mark the nuclei (step 1) and then conditionally dilate the marked image with respect to the image of all bright objects (step 2) to find only the ones with nuclei (step 3).
- Convex objects are closed to a convex disk.
- Mark the concave objects by a closing residue operation (step 4).

3.10 Prunings on the thinning of a daisy

input image: [15]

- It is a binary image which contains an arbitrary shaped object.

goal

- Determine the skeleton image without noisy small branches.

morphological procedure:

The following morphological operations use the structuring elements of the Golay alphabet. When the Golay symbol appears inside a brace, for example $\{L\}$, the sequence $\{L\}$ has the successive rotations of L as its elements.

1. $B_1 = B_0 \bigcirc \{L\}(\text{iteration} = N_1)$:
   Thins the daisy to get the skeleton image. The symbol $\bigcirc$ represents a thinning operation defined as follows:
   $$A \bigcirc L = A - (A \circledast L)$$
   where L_0 is a set of pixel positions with 0 in L, and L_1 is a set of pixel positions with 1 in L. The hit–or–miss operation ($\circledast$) can be used to detect a portion of image which matches the pattern defined by the structuring element used in the operation. The thinning operation is applied N_1 times where N_1 is the half of the maximum width of the object to be thinned.

2. $B_2 = B_1 \bigcirc \{E\}(\text{iteration} = N_2)$:
   Prunes the branches shorter than N_2 pixels long. This also shortens the main skeleton.

3. $B_3 = B_2 \circledast \{E\}$:
   Detects the end points of the main skeleton.

4. $B_4 = B_3 \oplus|_{B_1} \text{disk}(1)$:
   Recovers the shortened main skeleton by conditionally dilating the end points found in step 3 with respect to the original skeleton image. The conditional dilation operation is applied N_2 times since we know the main skeleton was shortened N_2 times in step 2.

5. $B_5 = B_2 \vee B_4$:
   Combines the pruned skeleton with the lengthened end points image.

reasoning and knowledge

- To shorten the branches in the skeleton image, perform a thinning operation with structuring elements which can detect only the end points of a line. It will also shorten the longer branches.

- If we conditionally dilate the above shortened image, we get the original image back. We want to lengthen only the long branches left in the shortened image.

- The end points of branches can be detected by a pattern matching using a hit–or–miss operation.

3.11 Neighbor Analysis

input image: [15]

- It is a grayscale image of polished section of polycrystalline ceramic (bright) seen in a rectangular mask. The bright region can have small dark noisy spots.
- The boundaries of bright grains are relatively dark and are of known thickness. The boundaries are quite homogeneously relatively dark. The grains can have some dark noise inside them.

goal

- Find grains whose boundaries are completely shown in the image. Make the boundaries one pixel thick.

morphological procedure:

1. $B_1 = (G_0 \geq 0) \ominus \text{disk}(2)$:
   Mask of whole image shrunk 2 pixels from the boundary of the image.
2. $B_2 = (B_1 \oplus \text{disk}(1)) - B_1$:
   Extracts the boundary of input image.
3. $B_3 = G_0 > \text{thr}_1$:
   Detects the bright grains. The detected grains can have holes in them because of dark noise.
4. $B_4 = (\text{skiz}(B_3^c))^c$:
   The skiz operation is defined as $\text{skiz}(A) = [A \bigcirc \{L\}] \bigcirc \{E\}$ and it detects skeletons without short branches. The skiz of dark areas (B_3^c) will give a single pixel wide grain boundaries. It does not contain the skeleton of dark noise because the noisy spot does not touch two different grains. Thus the complement of the skiz extracts zones of influence of each bright grains.
5. $B_5 = B_4 \wedge B_1$:
   Take out the boundary effect of the input image.
6. $B_6 = B_2 \oplus |_{B_5} \text{disk}(1)$:
   Extracts grains touching the image boundary (B_2).
7. $B_7 = B_5 - B_6$:
   Detects grains whose boundaries are completely shown in the image.

reasoning and knowledge

- The grains touching the image boundary are not completely shown in the image.
- The boundaries of grains can be thinned to a single pixel by a skiz operation.
- The skiz of an isolated blob is a null image.

