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Analysis and Solutions of The Three Point Perspective Pose Estimation Problem

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ABSTRACT

In this paper, the major direct solutions to the three point perspective pose estimation problems are reviewed from a unified perspective beginning with the first solution which was published in 1841 by a German mathematician, continuing through the solutions published in the German and then American photogrammetry literature, and most recently in the current computer vision literature. The numerical stability of these three point perspective solutions are discussed. We show that even in case where the solution is not near the geometric unstable region considerable care must be exercised in the calculation. Depending on the order of the substitutions utilized, the relative error can change over a thousand to one. This difference is due entirely to the way the calculations are performed and not due to any geometric structural instability of any problem instance. We present an analysis method which produces a numerically stable calculation.

1. Introduction

Given the perspective projection of three points constituting the vertices of a known triangle in 3D space, it is possible to determine the position of each of the vertices. There may be as many as four possible solutions for point positions in front of the center of perspectivity and four corresponding solutions whose point positions are behind the center of perspectivity. In photogrammetry, this problem is called the three point space resection problem. This problem is an important problem in photogrammetry as well as in computer vision. It was solved by a direct solution first by a German mathematician in 1841 and then refined by German photogrammatrists in 1904 and 1925. Then it was independently solved by an American photogrammatrist in 1949.

In this paper, first, we give a consistent treatment of all the major direct solutions to the three point pose estimation problem. There is a bit of mathematical tedium in describing the various solutions, and perhaps it is worthwhile to put them all in one place so that another vision researcher can be saved from having to redo the tedium himself or herself. Second, we discuss the numerical stability of each of these solutions. Some of solutions have a singular point or region during the derivation. Calculation near the singularity will cause serious error. However, the probability of occurrence is very scarce. Each of the six solutions begins from three equations generated by the law of cosines.

We show that the order of using these equations to derive the final solution affects the accuracy of numerical results. In order to determine how to obtain a reasonably good accuracy by choice of appropriate equation the best accuracy from different manipulation order, we review some analysis methods to analyze the first solution. The analysis methods focus on the rounding error and the characteristic zero of a derived polynomial. We discuss an experimental protocol and summarize the results. The results show that our stability analysis technique is effective in determining equation order manipulation.

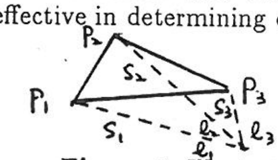

Figure 1 *Illustrates the geometry of the three point space resection problem.*

2. The Problem Definition

Grunert (1841) appears to have been the first one to solve the problem. The solution he gives is outlined by Müller (1925). The problem can be set up in the following way which is illustrated in Figure 1.

Let the unknown positions of the three points of the known triangle be p_1, p_2, and p_3; $p_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}, i = 1,2,3$. Let the known side lengths of the triangle be $a = ||p_2 - p_3||, b = ||p_1 - p_3||, c = ||p_1 - p_2||$. We take the origin of the camera coordinate frame to be the center of perspectivity and the image projection plane to be a distance f in front of the center of perspectivity. Let the observed perspective projection of p_1, p_2, p_3 be q_1, q_2, q_3, respectively; $q_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}, i = 1,2,3$. By the perspective equations, $u_i = f\frac{x_i}{z_i}$, $v_i = f\frac{y_i}{z_i}$. The unit vectors j_1, j_2, j_3 pointing from the center of perspectivity to the observed points p_1, p_2, p_3 are given by $\frac{1}{\sqrt{u_i^2+v_i^2+f^2}} \begin{pmatrix} u_i \\ v_i \\ f \end{pmatrix}, i = 1,2,3$ respectively. The center of perspectivity together with the three points of the 3D triangle form a tetrahedron. Let the angles at the center of perspectivity opposite sides a, b, c be α, β, and γ. These angles are given by $\cos\alpha = j_2 \cdot j_3$, $\cos\beta = j_1 \cdot j_3$, $\cos\gamma = j_1 \cdot j_2$.

Let the unknown distances of the points p_1, p_2, p_3 from the center of perspectivity be s_1, s_2, and s_3, respectively. Then $s_i = ||p_i||, i = 1,2,3$. To determine the position of the points p_1, p_2, p_3 with respect to the camera reference frame, it is sufficient to determine s_1, s_2, and s_3 since $p_i = s_i j_i$, $i = 1,2,3$.

