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B-Code Dilation and Structuring Element Decomposition for Restricted Convex Shapes

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ABSTRACT

A convex, filled polygonal shape in $\mathbf{R} \times \mathbf{R}$ can be uniquely represented in the discrete $\mathbf{Z} \times \mathbf{Z}$ domain by the set of all the lattice points lying in its interior and on its edges. We define a ***restricted convex shape*** as the discrete four connected set of points representing any convex, filled polygon whose vertices lie on the lattice points and whose interior angles are multiples of 45^0. In this paper we introduce the Boundary Code (B-Code), and we express the morphological dilation operation on the restricted convex shapes with structuring elements that are also restricted convex shapes. The algorithm for this operation is of $O(1)$ complexity and hence is independent of the size of the object. Further, we show that the algorithm for the n-fold dilation is of $O(1)$ complexity. We prove that there is an unique set of thirteen shapes $\{K_1, K_2, \ldots, K_{13}\}$ such that any given restricted convex shape, K, is expressible as $K = K_1^{n_1} \oplus K_2^{n_2} \oplus \cdots \oplus K_{13}^{n_{13}}$ where $K_i^{n_i}$ represents the n_i-fold dilation of K_i. We also derive a finite step algorithm to find this decomposition.

1. Introduction

The concepts of mathematical morphology have been used for shape description Ghosh[1988], Ghosh and Haralick[1990] , Pitas et al.[1990]. Shapes or objects can be described in terms of simpler, better characterized, underlying parts. A morphological description of a shape usually expresses a shape by decomposing it into an equivalent series of dilations of simpler parts. Simpler parts in the case of binary shapes can be disks, lines, rectangles etc. of various sizes. When a shape is expressible as a dilation of two other simpler shapes, it means that the original shape can be described as the area marked out when one of the parts is held fixed and the other is swept over the first.

Binary shape are usually represented as the sets of all the points constituting them. These shapes are completely characterized by their boundaries and many efficient representation schemes for representing border information have been presented, Freeman[1974]. Boundary representations make explicit many important features such as the vertices, edge lengths, etc. If these features are used by shape description algorithms, the use of the boundary representation will make the extraction of the description from the representation much more efficient.

Algorithms that perform morphological operations using object outlines in the continuous domain have been proposed Ghosh[1986]. However, the equivalence of these approaches to the existing set theoretic definitions has not been proved.

Morphological operations on machines specialized to perform these operations are limited by the maximum size of the structuring elements that the machine allows. If a morphological operation has to be performed with a structuring element larger than the maximum allowable size, the structuring element has to be decomposed into smaller ones. The new structuring elements have to be such that (i) each of them can be handled by the hardware, and (ii) the dilation of all of them is the original structuring element.

From the above discussion we can see that structuring element decomposition is an important problem from both points of view – shape description and hardware implementation of morphological operations. Several algorithms to find such decompositions have been presented in the literature . These algorithms work either on shapes represented as sets or as their outlines in the continuous domain and have a time complexity of $O(n^2)$. In this paper we introduce the concept of B-Codes which is a boundary representation for binary shapes. B-Codes are then used to perform morphology and decomposition. We also prove that results obtained by B-Code morphology is equivalent to performing the same operations using the standard set theoretic operations. Furthermore, we show that the time complexity of our algorithms for decomposition and morphology are $O(1)$.

In section 2 the related literature has been discussed. In section 3 we set the stage by giving all the definitions and notations. In section 4 we define restricted convex shapes and B-Codes . B-Code dilation and n-fold dilation is discussed in section 5. The proof of the algorithm is given in 6. The algorithm for structuring element decomposition has been given in section 7. Computational complexity of the algorithms has been considered in section 8. Finally, a summary of the presented work and future work directions towards generalizing the algorithms for any convex and non convex shape has been considered in section 9.

2. Literature Survey

The paper on structuring element decomposition by Zhuang and Haralick[1986] presented the theory and algorithm for decomposing a binary structuring element as the dilations of two point structuring elements. This algorithm is for an arbitrary binary structuring element and the time complexity is proportional to the square of number of foreground pixels in the structuring element. Gong[1988] presents an $O(n^2)$ algorithm for decomposing a binary 2-D image. Here the n is the number of points in the decomposed sequence. Gosh [1988] discusses the decomposition of convex shapes in $\mathbf{R}^2$ and $\mathbf{R}^3$. Gosh and Haralick[1989] present an algorithm for decomposing a convex shape in $\mathbf{R}^2$ into triangles and straight lines. They also give a way of decomposing binary structuring elements in $\mathbf{Z}^2$ by going into the $\mathbf{R}^2$ domain by taking the convex hull of the structuring element, then decomposing in the $\mathbf{R}^2$ domain, and finally comming back to the discrete domain by sampling. Xu [1989] presents an algorithm for decomposing a class of chain coded convex shape into dilations of a structuring elements from a finite set of structuring elements but does not provide any type of basis set. Pitas et al.[1990] use morphological shape decomposition techniques for shape description of real, binary images. None of the above papers have provided an algebra for dilation of restricted convex shapes in terms of their B-Code. In this paper we provide a tecnique way for performing dilation of B-coded structuring elements without going back into the discrete domain. Also, we provide an algorithm for decomposing the B-coded structuring element into dilations of structuring elements from a primitive set of 13 structuring elements. The time complexity is independent of the size. In fact it is constant time.

3. Notations and Definitions

In order to present our work, we need to introduce the following definitions:

A *binary discrete image* is a function $g : \mathbf{Z}^2 \rightarrow \mathbf{Z}$ such that the function g can take only two values : 0 and 1. The lattice points $(i, j) \in \mathbf{Z}^2$ corresponding to the the points where g takes up the value 1 are called *foreground* points, or pixels and *background* pixels otherwise. Two foreground points (i,j) and (k,l) are said to be *4-neighbours* if and only if (k,l) is one of $\{(i+1,j),(i-1,j),(i,j+1),(i,j-1),(i,j)\}$. The foreground points (i,j) and (k,l) are said to be *8-neighbours* if and only if (k,l) is one of $\{(i+1,j),(i-i,j),(i,j+1),(i,j-1),(i,j),(i+1,j+1),(i+1,j-1),(i-1,j+1)\}$. The points x_i and x_j are said to be *4-connected* if and only if there is a sequence of foreground points $\{ x_i, \ldots, x_j \}$ such that x_k and x_{k+1} are 4-neighbours. Two foreground points x_i and x_j are said to be *8-connected* if and only if there is a sequence of foreground points $\{ x_i, \ldots, x_j \}$ such that x_k and x_{k+1} are 8-neighbours.