3.12 Defect lines detection

input image: [15]

- It is a binary image with black stripes. The stripes have a specified width and jaggedness. Most of the stripes start from one side of the image and end at some other side of the image.
- Some of the stripes are broken inside the image. Consecutive broken stripes form an imaginary line in the image if we suppose connecting the end points of the broken stripes.
- There could be several such imaginary lines formed by different groups of consecutive broken stripes.

goal

- Find the regions where the stripes are disconnected. Connect the end points of those disconnected stripes.

morphological procedure:

1. $B_1 = B_0^c \bigcirc \{L\}(\text{iteration} = N_1)$:
   Homotopically thins the dark stripes to get the skeletons of them. Detects the end points of the main skeleton.
2. $B_2 = B_1 \bigcirc \{E\}(\text{iteration} = N_2)$:
   Prunes small branches of the detected skeletons.
3. $B_3 = B_2 \circledast \{E\}$:
   Detects the end points of pruned skeletons.
4. $B_4 = B_2 \vee (B_3 \oplus |_{B_1} \text{disk}(1))$:
   Recovers the shortened main skeletons by conditionally dilating the end points of the shortened main skeletons.
5. $B_5 = B_4 \circledast \{E\}$:
   Detects the end points of the main skeletons.
6. $B_6 = (B_4 - (B_5 \oplus \text{disk}(1))) \vee B_5$:
   Separates the end points from the main skeletons.
7. $B_7 = (\text{skiz}(B_6^c))^c$:
   Determines zones of influence of end points as well as zones of influence of main skeletons.
8. $B_8 = B_5 \oplus |_{B_7} \text{disk}(1)$:
   Extracts only the zones of influence of end points by conditionally dilating the end points with respect to the zones of influence image obtained in step 7.
9. $B_9 = B_8 \oplus \text{disk}(1)$:
   Connects the detected zones of influence of end points.

10. $B_{10} = B_9 \bigcirc \{L\}$(iteration $= N_3$):
    Thins the detected zones of influence to get the imaginary line connecting the end points.

reasoning and knowledge

- The overall plan is to detect the end points of each of the broken stripes and connect them (restatement of the goal).
- The end points of the broken stripes can be detected by finding the end points of the longer lines in the skeleton image of the stripes (step 5).
- We can predict that the skeleton of the stripes will have small branches because of the jaggedness of the stripes.
- Use the procedure described in section 3.10 to extract only the main skeletons (step 2 to 4).
- There could be several regions of broken stripes. Since they are not guaranteed to be separated father than the distance between the end points inside one such region, a dilation to connect the end points followed by a thinning may not work.
- The end points of one group of consecutive broken stripes are next to each other. Thus, if we compute the zones of influence image from the image of skeletons and separated end points zones of influence of a single group of end points will be next to each other (step 6 and 7).
- The end points of a line can be separated from the line by subtracting the dilated end points from the line and ORing with the end points (step 6).
- Fill in each zones of influence of end points by a conditional dilation operation (step 8).
- Connect zones of influence of end points next to each other by a dilation operation (step 9).

3.13 Discrimination between cells and artifacts

input image: [13, 15]

- It consists of relatively bright particles in a dark background. The particles are generally round. Some of them are nonconvex and relatively large.
- Some of the particles are overlapping each other. Some have holes.

goal

- Find only the large roundly shaped particles. If a particle is round, the curvature of its boundary is bounded.

morphological procedure:

1. $B_1 = G_0 > thr_1$:
   Detects bright objects.

2. $B_2 = \text{conditional\_bisector}(B_1, 2)$:
The conditional bisector is defined as follows:

$$\text{conditional\_bisector}(A, N) = \bigcup_i [(A \oplus \text{disk(i)}) - ((A \ominus \text{disk(i + 1)}) \oplus \text{disk(N)})]$$

It can detect the local peaks in the distance transformed image. It does not detect the ridges. If a blob is not round, the conditional bisector detects more than one isolated point inside the blob.

3. $B_3 = B_1 - B_2$:
Make holes inside blobs detected in step 1.

4. $B_4 = \text{skiz}(B_3)$:
Gets the skiz of the bright blobs with holes. For a blob with single hole, the skiz is a single loop inside the blob. For a blob with more than one hole, the skiz consists of more than one loop connected each other by a line. Thus, if a blob is not round, the skiz inside the blob includes points where more than three lines meet.

5. $B_5 = (B_1 \oplus \text{disk}(2)) - B_1$:
Detects boundaries of the bright blobs.