3. The Solutions

Grunert (1841) proceeded in the following way. By the law of cosines,

$$s_2^2 + s_3^2 - 2s_2 s_3 \cos\alpha = a^2 \tag{1}$$

$$s_1^2 + s_3^2 - 2s_1 s_3 \cos\beta = b^2 \tag{2}$$

$$s_1^2 + s_2^2 - 2s_1 s_2 \cos\gamma = c^2 \tag{3}$$

$$\text{Let } s_2 = us_1 \text{ and } s_3 = vs_1. \tag{4}$$

Then,

$$s_1^2 = \frac{a^2}{u^2 + v^2 - 2uv\cos\alpha} = \frac{b^2}{1 + v^2 - 2v\cos\beta} = \frac{c^2}{1 + u^2 - 2u\cos\gamma} \tag{5}$$

from which

$$u^2 + \frac{b^2 - a^2}{b^2}v^2 - 2uv\cos\alpha + \frac{2a^2}{b^2}v\cos\beta - \frac{a^2}{b^2} = 0 \tag{6}$$

$$u^2 - \frac{c^2}{b^2}v^2 + 2v\frac{c^2}{b^2}\cos\beta - 2u\cos\gamma + \frac{b^2 - c^2}{b^2} = 0. \tag{7}$$

From (6)

$$u^2 = -\frac{b^2 - a^2}{b^2}v^2 + 2uv\cos\alpha - \frac{2a^2}{b^2}v\cos\beta + \frac{a^2}{b^2}.$$

This expression for u^2 can be substituted into (7). This permits a solution for u to be obtained in terms of v.

$$u = \frac{(-1 + \frac{a^2-c^2}{b^2})v^2 - 2(\frac{a^2-c^2}{b^2})\cos\beta v + 1 + \frac{a^2-c^2}{b^2}}{2(\cos\gamma - v\cos\alpha)} \tag{8}$$

This expression for u can then be substituted back into (6) to obtain a fourth order polynomial in v.

$$A_4 v^4 + A_3 v^3 + A_2 v^2 + A_1 v + A_0 = 0 \tag{9}$$

This fourth order polynomial equation can have as many as four real roots.

Finsterwalder (1903) as summarized by Finsterwalder and Scheufele (1937) proceeded in a manner which required only finding a root of a cubic polynomial and the roots of two quadratic polynomials rather than finding all the roots of a fourth order polynomial. Finsterwalder multiplies (7) by λ and adds the result to (6) to produce

$$Au^2 + 2Buv + Cv^2 + 2Du + 2Ev + F = 0 \tag{10}$$

where the coefficients depend on λ. $A = 1 + \lambda$, $B = -\cos\alpha$, $C = \frac{b^2-a^2}{b^2} - \lambda\frac{c^2}{b^2}$, $D = -\lambda\cos\gamma$, $E = \left(\frac{a^2}{b^2} + \lambda\frac{c^2}{b^2}\right)\cos\beta$, $F = \frac{-a^2}{b^2} + \lambda\left(\frac{b^2-c^2}{b^2}\right)$.

Finsterwalder considers this as a quadratic equation in v. Solving for v,

$$v = \frac{-(Bu + E) \pm \sqrt{(B^2 - AC)u^2 + 2(BE - CD)u + E^2 - CF}}{C} \tag{11}$$

Now Finsterwalder asks, can a value for λ be found which makes $(B^2 - AC)u^2 + 2(BE - CD)u + E^2 - CF$ be a perfect square. For in this case v can be expressed as a first order polynomial in terms of u. The geometric meaning of this case is that the solution to (10) corresponds to two intersecting lines. This first order polynomial can then be substituted back into (6) or (7) either one of which yields a quadratic equation which can be solved for u, and then using the just determined value for u in the first order expression for v, a value for v can be determined. Four solutions are produced since there are two first order expressions for v and when each of them is substituted back into (6) or (7) the resulting quadratic in u has two solutions.

The value of λ which produces a perfect square satisfies

$$G\lambda^3 + H\lambda^2 + I\lambda + J = 0 \tag{12}$$

where $G = c^2(c^2\sin^2\beta - b^2\sin^2\gamma)$, $H = b^2(b^2 - a^2)\sin^2\gamma + c^2(c^2 + 2a^2)\sin^2\beta + 2b^2c^2(-1 + \cos\alpha\cos\beta\cos\gamma)$, $I = b^2(b^2 - c^2)\sin^2\alpha + a^2(a^2 + 2c^2)\sin^2\beta + 2a^2b^2(-1 + \cos\alpha\cos\beta\cos\gamma)$, $J = a^2(a^2\sin^2\beta - b^2\sin^2\alpha)$. Solve this equation for any root λ_0. This determines the coefficients A, B, C, D, E and F. The v of (11) can be in terms of the first order variable u. There are two equations. Substituting v back into (7) and simplifying there results a second order equation $Uu^2 + Vu + W = 0$. The numerically stable way to calculate u is to compute the smaller root in terms of the larger root. Thus, $u_{large} = \frac{-sgn(V)}{U}\left[|V| + \sqrt{V^2 - UW}\right]$ and $u_{small} = \frac{W}{U u_{large}}$.