A set of foreground points is a *4-connected component* if has the property that all pairs of points in the set are 4-connected. A set of foreground points is an *8-connected component* if has the property that all pairs of points in the set are 8-connected.

We define a 4 or 8 connected component as *convex* if and only if all the lattice points lying inside or on the convex hull of the foreground points are members of set of foreground points constituting the connected component. This definition directly implies that a convex connected component has no holes.

Dilation is the morphological transformation which combines two sets using vector addition of set elements. If A and B are sets in N-space (E^N) with elements a and b, respectively, $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$ being N-tuples of element coordinates, then the dilation of A by B is the set of all possible vector sums of pairs of elements, one comming from A and one comming from B. Rigorously, the dilation of A by B is denoted by $A \oplus B$ and is defined by $A \oplus B = \{c \in E^N \mid c = a + b \text{ for some } a \in A \text{ and } b \in B\}$.

Structuring element decomposition problem is defined as follows: given a set $A \subset \mathbf{Z} \times \mathbf{Z}$ determine the smallest N and the corresponding structuring elements $K_1, K_2, \ldots, K_N$ such that $A = K_1 \oplus K_2 \oplus \cdots \oplus K_N$.

4. Restricted Convex Shapes and B-Codes

A *restricted convex shape* is defined as a convex 4-connected component whose convex hull has boundary lines oriented only at angles $0^0, 45^0, 90^0, 135^0$ with respect to the positive x-axis.

Boundary Code or B-Code is a notation for representing 4 or 8 connected components in terms of their boundary lattice points. Only a starting boundary point is represented explicitly, while the rest of the boundary points are represented as successive displacements from a boundary point to one of its neighbours along fixed set of possible directions. If the successive displacements happen to be in the same direction, it is encoded as the direction followed by the number of successive moves in that direction. The formal notation to represent a convex connected component A is given below $A =< (i_A, j_A) \mid (d_1 : n_1)(d_2 : n_2) \ldots (d_m : n_m) >$ where (i_A, j_A) is the starting boundary lattice point, and the ordered pairs following the vertical bar describes each successive displacement. The integer $d_i \in \{0^0, 45^0, 90^0, 135^0, 180^0, 225^0, 270^0, 315^0\}$ gives the direction and the the integer n_i following the colon sign is the number of successive moves in that direction.

Let $d_0, d_1, \ldots, d_7$ be vectors given by $d_0 = (1,0), d_1 = (1,1), d_2 = (0,1), d_3 = (-1,1), d_4 = (-1,0), d_5 = (-1,-1), d_6 = (0,-1), d_7 = (1,-1)$ Let an restricted convex shape, A, be given by $A = \langle (i_A, j_A) \mid (d_0 : n_0^A)(d_1 : n_1^A) \cdots (d_7 : n_7^A) \rangle$ The eight vertices of A are $V_0^A, V_1^A, \ldots, V_7^A$, where $V_0^A = (i_A, j_A)$ and rest of the vertices are given by $V_{i+1}^A = V_i^A + n_i^A d_i$. We define A^* as the set of vertices of A: $A^* = \{V_0^A, V_1^A, \ldots, V_7^A\}$. See figure 1 for examples of restricted convex shapes and their B-Codes.

The convex hull of A is denoted by Hull$[A]$, and is defined as

$$\text{Hull}[A] = \{d \in \mathbf{R}^2 \mid d = \sum \alpha_i V_i^A \text{ where } \alpha_i \in \mathbf{R}, \alpha_i \geq 0, \sum \alpha_i = 1, V_i^A \in A^*\} = \text{Hull}[A^*]$$

We define a sampling set $\mathbf{S}$ as the set of all lattice points ,i.e., $\mathbf{S} = \{s \mid s \in \mathbf{Z}^2\}$. The set of all lattice points contained in A can be defined using the sampling set $\mathbf{S}$ as $\text{Hull}[A^*] \cap \mathbf{S} = \{d \in \mathbf{Z}^2 \mid d = \sum \alpha_i V_i^A \text{ where } \alpha_i \in \mathbf{R}, \alpha_i \geq 0, \sum \alpha_i = 1, V_i^A \in A^*\}$ Since the set vertices of A, i.e. A^*, are lattice points, it can be easily seen that the convex hull of A^* and $\text{Hull}[A^*] \cap \mathbf{S}$ are equal to Hull$[A]$. That is $\text{Hull}[A^*] = \text{Hull}[\,\text{Hull}[A^*] \cap \mathbf{S}\,] = \text{Hull}[A]$. Since a B-Code shape $A = \langle (i_A, j_A) \mid (d_0 : n_0^A)(d_1 : n_1^A) \cdots (d_7 : n_7^A) \rangle$ is a closed contour, it inherits the following properties of closed contours:

$$\begin{aligned} n_0^A + n_1^A + n_7^A &= n_3^A + n_4^A + n_5^A \\ n_1^A + n_2^A + n_3^A &= n_5^A + n_6^A + n_7^A \end{aligned} \tag{1}$$

where $n_i^A \geq 0$ for $0 \leq i \leq 7$. The second two equations come about form the noting that V_7^A is equal to V_8^A and substituting for the d_i's.

5. B-Code Dilation

Let A and B be two restricted convex shape defined by $A = \langle (i_A, j_A) \mid (d_0 : n_0^A)(d_1 : n_1^A) \cdots (d_7 : n_7^A) \rangle$ and $B = \langle (i_B, j_B) \mid (d_0 : n_0^B)(d_1 : n_1^B) \cdots (d_7 : n_7^B) \rangle$. The B-Code dilation of A and B is then a restricted convex shape D given by

$$D = \langle (i_A, j_A) + (i_B, j_B) \mid (d_0 : n_0^A + n_0^B)(d_1 : n_1^A + n_1^B) \cdots (d_7 : n_7^A + n_7^B) \rangle \tag{2}$$

The following proposition states the exact relations that have to be proved.