6. $B_6 = B_5 \bigcirc \{L\}(\text{iteration} = N_1)$:
Thin the boundaries of the bright blobs. If a blob is not round, the thinned boundary includes small branches because of its jaggedness.

7. $B_7 = B_4 \vee B_6$:
Combines the skiz image with the boundary image.

8. $B_8 = B_7 \circledast \{F\}$:
Finds points where three lines meet by a pattern matching operation (hit–or–miss).

9. $B_9 = B_8 \oplus|_{B_1} \text{disk}(1)$:
Extract blobs marked by step 8.

10. $B_{10} = B_1 - B_9$:
Extract only the round blobs.

reasoning and knowledge

- Mark the overlapping particles, non-round shaped particles, and the particles with holes in them. We can accomplish the task if we can mark all the particles that we do not want to detect.

- If two round blobs overlap each other, a conditional bisector operation results in two points inside the overlapped particles (step 2).

- Thus, if we take the skiz of the overlapping blobs without its conditional bisector, it gives two loops around each points detected by the conditional bisector operation connected together by a line segment (step 4).
- If the boundary of a particle is not round, its boundary obtained by a dilation residue operation will have some jaggedness (step 5).
- If we thin a jagged object, the thinned image will contain small branches (step 6).
- The points where three lines meet can be detected by a hit–or–miss operation (step 8).

3.14 Inspection of hybrid circuits

input image: [19]

- It is a grayscale image of hybrid circuit boards. It shows relatively bright stripes of all different widths in a dark background. The image is quite noisy.

goal

- Detect only the bright thin stripes.

morphological procedure:

1. $G_1 = \text{isotropic\_filter}(G_0, N_1, N_2)$:
   Remove noise. The isotropic filtering can be done by an alternating application of opening and closing by balls of increasing size. Thus, isotropic_filter (I, N_1, N_2) is implemented as follows:

   $I^{N_1} = I$
   for each $i = N_1, \cdots, N_2$
   $\quad A = I^i \circ \text{ball}_i$
   $\quad I^{i+1} = A \bullet \text{ball}_i$

2. $B_2 = G_1 > \text{thr}$:
   Detects bright areas.
3. $B_3 = B_2 - (B_2 \bullet \text{disk}(6))$:
   Removes larger blobs which can be closed to the disk of radius 6 pixels.
4. $B_4 = B_3 \circ \text{disk}(2)$:
   Removes small blobs smaller than the disk of radius 2 pixels.

reasoning and knowledge

- Detect large blobs by a closing operation.
- Removes small objects by an opening operation.
- Combine the above two operations to detect blobs with certain size.

3.15 Feature detection in Dental Image

input image: [5]

- It is a grayscale image containing several teeth shown relatively bright in a dark background. It also contains relatively bright gums near to one end of each tooth and between a pair of neighboring teeth.
- Within each tooth the grey tone is approximately constant. Eah tooth can have fillings or metal caps resulting in a very bright region inside the tooth.
- A tooth can have enamel at its vertical boundary which is shown brighter than the intensity of tooth. The shape of enamel region is sharp thin triangle which is vertically oriented between the background and tooth.

goal

- Find the starting position of the enamel region close to the gum region in between the teeth.

morphological procedure:

1. $G_1 = G_0 \bullet \text{rod}(r_1)$:
   Removes small grey tone pits.
2. $B_2 = (G_1 - G_1 \circ \text{rod}(r_2)) > thr_1$:
   Detects relatively bright region which cannot contain the rod.
3. $B_3 = B_2 \circ \text{box}(w_1, h_1)$:
   Removes artifacts which cannot contain the box(w_1, h_1). The box chosen here is a vertically long and thin that can be contained in the detected enamel region.
4. $B_4 = B_3 - B_3 \circ \text{box}(w_2, h_2)$:
   Finds corners by an opening residue with a box structuring element of small width.
5. $B_5 = (G_0 < thr_2) \oplus \text{disk}(r_3)$:
   Detects the dark background and expand it a little bit by dilating it with a small disk.
6. $B_6 = B_4 \wedge B_5$:
   Detects only the corner points touching the background.