Merritt (1949) unaware of the German solutions also obtained a fourth order polynomial. Smith (1965) gives the following derivation for Merritt's polynomial. He multiplies (1) by b^2, multiplies (2) by a^2 and subtracts to obtain one equation. Similarly, he multiplies (1) by c^2, multiplies (3) by a^2 and subtracts to obtain the other. Then using the substitution of (4) we obtain the following two equations.

$$-b^2u^2 + (a^2 - b^2)v^2 - 2a^2\cos\beta v + 2b^2\cos\alpha uv + a^2 = 0 \quad (13)$$

$$(a^2 - c^2)u^2 - c^2v^2 - 2a^2\cos\gamma u + 2c^2\cos\alpha uv + a^2 = 0. \quad (14)$$

From (13), $v^2 = \frac{2a^2\cos\beta v - 2b^2\cos\alpha uv + b^2u^2 - a^2}{a^2 - b^2}$. Substituting this expression for v^2 into (14) and simplifying to obtain a eqaution in terms of u^2, u, v. Solving for v and substituting this expression for v into (14) produces the fourth order polynomial equation

$$B_4u^4 + B_3u^3 + B_2u^2 + B_1u + B_0 = 0 \quad (15)$$

Fischler and Bolles (1981) were apparently not aware of the eariler American or earlier German solutions to the problem. From (5), they obtain

$$\left(1 - \frac{a^2}{b^2}\right)v^2 + 2\left(\frac{a^2}{b^2}\cos\beta - \cos\alpha u\right)v + u^2 - \frac{a^2}{b^2} = 0 \quad (16)$$

$$v^2 + 2(-\cos\alpha u)v + \left(1 - \frac{a^2}{c^2}\right)u^2 + \frac{2a^2}{c^2}\cos\gamma u - \frac{a^2}{c^2} = 0. \quad (17)$$

(16) is identical to (6) but (17) is different from (7) since it arises by manipulating a different pair of equations than was used to obtain (6).

Multiplying (16) by $(1 - \frac{a^2}{c^2})u^2 + \frac{2a^2}{c^2}\cos\gamma u - \frac{a^2}{c^2}$, and multiplying (17) by $u^2 - \frac{a^2}{b^2}$ and subtracting to produce one equation. Similarly, multiplying (17) by $\left(1 - \frac{a^2}{b^2}\right)$ and subtracting from (16) to produce the other. Finally, multiplying the first equation by $2c^2(\cos\alpha u - \cos\beta)$, multiplying the second equation by $[(a^2 - b^2 - c^2)u^2 + 2(b^2 - a^2)\cos\gamma u + (a^2 - b^2 + c^2)]$ and subtracting to eliminate v. This produces the fourth order polynomial equation

$$D_4u^4 + D_3u^3 + D_2u^2 + D_1u + D_0 = 0 \quad (18)$$

where $D_4 = 4b^2c^2\cos^2\alpha - (a^2 - b^2 - c^2)^2$, $D_3 = -4c^2(a^2 + b^2 - c^2)\cos\alpha\cos\beta - 8b^2c^2\cos^2\alpha\cos\gamma + 4(a^2 - b^2 - c^2)(a^2 - b^2)\cos\gamma$, $D_2 = 4c^2(a^2 - c^2)\cos^2\beta + 8c^2(a^2 + b^2)\cos\alpha\cos\beta\cos\gamma + 4c^2(b^2 - c^2)\cos^2\alpha - 2(a^2 - b^2 - c^2)(a^2 - b^2 + c^2) - 4(a^2 - b^2)^2\cos^2\gamma$, $D_1 = -8a^2c^2\cos^2\beta\cos\gamma - 4c^2(b^2 - c^2)\cos\alpha\cos\beta - 4a^2c^2\cos\alpha\cos\beta + 4(a^2 - b^2)(a^2 - b^2 + c^2)\cos\gamma$, $D_0 = 4a^2c^2\cos^2\beta - (a^2 - b^2 + c^2)^2$. Corresponding to each of the four roots of (18) for u there is an associated value for v through (16) or (17).

Grafarend, Lohse, and Schaffrim (1988) aware of all the previous work, except for the Fischler-Bolles solution, proceed in the following way. They begin with equations (1),(2), and (3) and seek to reduce them to a homogeneous form. After multiplying (3) by $\frac{-a^2}{c^2}$ and adding the result to (1) there results one equation. After multiplying (3) by $-\frac{b^2}{c^2}$ and adding the result to (2), there results the other. Next they use the same idea as Finsterwalder. They multiply the first equation by λ and from it subtract the second equation to produce

$$\begin{pmatrix} s_1 & s_2 & s_3 \end{pmatrix} A \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = 0 \quad (19)$$

where $A = \begin{pmatrix} \frac{a^2 - \lambda(b^2 - c^2)}{c^2} & \frac{(\lambda b^2 - a^2)\cos\gamma}{c^2} & -\lambda\cos\beta \\ \frac{(\lambda b^2 - a^2)\cos\gamma}{c^2} & \frac{a^2 - c^2 - \lambda b^2}{c^2} & \cos\alpha \\ -\lambda\cos\beta & \cos\alpha & -1 + \lambda \end{pmatrix}$.

Now, as Finsterwalder did, they seek a value of λ which makes the determinant of A zero. Setting the determinant of A to zero produces a cubic for λ. For this value of λ the solution to (19) becomes a pair of planes intersecting at the origin.