Proposition: Hull$[D]$ = Hull$[($ Hull$[A] \cap S$ $) \oplus ($ Hull$[B] \cap S$ $)]$

Proof : Since the vertices of A, B, and D are latice points, proposition becomes Hull$[D^*]$ = Hull$[($ Hull$[A^*] \cap S$ $) \oplus ($ Hull$[B^*] \cap S$ $)]$. It follows that Hull$[D^*]$ = Hull$[($ Hull$[A^*]) \oplus ($ Hull$[B^*])]$. From Ghosh and Haralick[1989] we have Hull$[A^*] \oplus$ Hull$[B^*]$ = Hull$[A^* \oplus B^*]$. The proposition now becomes, Hull$[D^*]$ = Hull$[$ Hull$[A^* \oplus B^*]]$. That is, we now have to prove that Hull$[D^*]$ = Hull$[A^* \oplus B^*]$. Let $V(x)$ and $V(y)$ denote the x and y coordinates of the lattice point V. Further, let

$$\begin{aligned}
A^*_{d_0 min} &= \{V_i^A \mid V_i^A(x) = \min_j V_j^A(x), V_i^A \in A^*\} = \{V_6^A, V_7^A\} \\
A^*_{d_0 max} &= \{V_i^A \mid V_i^A(x) = \max_j V_j^A(x), V_i^A \in A^*\} = \{V_2^A, V_3^A\} \\
A^*_{d_2 min} &= \{V_i^A \mid V_i^A(y) = \min_j V_j^A(y), V_i^A \in A^*\} = \{V_0^A, V_1^A\} \\
A^*_{d_2 max} &= \{V_i^A \mid V_i^A(y) = \max_j V_j^A(y), V_i^A \in A^*\} = \{V_4^A, V_5^A\} \\
A^*_{d_1 max} &= \{V_i^A \mid V_i^A(x) + V_i^A(y) = \max_j V_j^A(x) + V_j^A(y), V_i^A \in A^*\} = \{V_3^A, V_4^A\} \\
A^*_{d_1 min} &= \{V_i^A \mid V_i^A(x) + V_i^A(y) = \min_j V_j^A(x) + V_j^A(y), V_i^A \in A^*\} = \{V_0^A, V_7^A\} \\
A^*_{d_3 max} &= \{V_i^A \mid V_i^A(y) - V_i^A(x) = \max_j V_j^A(y) - V_j^A(x), V_i^A \in A^*\} = \{V_5^A, V_6^A\} \\
A^*_{d_3 min} &= \{V_i^A \mid V_i^A(y) - V_i^A(x) = \min_j V_j^A(y) - V_j^A(x), V_i^A \in A^*\} = \{V_1^A, V_2^A\}
\end{aligned}$$

Let $\Gamma = A^* \oplus B^* = \{\gamma \mid \gamma = V_i^A + V_j^B, V_i^A \in A^*, V_j^B \in B^*\}$ and let

$$\begin{aligned}
\Gamma^*_{d_0 min} &= \{\gamma \mid \gamma(x) = \min_k \gamma(x), \gamma \in \Gamma\} \\
&= \{\gamma \mid \gamma(x) = \min_j \min_i \{V_i^A(x) + V_j^B(x)\}, V_i^A \in A^*, V_j^B \in B^*\} \\
&= \{\gamma \mid \gamma(x) = \min_i \{V_i^A(x)\} + \min_j \{V_j^B(x)\}, V_i^A \in A^*, V_j^B \in B^*\} \\
&= \{\gamma \mid \gamma = p + q, p \in A^*_{d_0 min}, q \in B^*_{d_0 min}\} \\
&= A^*_{d_0 min} \oplus B^*_{d_0 min} = \{V_6^A, V_7^A\} \oplus \{V_6^B, V_7^B\}
\end{aligned}$$

We can easily see that there are four elements in $\Gamma^*_{d_0 min}$ and all of them lie on a straight line perpendicular to the x axis. Similarly we can show that $\Gamma^*_{d_0 min} = \{V_6^A, V_7^A\} \oplus \{V_6^B, V_7^B\}$; $\Gamma^*_{d_0 max} = \{V_2^A, V_3^A\} \oplus \{V_2^B, V_3^B\}$; $\Gamma^*_{d_2 min} = \{V_0^A, V_1^A\} \oplus \{V_0^B, V_1^B\}$; $\Gamma^*_{d_2 max} = \{V_4^A, V_5^A\} \oplus \{V_4^B, V_5^B\}$; $\Gamma^*_{d_1 max} = \{V_3^A, V_4^A\} \oplus \{V_3^B, V_4^B\}$; $\Gamma^*_{d_1 min} = \{V_0^A, V_7^A\} \oplus \{V_0^B, V_7^B\}$; $\Gamma^*_{d_3 max} = \{V_5^A, V_6^A\} \oplus \{V_5^B, V_6^B\}$; and $\Gamma^*_{d_3 min} = \{V_1^A, V_2^A\} \oplus \{V_1^B, V_2^B\}$.

Consider $\Gamma^*_{d_2max} = A^*_{d_2max} \tilde{\oplus} B^*_{d_2max} = \{V_4^A, V_5^A\} \oplus \{V_4^B, V_5^B\}$. This set has a four elements with same $\gamma(x)$. Hence these four points lie on a straight line segment parallel to the x axis. This segment is the uppermost bounding line of Hull[Γ]. Since the elements of $\Gamma^*_{d_2max}$ lie on a line, Hull[$\Gamma^*_{d_2max}$] is completely defined by the segment's end points. Thus Hull[$\Gamma^*_{d_2max}$] = Hull[γ_1, γ_2] where $\gamma_1 = \{\gamma \mid \gamma(x) = \min_j \gamma(x)\} = \{V_4^A + V_4^B\}$ and $\gamma_2 = \{\gamma \mid \gamma(x) = \max_j \gamma(x)\} = \{V_5^A + V_5^B\}$. Thus Hull[Γ] is bounded on the top by a line segment parallel to the x axis and given by the set of two end points – $\{(V_4^A + V_4^B), (V_5^A + V_5^B)\}$. i.e. Hull[$\Gamma^*_{d_2max}$] = Hull[$(V_4^A + V_4^B), (V_5^A + V_5^B)$]. Similarly the bounding segment on the extreme right is perpendicular to the x axis and is given by Hull[$\Gamma^*_{d_0max}$] = Hull[$(V_2^A + V_2^B), (V_3^A + V_3^B)$] The bounding segment on the upper right corner is given by Hull[$\Gamma^*_{d_1max}$] = Hull[$(V_3^A + V_3^B), (V_4^A + V_4^B)$]

Since the point $(V_3^A + V_3^B)$ is common in Hull[$\Gamma^*_{d_0max}$] and Hull[$\Gamma^*_{d_1max}$] and the point $(V_4^A + V_4^B)$ is common in Hull[$\Gamma^*_{d_1max}$] and Hull[$\Gamma^*_{d_2max}$] , the only bounding segment in the upper right corner is given by Hull[$\Gamma^*_{d_1max}$]. Similarly we can prove that the other bounding segments of Γ are

(*i*) $\text{Hull}[\Gamma^*_{d_2max}] = \text{Hull}[(V_4^A + V_4^B), (V_5^A + V_5^B)]$