$$
\begin{aligned}
G_a &= G_0 \bullet \text{rod}(r_1) \\
B_b &= [(G_a - G_a \circ \text{rod}(r_2)) > thr_1] \circ \text{box}(w_1, h_1) \\
B_c &= (B_b - B_b \circ \text{box}(w_2, h_2)) \wedge [(G_0 < thr_2) \oplus \text{disk}(r_3)]
\end{aligned}
$$

reasoning and knowledge

- The plan is finding the grey tone hill oriented along the column axis and then extracting the corners close to the background or to the gum area.
- Remove small relatively dark pits by a closing operation (step 1). Since we are interested in the sharp corners of the relatively bright region, we do not open the image to remove the small relatively bright peaks.
- After finding the relatively bright areas (step 2), remove noisy small artifacts that does not conform to the shape we are interested in by an opening operation (step 3). Use thin vertical box for the opening to keep the corners intact but to remove the artifacts.
- The sharp corner of the vertically thin triangle can be detected by an opening residue with a horizontally thin but short line segment (step 4).
- The corners touching the background or near the background can be extracted by expanding the background a little bit and ANDing it with the corner image (step 5 and 6).

3.16 Detecting occluded object

input image: [2]

- It is a grayscale image of a single target which is relatively dark and roughly oval in shape. The dimension of the target and its aspect ratio are approximately known.
- Parts of the target can be obscured by brighter areas of approximately known size.
- The background is relatively brighter than the target but may not be of uniform brightness.
- The image may contain areas of bright glint which are relatively brighter than their neighboring pixels. The size of glint area is small and approximately known. Areas of this type may aggregate to form larger regions of bright glint but no more than about 25% of the target area..
- Very small dark areas are present in the image and commonly occur near glint.

goal

- Detect the target with occlusions removed.

morphological procedure:

1. $B_1 = G_0 > thr_1$:
   Detects absolutely bright glint.
2. $G_2 = G_0 \bullet \text{brick}(w_1, h_1)$:
   Eliminates small dark regions.
3. $B_3 = [G_2 - (G_2 \circ \text{brick}(w_2, h_2))] > thr_2$:
   Detects small relatively bright regions (glint areas).

4. $G_4 = [(B_1 \vee B_3) \oplus \text{box}(3,3)] > 0$:
   Forms mask of absolutely bright or locally relatively bright glint areas.

5. $G_5 = G_2 \wedge G_4^c$:
   Replaces the grayscale values of the pixels in the glint area with zero in the image free of small dark regions.

6. $G_6 = \text{fill}(G_5, G_4)$:
   Fills in glint areas with the grayscale values surrounding them. The morphological fill operation, fill(f, g), can be defined iteratively as follows:

$$
\begin{aligned}
G^0 &= g \\
F^0 &= f \\
G^i &= G^{i-1} \ominus \text{rod}(1) \\
F^i &= F^{i-1} \vee \left[(F^{i-1} \oplus \text{rod}(1)) \wedge G^{i-1}\right] \\
\text{fill}(f,g) &= F^n \quad \text{where} \quad G^{n+1} = 0
\end{aligned}
$$

7. $G_7 = G_6 \circ \text{brick}(w_3, h_3)$:
   Removes the relatively bright areas occluding the target.

8. $B_8 = [G_7 - (G_7 \bullet \text{brick}(w_4, h_4))] > thr_3$:
   Detects the large relatively dark region, which constitutes the target.

reasoning and knowledge

- Overall plan is to remove unwanted noise and glints, to remove occlusion, and to detect the targets.
- Detect absolutely bright and larger glints and small relatively bright glints by the thresholding and the thresholding of the opening residue operation respectively (step 1, 3, and 4).
- Fill in the glint areas with the grayscale values of the pixels surrounding them (step 5 and 6). A simple opening will do the similar job, but it will also destroy the other areas.

4 Summary and comments

In this paper, we analyzed a small set of vision algorithms where the mathematical morphology is applied to the solution of machine vision tasks. However, it is still an open question to find and explicitly describe the functional descriptions of all the vision tasks that the morphological procedures can solve. One should first look for the solution from the theory side on the mathematical morphology such as [11, 15, 17, 6, 8, 10, 18]. One can also examine the vision algorithms already developed by vision experts such as the ones listed in this paper. This helps us understand the main role of the morphological operator in the algorithm sequence and the context in which the operator appears. On the experimental side, we are currently designing a system that can propose a morphological procedure and try to interpret them in a controlled search environment. This research will help us achieve our goal of automatically producing morphological vision algorithms.

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