They let $p = s_2/s_1$ and $q = s_3/s_1$ and rewrite the homogeneous (19) in s_1, s_2, and s_3 as a non-homogeneous equation in p and q.

$$(a^2 - c^2 - \lambda b^2)p^2 + 2c^2\cos\alpha pq + c^2(-1 + \lambda)q^2 + 2(-a^2 + \lambda b^2)\cos\gamma p - 2\lambda c^2\cos\beta q + a^2 - \lambda(b^2 - c^2) = 0 \quad (20)$$

Now since $|A| = 0$, and assuming $\begin{vmatrix} c^2\cos\alpha & c^2(-1+\lambda) \\ a^2 - c^2 - \lambda b^2 & c^2\cos\alpha \end{vmatrix} \neq 0$ a value for (p, q) exists such that (20) can be written in the homogeneous form, rotating the coordinate system by an angle, and then generating a pair of straight lines intercepting in p, q plane to solve λ. Finally, they solve p and q, and then s_1, s_2, s_3.

However, a simple method is proposed by Lohse [Lohse 1989]. Instead of translating and rotating (20). One can solve the quadratic equation in (20) to get p and q relation by using different λ. Once the relation of p and q obtained it can be substituted into (19) to solve s_1. There are 15 possible solutions. Since we are only interested in real solution, we only use real λ to solve (20).

Linnainmaa, Harwood, and Davis (1988) give another direct solution. They begin with (1), (2), and (3) and make a change of variables $s_2 = u + \cos\gamma s_1$ $s_3 = v + \cos\beta s_1$. (2) and (3) become $(1 - \cos^2\beta)s_1^2 + v^2 = b^2$ $(1 - \cos^2\gamma)s_1^2 + u^2 = c^2$. Substituting above equations into (1) there results

$$s_1^2(2\cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma + 2\cos^2\beta - 2) - 2\cos\alpha uv + c^2 + b^2 - a^2 + 2us_1(\cos\gamma - \cos\alpha\cos\beta) + 2vs_1(\cos\beta - \cos\alpha\cos\gamma) = 0. \quad (21)$$

Letting $q_1 = 1 - \cos^2\gamma$, $q_2 = 1 - \cos^2\beta$, $q_3 = 2(\cos^2\gamma - \cos\alpha\cos\beta\cos\gamma + \cos^2\beta - 1)$, $q_4 = c^2 + b^2 - a^2$, $q_5 =$

$2(\cos\alpha\cos\beta - \cos\gamma)$, $q_6 = 2(\cos\alpha\cos\gamma - \cos\beta)$, there results

$$q_1 s_1^2 + u^2 = c^2 \quad (22)$$

$$q_2 s_1^2 + v^2 = b^2 \quad (23)$$

$$q_3 s_1^2 - 2\cos\alpha uv + q_4 = q_5 u s_1 + q_6 v s_1. \quad (24)$$

Then they square (24) and simplify, obtaining

$$r_1 s_1^4 + r_2 s_1^2 + r_3 = (r_4 s_1^2 + r_5) uv \quad (25)$$

where $r_1 = q_3^2 + 4q_1q_2\cos^2\alpha + q_1q_5^2 + q_2q_6^2$, $r_2 = 2q_3q_4 - 4(c^2q_2 + b^2q_1)\cos^2\alpha - c^2q_5 - b^2q_6$, $r_3 = q_4^2 + 4\cos^2\alpha b^2c^2$, $r_4 = 4\cos\alpha q_3 + 2q_5q_6$, $r_5 = 4\cos\alpha q_4$.

Then to eliminate the uv term, they square (25) and simplify to obtain

$$t_1 s_1^8 + t_2 s_1^6 + t_3 s_1^4 + t_4 s_1^2 + t_5 = 0 \quad (26)$$

where $t_1 = r_1^2 - r_4^2 q_1 q_2$, $t_2 = 2r_1r_2 + (b^2q_1 + c^2q_2)r_4^2 - 2r_4r_5q_1q_2$, $t_3 = r_2^2 - b^2c^2r_4^2 + 2r_4r_5b^2q_1 + 2r_4r_5c^2q_2 - r_5^2q_1q_2 + 2r_1r_3$, $t_4 = 2r_2r_3 - 2b^2c^2r_4r_5 + b^2q_1r_5^2 + c^2q_2r_5^2$, $t_5 = r_3^2 - r_5^2b^2c^2$.

(26) is considered as a 4th degree equation in s_1^2. Since s_1 must be positive, there are at most 4 solutions to (26). Once a value for s_1 has been determined, (22) and (23) can be solved for two values of u and v. Once s_1, u, v are solved, the s_2 and s_3 can immediately solved.

4. The Numerical Accuracy of the Solutions

4.1 The Problem Definition

The behavior of numerical calculations are different for the different solution techniques. In this section we present several analysis methods to compare the performance of the six solutions. In the first solution we form a pair of equations, (6) and (7) from (5). But we still can form another two pairs of equations from (5). In further manipulations we still can change the order of the equation manipulation or variables. These changes in the order of equation manipulation may affect the numerical accuracy of final results. The summary of the features and the singularity of six solutions is listed in Table I. In the experiments we show the effects of the order of equation manipulation by preordering the three input corresponding 2D perspective projection 3D points in the six different possible permutation.