(*ii*) $\text{Hull}[\Gamma^*_{d_0max}] = \text{Hull}[(V_2^A + V_2^B), (V_3^A + V_3^B)]$

(*iii*) $\text{Hull}[\Gamma^*_{d_0min}] = \text{Hull}[(V_6^A + V_6^B), (V_7^A + V_7^B)]$

(*iv*) $\text{Hull}[\Gamma^*_{d_2min}] = \text{Hull}[(V_0^A + V_0^B), (V_1^A + V_1^B)]$

(*v*) $\text{Hull}[\Gamma^*_{d_1max}] = \text{Hull}[(V_3^A + V_3^B), (V_4^A + V_4^B)]$

(*vi*) $\text{Hull}[\Gamma^*_{d_3max}] = \text{Hull}[(V_5^A + V_5^B), (V_6^A + V_6^B)]$

(*vii*) $\text{Hull}[\Gamma^*_{d_1min}] = \text{Hull}[(V_0^A + V_0^B), (V_7^A + V_7^B)]$

(*viii*) $\text{Hull}[\Gamma^*_{d_3min}] = \text{Hull}[(V_1^A + V_1^B), (V_2^A + V_2^B)]$

Since the bounding segments of Hull[Γ] are given by the above set of equations, Hull[Γ] is the convex hull of the bounding segments. That is.

$$\begin{aligned}\text{Hull}[\Gamma] &= \text{Hull}[(V_0^A + V_0^B), (V_1^A + V_1^B), \ldots, (V_7^A + V_7^A)] \\ &= \text{Hull}[V_0^D, V_1^D, \ldots, V_7^D] \\ &= \text{Hull}[D^*]\end{aligned}$$

We have thus proved that $\text{Hull}[A^* \oplus B^*] = \text{Hull}[D^*]$.

From this it follows that the n-fold B-Code dilation of A, is given by

$$\begin{aligned}(\oplus A)^n &= A \oplus A \oplus \ldots \oplus A \quad \text{n times} \\ &= \langle (ni_A . nj_A) \mid (d_0 : nn_0^A)(d_1 : nn_1^A) \cdots (d_7 : nn_7^A) \rangle\end{aligned} \tag{3}$$

6. Proof of the Decomposition Algorithm

Proposition 1: There exists a unique set of 13 restricted convex shape $\{K_1, K_2, \ldots, K_{13}\}$ such that any restricted convex shape A is decomposable as $A = K_0 \oplus K_1^{k_1} \oplus K_2^{k_2} \oplus \cdots \oplus K_{13}^{k_{13}}$, where k_i can be zero or positive integers, restricted convex shape K_0 is given by the B-Code $\langle (i_A, j_A) \mid \rangle$, and the $K_i, i \leq 1 \leq 13$ are as given below.

$$\begin{aligned}
K_1 &= \langle (0,0) \mid (d_0:1)(d_4:1) \rangle \\
K_2 &= \langle (0,0) \mid (d_1:1)(d_5:1) \rangle \\
K_3 &= \langle (0,0) \mid (d_2:1)(d_6:1) \rangle \\
K_4 &= \langle (0,0) \mid (d_3:1)(d_7:1) \rangle \\
K_5 &= \langle (0,0) \mid (d_0:1)(d_2:1)(d_5:1) \rangle \\
K_6 &= \langle (0,0) \mid (d_0:1)(d_3:1)(d_6:1) \rangle \\
K_7 &= \langle (0,0) \mid (d_1:1)(d_4:1)(d_6:1) \rangle \\
K_8 &= \langle (0,0) \mid (d_2:1)(d_4:1)(d_7:1) \rangle \\
K_9 &= \langle (0,0) \mid (d_0:2)(d_3:1)(d_5:1) \rangle \\
K_{10} &= \langle (0,0) \mid (d_1:1)(d_4:2)(d_7:1) \rangle \\
K_{11} &= \langle (0,0) \mid (d_1:1)(d_3:1)(d_6:2) \rangle \\
K_{12} &= \langle (0,0) \mid (d_2:2)(d_5:1)(d_7:1) \rangle \\
K_{13} &= \langle (0,0) \mid (d_1:1)(d_3:1)(d_5:1)(d_7:1) \rangle
\end{aligned} \tag{4}$$

These shapes are shown in figure 2.

Proof :

From the dilation rule we see that it is sufficient to prove that there exists a set of k_i that satisfy the following relations

$$\begin{aligned}
n_0^A &= k_1 + k_5 + k_6 + 2k_9 \\
n_1^A &= k_2 + k_7 + k_{10} + k_{11} + k_{13} \\
n_2^A &= k_3 + k_5 + k_8 + 2k_{12} \\
n_3^A &= k_4 + k_6 + k_9 + k_{11} + k_{13} \\
n_4^A &= k_1 + k_7 + k_8 + 2k_{10} \\
n_5^A &= k_2 + k_5 + k_9 + k_{12} + k_{13} \\
n_6^A &= k_3 + k_6 + k_7 + 2k_{11} \\
n_7^A &= k_4 + k_8 + k_{10} + k_{12} + k_{13}
\end{aligned} \tag{5}$$

We can also see that the k_i chosen in such a manner satisfy the equations in (1).

Proposition 2: Any restricted convex shape $A = \langle (i_A, j_A) \mid (d_0 : n_0^A)(d_1 : n_1^A) \cdots (d_7 : n_7^A) \rangle$ can be decomposed as $A = A^{(0)} \oplus K_0$, where $A^{(0)}$ is an restricted convex shape given by the B-Code $\langle (0,0) \mid (d_0 : n_0^A)(d_1 : n_1^A) \cdots (d_7 : n_7^A) \rangle$ and $K_0 = \langle (i_A, j_A) \rangle$

Proof: The above proposition follows immediately from the rule for B-Code dilations.

$$(i_{A^{(0)}}, j_{A^{(0)}}) + (i_{K_0}, j_{K_0}) = (0,0) + (i_A, j_A) = (i_A, j_A)$$

And from the definition it follows $n_i^{A^{(0)}} = n_i^A$. Since $n_i^{A^{(0)}} = n_i^A$, $A^{(0)}$ is just a translated version of A. Hence it is an restricted convex shape satisfying the necessary and sufficient conditions given by equations (1).

Proposition 3: $A^{(0)} = A^{(1)} \oplus K_1^{k_1}$ where $K_1 = \langle (0,0) \mid (d_0 : 1)(d_4 : 1) \rangle$ and $k_1 = \min[n_0^{A^{(0)}}, n_4^{A^{(0)}}]$. and $A^{(1)}$ is an restricted convex shape with

$$\begin{aligned}
(i_{A^{(1)}}, j_{A^{(1)}}) &= (0.0) \\
n_0^{A^{(1)}} &= \begin{cases} n_0^{A^{(0)}} - n_4^{A^{(0)}}, & \text{if } n_0^A \geq n_4^A; \\ 0, & \text{otherwise.} \end{cases} \\
n_4^{A^{(1)}} &= \begin{cases} n_4^{A^{(0)}} - n_0^{A^{(0)}}, & \text{if } n_4^A > n_0^A; \\ 0, & \text{otherwise.} \end{cases} \\
n_i^{A^{(1)}} &= n_i^{A^{(0)}} \text{ if } i \neq 0 \text{ or } 4
\end{aligned}$$

Proof:

Case 1: $n_0^{A^{(0)}} \geq n_4^{A^{(0)}}$ From the dilation rule it follows that $n_0^{A^{(1)}} + k_1(n_0^{K_1}) = [n_0^{A^{(0)}} - n_4^{A^{(0)}}] + \min[n_0^{A^{(0)}}, n_4^{A^{(0)}}] = n_0^{A^{(0)}}$ and $n_4^{A^{(1)}} + k_1(n_4^{K_1}) = 0 + \min[n_0^{A^{(0)}}, n_4^{A^{(0)}}] = n_4^{A^{(0)}}$ From the definition we have $n_i^{A^{(1)}} + k_1(n_i^{K_1}) = n_i^{A^{(0)}}$ if $i \neq 0$ or 4; $(i_{A^{(1)}}, j_{A^{(1)}}) = (i_{A^{(0)}}, j_{A^{(0)}}) = (0,0)$. To prove that $A^{(1)}$ is an restricted convex shape we have to show that the $n_i^{A^{(1)}}$ satisfy the conditions given by the equations in (1). That is, it is sufficient to show that

$$n_0^{A^{(1)}} + n_1^{A^{(1)}} + n_7^{A^{(1)}} = n_3^{A^{(1)}} + n_4^{A^{(1)}} + n_5^{A^{(1)}} \tag{6}$$

$$n_1^{A^{(1)}} + n_2^{A^{(1)}} + n_3^{A^{(1)}} = n_5^{A^{(1)}} + n_6^{A^{(1)}} + n_7^{A^{(1)}} \tag{7}$$

Equation (6) can be rewritten using the definition of $n_i^{A^{(1)}}$ as follows

$$[n_0^{A^{(0)}} - n_4^{A^{(0)}}] + n_1^{A^{(0)}} + n_7^{A^{(0)}} = n_3^{A^{(0)}} + 0 + n_5^{A^{(0)}}$$

that is,

$$n_0^{A^{(0)}} + n_1^{A^{(0)}} + n_7^{A^{(0)}} = n_3^{A^{(0)}} + n_4^{A^{(0)}} + n_5^{A^{(0)}}$$

But the above equation holds since $A^{(0)}$ is an restricted convex shape. Similarly we show that equation (7) holds. QED.

Case 2: $n_4^{A^{(0)}} > n_0^{A^{(0)}}$.

The proof is similar to that of case 1. From the dilation rule it again follows that $n_4^{A^{(1)}} + k_1(n_4^{K_1}) = [n_4^{A^{(0)}} - n_0^{A^{(0)}}] + \min[n_0^{A^{(0)}}, n_4^{A^{(0)}}] = n_4^{A^{(0)}}$ and $n_0^{A^{(1)}} + k_1(n_0^{K_1}) = 0 + \min[n_0^{A^{(0)}}, n_4^{A^{(0)}}] = n_0^{A^{(0)}}$ From the definition we have

$$n_i^{A^{(1)}} + k_1(n_i^{K_1}) = n_i^{A^{(0)}} \text{ if } i \neq 0 \text{ or } 4$$

$$(i_{A^{(1)}}, j_{A^{(1)}}) = (i_{A^{(0)}}, j_{A^{(0)}}) = (0,0)$$

To prove that $A^{(1)}$ is an restricted convex shape we have to show that the $n_i^{A^{(1)}}$ satisfy the conditions given by the equations in (1). That is, it is sufficient to show that

$$n_0^{A^{(1)}} + n_1^{A^{(1)}} + n_7^{A^{(1)}} = n_3^{A^{(1)}} + n_4^{A^{(1)}} + n_5^{A^{(7)}} \tag{8}$$

$$n_1^{A^{(1)}} + n_2^{A^{(1)}} + n_3^{A^{(1)}} = n_5^{A^{(1)}} + n_6^{A^{(1)}} + n_7^{A^{(8)}} \tag{9}$$

Equation (8) can be rewritten using the definition of $n_i^{A^{(1)}}$ as follows

$$0 + n_1^{A^{(0)}} + n_7^{A^{(0)}} = n_3^{A^{(0)}} + [n_4^{A^{(0)}} - n_0^{A^{(0)}}] + n_5^{A^{(0)}}$$

that is,

$$n_0^{A^{(0)}} + n_1^{A^{(0)}} + n_7^{A^{(0)}} = n_3^{A^{(0)}} + n_4^{A^{(0)}} + n_5^{A^{(0)}}$$

But the above equation holds since $A^{(0)}$ is an restricted convex shape. Similarly we show that equation (9) holds. QED.

Proposition 4: $A^{(1)} = A^{(2)} \oplus K_2^{k_2}$ where $K_2 = \langle (0,0) \mid (d_1 : 1)(d_5 : 1) \rangle$ and $k_2 = \min[n_1^{A^{(1)}}, n_5^{A^{(1)}}]$ and $A^{(1)}$ is an restricted convex shape with

$$(i_{A^{(2)}}, j_{A^{(2)}}) = (0,0)$$

$$n_1^{A^{(2)}} = \begin{cases} n_1^{A^{(1)}} - n_5^{A^{(1)}}, & \text{if } n_1^{A^{(1)}} \geq n_5^{A^{(1)}}; \\ 0, & \text{otherwise.} \end{cases}$$

$$n_5^{A^{(2)}} = \begin{cases} n_5^{A^{(1)}} - n_1^{A^{(1)}}, & \text{if } n_5^{A^{(1)}} > n_1^{A^{(1)}}; \\ 0, & \text{otherwise.} \end{cases}$$

$$n_i^{A^{(2)}} = n_i^{A^{(1)}} \text{ if } i \neq 1 \text{ or } 5$$

Proof : The proof is similar to that of Proposition 3. QED.