4.2 The Analysis

In this section we present several analysis methods to compare the performance of the six solutions. They include the histogram of the mean absolute error which will be defined in the experiment section, the numerical relative and absolute errors, and the drift of polynomial zeros. We are mainly concerned about how the manipulation order affects the rounding error propagation and the computed roots of the polynomial. Since both absolute rounding error and relative rounding error may affect the final accuracy, we consider both factors.

4.2.1 The Sensitivity Analysis of Polynomial Zeros

The global accuracy is affected by the side lengths, the angles at the center of perspectivity with respect to side lengths, and the permutation order in which the input data is given. These effects appear in the coefficients of the computed polynomial and affect the stability of the zeros of the polynomial. The sensitivity of the zeros of a polynomial with respect to a change in the coefficients is best derived by assuming the zero location is a function of the coefficients (Vlach and Singhal 1983). Thus for j-th zero z_j of the polynomial $P(a_0, a_1, ..., a_n, x) = a_n x^n + ... + a_1 x + a_0$ we represent $P(a_0, a_1, ..., a_n, x(a_0, a_1, ..., a_n))|_{x=z_j} = 0$. Differentiating with respect to a_i gives $\frac{\partial P}{\partial a_i} + \frac{\partial P}{\partial x}\frac{dx}{da_i}|_{x=z_j} = 0$. Rearranging the equation gives $\frac{dx}{da_i}|_{x=z_j} = \frac{-\frac{\partial P}{\partial a_i}}{\frac{\partial P}{\partial x}}|_{x=z_j}$, where $a_0, a_1, ..., a_n$ are the coefficients of the polynomial, z_j is the j-th zero of polynomial.

Consider the total sensitivity, S, of all the coefficients on a particular zero. We have $S = \frac{dx}{da}|_{x=z_j} = \sum_{i=1}^{n} \frac{-\frac{\partial P}{\partial a_i}}{\frac{\partial P}{\partial x}}_{x=z_j}$. To avoid the cancellation among positive and negative terms, we take the absolute value of each term and consider the worst case. We express the worst sensitivity S_w by $S_w = \sum_{i=1}^{n} \left| \frac{-\frac{\partial P}{\partial a_i}}{\frac{\partial P}{\partial x}} \right|_{x=z_j}$. A large sensitivity of the zero with respect to the coefficients may lead to a large error in the final result. Laguerre's method is used to find the zeros of polynomial. The accuracy for the iterative stop criterion is the rounding error of the machine.

4.2.2 The Numerical Stability

In order to study how large a rounding absolute error can be produced by the mathematical operation, we will calculate the worst absolute and relative error for each kind of arithmetic operation. Hence, the rounding error produced by a mathematical operation on two numbers which themselves have rounding error or truncation error can be found in (Wilkinson 1963). We define a sequence $< OP_1, OP_2, ..., OP_{n-1} >$ of binary mathematical operators from the class of addition, subtraction, multiplication and division applied to a series of numbers $(x_1, x_2, ..., x_n)$ two at a time is given as $OP_{i=1}^{n-1}(x_i, x_{i+1}) f(\epsilon_{x_i}, \epsilon_{x_{i+1}}, \epsilon_r) = \hat{x}(1 + \epsilon_{total})$, where f is a function of $\epsilon_{x_i}, \epsilon_{x_{i+1}}$ *and* ϵ_r, $\hat{x}$ is the result of the operation assuming infinite precision computation and ϵ_{total} is the total relative error propagated from the first operation to the last operation. Hence, $\hat{x}(1+\epsilon_{total})$ is the result of the calculation using finite precision. Similarly, ϵ_{x_i} is the relative error of x_i; $\epsilon_{x_{i+1}}$ is the relative error of x_{i+1}.

We consider the worst case for each operation, i.e. $\epsilon_r = 0.5 \times 10^{1-d}$. Thus, the worst relative rounding error(ϵ_{wrre}) is expressed by $\epsilon_{wrre} = \epsilon_{total}$ and the worst absolute rounding error(ϵ_{ware}) is given by $\epsilon_{ware} = \hat{x} \times \epsilon_{total}$. The ϵ_{ware} and ϵ_{wrre} will be accumulated for each of the coefficient.