Proposition 5: $A^{(2)} = A^{(3)} \oplus K_3^{k_3}$ where $K_3 = \langle(0,0) \mid (d_2 : 1)(d_6 : 1)\rangle$ and $k_3 = \min[n_2^{A^{(2)}}, n_6^{A^{(2)}}]$ and $A^{(3)}$ is an restricted convex shape with

$$
\begin{aligned}
(i_{A^{(3)}}, j_{A^{(3)}}) &= (0,0) \\
n_1^{A^{(3)}} &= \begin{cases} n_2^{A^{(2)}} - n_6^{A^{(2)}}, & \text{if } n_2^{A^{(2)}} \geq n_6^{A^{(2)}}; \\ 0, & \text{otherwise.} \end{cases} \\
n_5^{A^{(3)}} &= \begin{cases} n_6^{A^{(2)}} - n_2^{A^{(2)}}, & \text{if } n_6^{A^{(2)}} > n_2^{A^{(1)}}; \\ 0, & \text{otherwise.} \end{cases} \\
n_i^{A^{(3)}} &= n_i^{A^{(2)}} \text{ if } i \neq 2 \text{ or } 6
\end{aligned}
$$

Proof : The proof is similar to that of Proposition 3. QED.

Proposition 6: $A^{(3)} = A^{(4)} \oplus K_4^{k_4}$ where $K_4 = \langle(0,0) \mid (d_3 : 1)(d_7 : 1)\rangle$ and $k_4 = \min[n_3^{A^{(3)}}, n_7^{A^{(3)}}]$ and $A^{(4)}$ is an restricted convex shape with

$$
\begin{aligned}
(i_{A^{(4)}}, j_{A^{(4)}}) &= (0,0) \\
n_3^{A^{(4)}} &= \begin{cases} n_3^{A^{(3)}} - n_7^{A^{(3)}}, & \text{if } n_3^{A^{(3)}} \geq n_7^{A^{(3)}}; \\ 0, & \text{otherwise.} \end{cases} \\
n_7^{A^{(4)}} &= \begin{cases} n_7^{A^{(3)}} - n_3^{A^{(3)}}, & \text{if } n_7^{A^{(3)}} > n_3^{A^{(3)}}; \\ 0, & \text{otherwise.} \end{cases} \\
n_i^{A^{(4)}} &= n_i^{A^{(3)}} \text{ if } i \neq 3 \text{ or } 7
\end{aligned}
$$

Proof : The proof is similar to that of Proposition 3. QED.

Now $A^{(4)}$ is a four sided restricted convex shape with the following properties,

$$
\begin{aligned}
n_i^{A^{(4)}} &\geq 0 \text{ for } 0 \leq i \leq 7) \\
n_0^{A^{(4)}} + n_1^{A^{(4)}} + n_7^{A^{(4)}} &= n_3^{A^{(4)}} + n_4^{A^{(4)}} + n_5^{A^{(4)}} \\
n_1^{A^{(4)}} + n_2^{A^{(4)}} + n_3^{A^{(4)}} &= n_5^{A^{(4)}} + n_6^{A^{(4)}} + n_7^{A^{(4)}} \\
n_0^{A^{(4)}} \text{ or } n_4^{A^{(4)}} &= 0 \\
n_1^{A^{(4)}} \text{ or } n_5^{A^{(4)}} &= 0 \\
n_2^{A^{(4)}} \text{ or } n_6^{A^{(4)}} &= 0 \\
n_3^{A^{(4)}} \text{ or } n_7^{A^{(4)}} &= 0
\end{aligned}
\qquad (10)
$$

It can be easily verified that there are only eight four sided restricted convex shape that satisfy the above equations. They are

$$\langle(0,0) \mid (d_0 : n_0^{A^{(4)}})(d_1 : n_1^{A^{(4)}})(d_3 : n_3^{A^{(4)}})(d_6 : n_6^{A^{(4)}})\rangle \qquad (i)$$
$$\langle(0,0) \mid (d_1 : n_1^{A^{(4)}})(d_2 : n_2^{A^{(4)}})(d_4 : n_4^{A^{(4)}})(d_7 : n_7^{A^{(4)}})\rangle \qquad (ii)$$
$$\langle(0,0) \mid (d_0 : n_0^{A^{(4)}})(d_2 : n_2^{A^{(4)}})(d_3 : n_3^{A^{(4)}})(d_5 : n_5^{A^{(4)}})\rangle \qquad (iii)$$
$$\langle(0,0) \mid (d_1 : n_1^{A^{(4)}})(d_3 : n_3^{A^{(4)}})(d_4 : n_4^{A^{(4)}})(d_6 : n_6^{A^{(4)}})\rangle \qquad (iv)$$
$$\langle(0,0) \mid (d_2 : n_2^{A^{(4)}})(d_4 : n_4^{A^{(4)}})(d_5 : n_5^{A^{(4)}})(d_7 : n_7^{A^{(4)}})\rangle \qquad (v)$$
$$\langle(0,0) \mid (d_0 : n_0^{A^{(4)}})(d_3 : n_3^{A^{(4)}})(d_5 : n_5^{A^{(4)}})(d_6 : n_6^{A^{(4)}})\rangle \qquad (vi)$$
$$\langle(0,0) \mid (d_1 : n_1^{A^{(4)}})(d_4 : n_4^{A^{(4)}})(d_6 : n_6^{A^{(4)}})(d_7 : n_7^{A^{(4)}})\rangle \qquad (vii)$$
$$\langle(0,0) \mid (d_0 : n_0^{A^{(4)}})(d_2 : n_2^{A^{(4)}})(d_5 : n_5^{A^{(4)}})(d_7 : n_7^{A^{(4)}})\rangle \qquad (viii)$$

Now we will solve for $n_i^{A^{(4)}}$ on a case by case basis.

Case (i) : $A^{(4)} = \langle(0,0) \mid (d_0 : n_0^{A^{(4)}})(d_1 : n_1^{A^{(4)}})(d_3 : n_3^{A^{(4)}})(d_6 : n_6^{A^{(4)}})\rangle$

From the set of equations in (5) $A^{(4)} = K_6^{k_6} \oplus K_{11}^{k_{11}}$ since the inclusion of any other restricted convex shape in the decomposition string will produce a non-zero $n_2^{A^{(4)}}, n_4^{A^{(4)}}, n_5^{A^{(4)}}$, or $n_7^{A^{(4)}}$. The following equations then hold $n_0^{A^{(4)}} = k_6$; $n_1^{A^{(4)}} = k_{11}$; $n_3^{A^{(4)}} = k_6 + k_{11}$; $n_6^{A^{(4)}} = k_6 + 2k_{11}$.

These equations provide the values for k_6 and k_{11}.

Similarly we can find out the decomposition for all the other seven cases. We give the results below. The values for k_i's can be also found in a similar fashion.

Case (ii) : $A^{(4)} = K_8^{k_8} \oplus K_{10}^{k_{10}}$.
Case (iii) : $A^{(4)} = K_5^{k_5} \oplus K_9^{k_9}$.
Case (iv) : $A^{(4)} = K_7^{k_7} \oplus K_{11}^{k_{11}}$.
Case (v) : $A^{(4)} = K_8^{k_8} \oplus K_{12}^{k_{12}}$.
Case (vi) : $A^{(4)} = K_6^{k_6} \oplus K_9^{k_9}$.
Case (vii) : $A^{(4)} = K_7^{k_7} \oplus K_{10}^{k_{10}}$.
Case (viii) : $A^{(4)} = K_5^{k_5} \oplus K_{12}^{k_{12}}$.

7. The Decomposition Algorithm

The decomposition proposition states that there exists a unique set of 13 restricted convex shape $\{K_1, K_2, \ldots, K_{13}\}$ such that any restricted convex shape $A = \langle (i_A, j_A) \mid (d_0 : n_0^A)(d_1 : n_1^A)\cdots(d_7 : n_7^A)\rangle$is decomposable as $A = K_0 \oplus K_1^{k_1} \oplus K_2^{k_2} \oplus \cdots \oplus K_{13}^{k_{13}}$ where k_i can be zero or positive integers, restricted convex shape K_0 is given by the B-Code $\langle (i_A, j_A) \mid \rangle$, and the $K_i, i \leq 1 \leq 13$ are as given in (4).

This section gives the details of the algorithm for finding the $K_0, k_1, k_2, \ldots, k_{13}$.

STEP 1 : Check if the B-Code of the restricted convex shape, A, satisfies the following requirements.

(i) $n_i^A \geq 0$ for $0 \leq i \leq 7$

(ii) for some $i, 0 \leq i \leq 7, n_i^A \neq 0$

(iii) $n_0^A + n_1^A + n_7^A = n_3^A + n_4^A + n_5^A$

(iv) $n_1^A + n_2^A + n_3^A = n_5^A + n_6^A + n_7^A$

(v) Set all k_i equal to zero.

If the conditions (i), (iii) and (iv) are not satisfied A is not an restricted convex shape. If (ii) is not satisfied, A is a trivial restricted convex shape, i.e., a solitary point in the $\mathbf{Z} \times \mathbf{Z}$ domain.

STEP 2: Decompose the restricted convex shape A such that

$$A = A^{(4)} \oplus K_0 \oplus K_1^{k_1} \oplus K_2^{k_2} \oplus K_3^{k_3} \oplus K_4^{k_4}$$

where K_i are as defined earlier, k_i are integers, and $A^{(4)}$ is another restricted convex shape. $K_i^{k_i}$ represents restricted convex shape K_i dilated k_i times. The K_i used here are lines at orientations of $0^0, 45^0, 90^0$, and 135^0.

Here the k_i are given as:

$$k_1 = \min[n_0^A, n_4^A]$$
$$k_2 = \min[n_1^A, n_5^A]$$
$$k_3 = \min[n_2^A, n_6^A]$$
$$k_4 = \min[n_3^A, n_7^A]$$

and K_0 is given as:

$$K_0 = < (i_A, j_A) >$$

The restricted convex shape $A^{(4)}$ is now given as:

$$A^{(4)} = < (0,0) | (d_0 : n_0^{A^{(4)}})(d_1 : n_1^{A^{(4)}}) \cdots (d_6 : n_7^{A^{(4)}}) >$$

where

$$n_i^{A^{(4)}} = \max\{(n^A(4)_i - n_{(i+4) \bmod 8}^{A^{(4)}}), 0\}$$

and

$$i = 0, 1, \ldots, 7$$

STEP 3: Count the number of $n_i^{A^{(4)}}$ that are non-zero. The count can be 0, 3, or 4. If count equals 0, goto Step 6; if count equals 3, goto Step 5; if count equals 4, goto Step 4.

STEP 4: $A^{(4)}$ is a 4-sided polygon. The appropriate decomposition is a dilation of two triangles.

IF ($n_0^{A^{(4)}} \neq 0$)

  IF ($n_1^{A^{(4)}} \neq 0$)

    It is case (i) : $A^{(4)} = K_6^{k_6} \oplus K_{11}^{k_{11}}$

    Do the following assignments:

    $k_6 = n_0^{A^{(4)}}$

    $k_{11} = n_{11}^{A^{(4)}}$

  ELSEIF ($n_2^{A^{(4)}} \neq 0$)

    IF ($n_3^{A^{(4)}} \neq 0$)

      It is case (iii) : $A^{(4)} = K_5^{k_5} \oplus K_9^{k_9}$

      Do the following assignments:

      $k_5 = n_2^{A^{(4)}}$

      $k_9 = n_3^{A^{(4)}}$

    ELSE

      It is case (viii) : $A^{(4)} = K_5^{k_5} \oplus K_{12}^{k_{12}}$

      Do the following assignments:

      $k_5 = n_0^{A^{(4)}}$

      $k_{12} = n_7^{A^{(4)}}$

  ELSE

    It is case (vi) : $A^{(4)} = K_6^{k_6} \oplus K_9^{k_9}$

    Do the following assignments:

    $k_6 = n_6^{A^{(4)}}$

    $k_9 = n_5^{A^{(4)}}$

ELSEIF ($n_1^{A^{(4)}} \neq 0$)

  IF ($n_2^{A^{(4)}} \neq 0$)

    It is case (ii) :$A^{(4)} = K_8^{k_8} \oplus K_{10}^{k_{10}}$

    Do the following assignments:

    $k_8 = n_2^{A^{(4)}}$

    $k_{10} = n_1^{A^{(4)}}$

  ELSEIF ($n_3^{A^{(4)}} \neq 0$)

    It is case (iv) : $A^{(4)} = K_7^{k_7} \oplus K_{11}^{k_{11}}$

    Do the following assignments:

    $k_7 = n_4^{A^{(4)}}$

    $k_{11} = n_3^{A^{(4)}}$

  ELSE

    It is case (vii) : $A^{(4)} = K_7^{k_7} \oplus K_{10}^{k_{10}}$

    Do the following assignments:

    $k_7 = n_6^{A^{(4)}}$

    $k_{10} = n_7^{A^{(4)}}$

ELSE

  It is case (v) : $A^{(4)} = K_8^{k_8} \oplus K_{12}^{k_{12}}$

  Do the following assignments:

  $k_8 = n_4^{A^{(4)}}$

  $k_{12} = n_5^{A^{(4)}}$

  GOTO Step 6

STEP 5: $A^{(4)}$ is a 3-sided polygon. Find which triangle $K_i, 5 \leq i \leq 12$ it is and the number of times (k_i) it is dilated.