4.2.3 Polynomial Zero Drift

The zero sensitivity helps us to understand how a permutation of the polynomial coefficients affects the zeros. The worst relative and absolute error provide a quantitative measurement of errors. The drift of a polynomial zero from its correct value depends on both sensitivity and error variation. Define the normalized sensitivity S^x_{ai} of zero with respect to a coefficient by $S^x_{a_i} = \frac{a}{x}\frac{\partial x}{\partial a_i}$ and the function x $x|_{x=z_j} = x(a_0, a_1, ..., a_n)$. Then, the worst normalized sensitivity (S_{wn}) is given by $S_{wn} = \sum_{i=0}^{n} |S^x_{a_i}|$. The polynomial zero drift can be expressed as $dx|_{x=z_j} = \sum_{i=0}^{n} \frac{\partial x}{\partial a_i} da_i$. Dividing both sides of the previous equation by x and in terms of normalized sensitivity we obtain $\frac{dx}{x}|_{x=z_j} = \sum_{i=0}^{n} S^x_{a_i}\frac{da_i}{a_i}$. Consider the worst absolute drift case due to the absolute rounding error (ϵ_{sware}) we have $\epsilon_{sware} = \sum_{i=0}^{n} |S \times \epsilon_{ware}|$ and the worst relative drift case due to the relative rounding error (ϵ_{swrre}) we have $\epsilon_{swrre} = \sum_{i=0}^{n} |S^x_{a_i} \times \epsilon_{wrre}|$. As discused above the final error is expected in proportion to the value of the worst drift ϵ_{swrre} and ϵ_{sware}.

5. The Experiments

To characterize the numerical sensitivity of each three point perspective solution, we performed experiments to compare the stability of six different 3 point resection solutions and analyze the Grunert solution.

5.1 Test Data Generation and Permutation

The coordinates of the vertices of the 3D triangle are randomly generated by a uniform random number generator. The range of the x, y, and z coordinates are within [-25, 25], [-25, 25], and $[f + a, b]$ respectively. Since the image plane is located in front of camera at the distance of focal length, f, the z coordinate must be larger than the focal length. So $a \geq 0$ and $b \geq f + a$. The a and b are used as parameters to test the solution under different sets of depth. To permute the test data assume the original order of vertices is 123 for vertex one, vertex two and vertex three, respectively, then the other five permutations are 312, 231, 132, 321, and 213. The permutation of triangle vertices means permuting in a consistent way to the 3D triangle side lengths, the 3D vertices and the corresponding 2D perspective projection vertices.

5.2 The Design of Experiments

We summarize the parameters in the experiments discussed in the section 4 and present the experimental procedures of experiments. The parameters and methods involved in accuracy and picking the best permutation are N_1, the number of trials = 10000, N_2, the number of trials = 100000, P, different number of precisions = 2, d_1, the number of the different depths = 2, d_2, the number of the different depths = 3, S_w, S_{wn}, ϵ_{ware}, ϵ_{wrre}, ϵ_{sware}, and ϵ_{swrre}.

5.2.1 The Design Procedures

The experimental procedures and the characteristics to be studied are itemized as follows:

Step 0. Do the following steps N times.

Step 1. Generate the coordinates of vertices of the 3D triangle. The values of z_i coordinate can be between 1 and 5, 5 and 20, or 25 and 75.

Step 2. For single and double precision do the calculation.

Step 3. Permutation of the vertices.

Step 4. For any of the resection techniques, determine the location of the 3D vertices if the calculation can succeed.

Step 4.1. For any calculation which has succeeded record the absolute distance error associated with each permutation. The mean absolute distance error is defined as $\epsilon_\mu = \sum_{i=1}^{n} \frac{\epsilon_i}{n}$ where n is the number of experiments and ϵ_i in the Euclidean distance between the three coordinates of the vertices and the computed coordinates of the vertices. The error standard deviation is expressed as $sd = \sqrt{\frac{\sum_{i=1}^{n} (\epsilon_i - \epsilon_\mu)^2}{(n-1)}}$

Step 5. The following procedures are only applied to Grunert's solution

Step 5.1. Calculate the sensitivity of zero w.r.t. each coefficient and total sensitivity for all coefficients based on the discussion in section 4.2.3.

Step 5.2. Calculate the worst absolute and relative rounding error for each coefficient based on the discussion in section 4.2.4. The number of significant digits is the same as the mantissa representation of machine for multiplication and division. For addition and subtraction the possible lost significant digits in operation must be checked.

Step 5.3. Calculate the polynomial zero drift.

Step 5.4. Record the values of S_w, S_{wn}, ϵ_{ware}, ϵ_{wrre}, ϵ_{swrre}, ϵ_{sware} for each permutation.

Step 5.5. Based on the smallest value of $ware$, $wrre$, S_w, S_{wn}, $sware$, or $swrre$ pick the corresponding error generated by the corresponding permutation.

Step 6. Redo the whole procedure again by changing N_1

to N_2 and d_1 to d_2 and use Grunert's solution only. If the largest absolute distance error is greater than 10^{-7} redo steps 5 and record the corresponding values for the large error cases.

6. Results and Discussion

The software is coded in the C language and the experiments are run on both a Sun 3/280 workstation and a Vax 8500 computer. Unless stated otherwise, the results in the following paragraphs are obtained from the Sun 3/280. Table II shows the results of random permutation of six different solutions. From Table II we find that Finsterwalder's solution gives the best accuracy and Merritt's solution gives the worst result. The reasons for the better results can be explained in terms of the order of polynomial and the complexity of computation. Finsterwalder's solution only needs to solve a third order polynomial, but Linnainmaa's solution generates an eighth order polynomial. The higher order polynomial calculation tends to be less numerically stable. However, Merritt's solution also converts the fourth order polynomial problem into a third order polynomial problem, but it gives a worse result. This is because the conversion process itself is not the most numerical stable.