IF ($n_0^{A^{(4)}} \neq 0$)

  IF ($n_2^{A^{(4)}} \neq 0$)

    $A^{(4)} = K_5^{k_5}$

    Do the following assignment:

    $k_5 = n_0^{A^{(4)}}$

  ELSEIF ($n_5^{A^{(4)}} \neq 0$)

    $A^{(4)} = K_9^{k_9}$

    Do the following assignment:

    $k_9 = n_5^{A^{(4)}}$

  ELSE

    $A^{(4)} = K_6^{k_6}$

    Do the following assignment:

    $k_6 = n_0^{A^{(4)}}$

ELSEIF ($n_1^{A^{(4)}} \neq 0$)

  IF ($n_7^{A^{(4)}} \neq 0$)

    $A^{(4)} = K_{10}^{k_{10}}$

    Do the following assignments:

    $k_{10} = n_1^{A^{(4)}}$

  ELSEIF ($n_3^{A^{(4)}} \neq 0$)

    $A^{(4)} = K_{11}^{k_{11}}$

    Do the following assignments:

    $k_{11} = n_1^{A^{(4)}}$

  ELSE

    $A^{(4)} = K_7^{k_7}$

    Do the following assignments:

    $k_7 = n_1^{A^{(4)}}$

ELSEIF ($n_4^{A^{(4)}} \neq 0$)

  $A^{(4)} = K_8^{k_8} \oplus K_{12}^{k_{12}}$

  Do the following assignments:

  $k_8 = n_4^{A^{(4)}}$

ELSE

  $A^{(4)} = K_{12}^{k_{12}}$

  Do the following assignments:

  $k_{12} = n_5^{A^{(4)}}$

STEP 6: STOP : The values for the k_i have been obtained.

8. Complexity of the Algorithms

- The algorithm given for the B-code dilation of the restricted convex shapes (2) consists of 10 additions only. Hence it is finite time.

- The algorithm for an n-fold dilation of restricted convex shapes(3) consists of 10 multiplications. Hence this too is finite time.
- The algorithm for the structuring element decomposition consists of only assignment statements and no loops. Thus this algorithm is also finite time.

9. Summary and Future Work

An technique for representing a restricted convex shape was introduced. Algorithms for morphological dilation, n-fold dilation, and structuring element decomposition were provided. The proofs of the algorithms were also provided and the time complexity of the algorithms was shown to be constant. Suggestions have been made as how the algorithm can be generalized to and arbitary 2-D convex shape and then to 3-d convex shape.

Many extensions to the work presented here are being tried out. Some of the salient ones are:

- An algorithm for B-Code erosion of restricted domains is being worked out. With that, opening and closing of restricted domains will automatically follow, and hence all the basic operations of Morphology will be defined on the restricted domains for shapes represented in B-Codes. Also, given the B-Codes, all these operations will be of constant time complexity.
- This algorithm can be generalized for the case of any convex figure. In that case the polygon edges can be at any angle. These angles can be defined in terms of the basic angles that can be formed by a vector starting from the origin and ending on any pixel (m, n) such that m and n are coprime.
- These algorithms can be extended to the case of 3-D restricted domain and then to any convex polyhedra. But a new datastructure will be necessary for representing the polyhedra.

10. References

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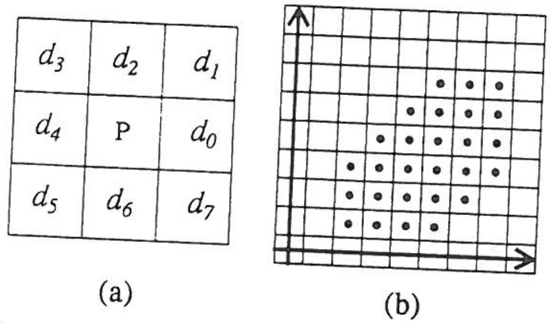

Figure 1: (a) The displacement directions from lattice point P. (b) A restricted convex shape corresponding to the B-Code:< (2,1) | $(d_0 : 3)(d_1 : 2)(d_2 : 3)(d_4 : 2)(d_5 : 3)(d_6 : 2)$ >

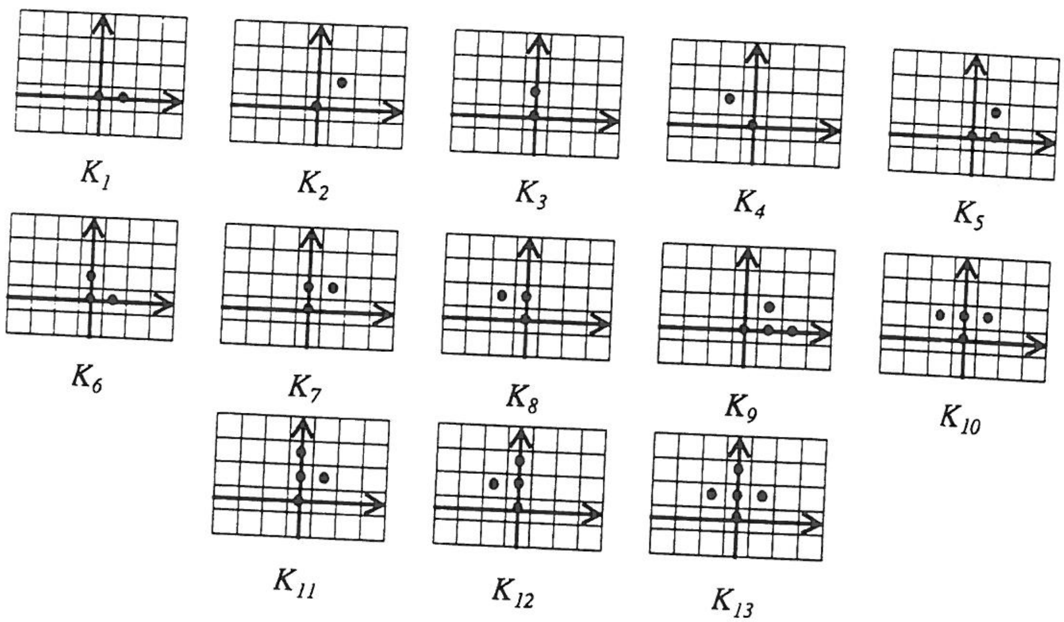

Figure 2: The thirteen basis shapes corresponding to equation 4.