Generally speaking, the double precision results are about 10^8 times better than the results of single precision. Table III shows the best mean absolute distance error and the worst mean absolute distance error of six permutations for the double precision. The best results are about 10^4 times better than the worst results. The best permutation of Finsterwalder's solution, Grunert's solution and Fischler's solution give the better accuracy.

Because Grunert's solution has the second best accuracy and most easy to analyze, we use it to demonstrate how analysis methods can discriminate the worst and the best from the six permutations. The analysis methods can be applied to the other solution techniques as well.

The fraction of times of six selection techniques select the data permutation giving the best (least) error to the worst (most) error for $1 < z < 5$ is plotted in the Fig. 2. Obviously the drift of zeros is not affected by the absolute error or the relative error. The worst sensitivity does not permit an accurate choice to be made for the picking order. The worst normalized sensitivity produces the best results and can effectively stabilize the calculation of the coefficients of the polynomial. The absolute drift of polynomial zeros is changed by both the absolute error of coefficients and the sensitivity of the polynomial zero with respect to the coefficients.

The comparisons of the mean absolute error of randomly order, the best, and the worst and of that picked by the $ware$, $wrre$, S_w, S_{wn}, $swrre$ and $sware$ for two different depths are shown in Table IV. The goal is to achieve the best accuracy. The accuracy of the best permutation is about a ten thousand times better than the accuracy obtained by the worst case and by a random permutation. The S_{wn}, $sware$, and $swrre$ methods is approximately 2 times worse than the accuracy of the best permutation. Hence, any of these three methods can be used to choose a permutation order which gives a reasonably good accuracy. However, the worst normalized sensitivity only involves the sensitivity calculation, so that it is a good method to quickly pick the right permutation. Although the histograms of probability of S_{wn}, sware, and swrre don't have very high population around the best pick, they still have a very accurate ADE compared to the best ADE. This reveals that in many cases the accuracy of six permutations are too close to be discriminated.

In order to study the frequency with which singularites may happen we pick the large error cases whose ADE is greater than 10^{-7}, run more trials and add different depths for Grunert's technique. There are two obvious singularities of the Grunert solution. The first singularity is related to a vanishing dominator in the formula for u, i.e. $\cos\gamma = v \cdot \cos\alpha$. The second problem might occur on a vanishing of the constant term A_0 in the polynomial, leading to $v = 0$ as a solution of the polynomial. In addition to singular cases we have to deal with very large errors in the vicinity of singular points in the parameter space. Our task is to define an objective function on the parameter space, which allows us to select a parametrization from the six possible parametrizations, which has the smallest absolute distance error to the exact solution.

Table V and Table VI whose results are obtained from the Vax 8500 running VMS operating system contain the statistics of the absolute distance error of the different picking methods for the three different depth cases. Table V is based on the sample of all 100000 experiments. and Table IV is based on the subsample of large error cases. The sample size for this cases is about 69 for the first depths, about 96 for the second depth and about 495 for the large depth. The experiment of one hundred thousands trials basically gives the similar results. The S_{wn}, $sware$, and $swrre$ picking methods are pretty good in select the right permutation. Table VI shows that these picking methods do quite well in the larger error cases. The similar results are also obtained in the Sun 3/280.

Table I

Authors	Features	Singularity
Grunert 1841	Direct solution, solve a fourth order polynomial	Yes
Finsterwalder 1903	Form a cubic polynomial and find the roots of two quadratics	Yes
Merritt 1949	Convert fourth order polynomial into cubic polynomial	Yes
Fischler and Bollers 1981	Another approach to form a fourth order polynomial	No
Linnainmaa et al. 1988	Generate an eighth order polynomial	No
Grafarrend et al. 1989	Form a cubic polynomial and find intersection of two quadratics	Yes

Table II

Algorithms	Precision	Mean Absolute Distance Error	Standard Deviation
Sol. 1 (Grunert)	D.P.	0.19e-08	0.16e-06
	S.P.	0.31e-01	0.88e-00
Sol. 2 (Finsterwalder)	D.P.	0.22e-10	0.90e-09
	S.P.	0.89e-02	0.51e-01
Sol. 3 (Merritt)	D.P.	0.11e-05	0.64e-04
	S.P.	0.28e-01	4.15e-00
Sol. 4 (Fischler)	D.P.	0.62e-08	0.59e-06
	S.P.	0.14e-01	0.34e-00
Sol. 5 (Linnainmaa)	D.P.	0.74e-07	0.61e-05
	S.P.	0.32e-01	0.82e-00
Sol. 6 (Grafarend)	D.P.	0.46e-08	0.43e-06
	S.P.	0.20e-01	0.75e-01

Table III The best and the worst mean absolute distance error in double precision.

Algorithms	The Best Mean Error	Standard Deviation	The Worst Mean Error	Standard Deviation
Sol1 (Grunert)	0.41e-12	0.90e-11	0.60e-08	0.25e-06
Sol2 (Finsterwalder)	0.34e-12	0.73e-11	0.20e-09	0.51e-08
Sol3 (Merritt)	0.26e-10	0.15e-08	0.18e-04	0.13e-02
Sol4 (Fischler)	0.69e-12	0.19e-10	0.13e-07	0.69e-06
Sol5 (Linnainmaa)	0.35e-11	0.24e-09	0.36e-06	0.23e-04
Sol6 (Grafarend)	0.44e-12	0.16e-10	0.88e-08	0.48e-06

Table IV The comparison of the mean absolute error of randomly order, the best and the worst and the mean absolute distance error picked by the *ware*, *wrre*, S_w, S_{wn}, *swrre* and *sware* for two different depths.

Picking methods	Mean Absolute Distance Error of Rounding Error	
Depth	1 < z < 5	5 < z < 20
Random order	0.19e-08	0.16e-06
The best	0.41e-12	0.19e-11
The worst	0.60e-08	0.87e-08
ware	0.99e-11	0.34e-09
wrre	0.40e-08	0.31e-08
Sw	0.15e-08	0.75e-09
Swn	0.89e-12	0.58e-11
swrre	0.90e-12	0.11e-10
sware	0.93e-12	0.11e-10

Table V The same as Table IV. But it runs 100000 trials and with three different depths.

Picking	Depth [1...5]		Depth [5...20]		Depth [25...75]	
	Mean ADE	Std. Dev.	Mean ADE	Std. Dev.	Mean ADE	Std. Dev.
Random	2.22e-07	6.38e-05	4.49e-09	6.11e-07	1.44e-07	1.72e-05
Best	6.69e-12	1.79e-09	2.01e-12	2.41e-10	4.18e-11	3.88e-09
Worst	1.69e-06	4.39e-04	7.14e-07	1.90e-04	1.61e-04	5.50e-02
WARE	1.06e-08	3.31e-06	4.49e-10	6.24e-08	3.70e-07	1.14e-04
WRRE	1.63e-06	4.39e-04	6.31e-07	1.88e-04	1.83e-06	3.81e-04
SW	5.99e-09	1.33e-06	5.98e-07	1.38e-04	1.34e-08	2.13e-06
SWN	9.13e-12	1.38e-09	3.76e-12	3.35e-10	2.43e-10	1.59e-08
SWARE	7.64e-12	1.30e-09	3.66e-12	4.17e-10	1.21e-10	1.13e-08
SWRRE	[illegible]	1.30e-09	4.16e-12	4.53e-10	1.21e-10	1.17e-08

Table VI The same as Table V. But it only considers large error cases.

Picking	Depth [1...5]		Depth [5...20]		Depth [25...75]	
	Mean ADE	Std. Dev.	Mean ADE	Std. Dev.	Mean ADE	Std. Dev.
Random	1.35e-05	6.35e-05	4.03e-06	2.37e-05	1.22e-04	2.43e-03
Best	7.23e-08	5.16e-07	2.43e-09	1.64e-08	1.37e-08	2.38e-07
Worst	1.18e-04	5.78e-04	2.59e-03	2.52e-02	5.43e-02	1.20e-00
WARE	6.76e-07	4.30e-06	2.94e-07	1.21e-06	9.56e-06	2.08e-04
WRRE	6.02e-05	3.18e-04	2.58e-03	2.52e-02	1.29e-04	2.43e-03
SW	6.21e-06	4.70e-05	1.14e-07	5.25e-07	1.09e-04	2.42e-03
SWN	5.42e-07	4.23e-06	8.02e-09	4.49e-08	2.18e-08	2.97e-07
SWARE	5.20e-07	4.23e-06	7.93e-09	4.43e-08	1.78e-08	2.37e-07
SWRRE	5.20e-07	4.23e-06	8.01e-09	4.49e-09	1.73e-08	2.87e-07

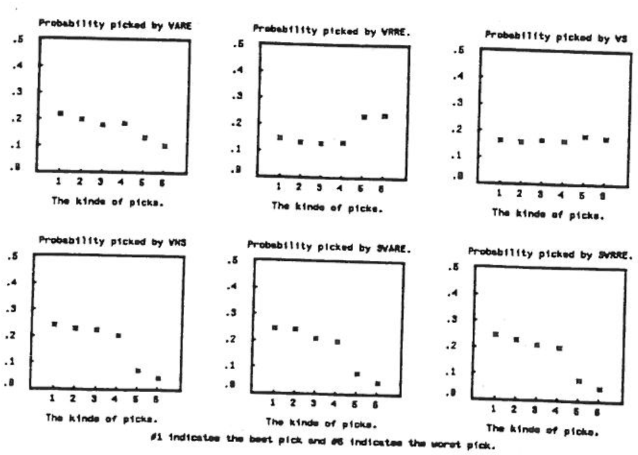

Figure 2. shows, for each of the six selection techniques, the fraction of times the technique selected the data permutation giving the best (least) error to the worst (most) error for all 10000 experimental cases for which the depth z is in the range $1 < z < 5$.

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