Regression analysis and automorphic orbits in free groups of Rank 2.

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Abstract

The main goal of this paper is to show that pattern recognition techniques can be successfully used in abstract algebra. We introduce a pattern recognition system to recognize words of minimal length in their automorphic orbits in free groups of rank 2. This system is based on linear regression and does not use any particular results from group theory. The corresponding classifier is very fast and surprisingly accurate.

1. Introduction

The field of pattern recognition has been actively developing for several decades. It has been successfully applied in a large number of diverse fields.

In this paper we show that pattern recognition techniques can be successfully used in abstract algebra and the theory of infinite groups in particular. The statistical approach gives one an exploratory methods which could be helpful in revealing hidden mathematical structures and formulating rigorous mathematical hypotheses. Our philosophy here that if irregular or non-random behavior has been observed during an experiment then there must be a pure mathematical reason behind this phenomenon, which can be uncovered by a proper statistical analysis. The discovered knowledge can be of great interest to mathematicians. In addition, one can try to develop new (perhaps probabilistic) methods to solve hard combinatorial problems in algebra.

Note that this is a very novel application area of pattern recognition. Some of the previous work can be found in [2, 4].

We start by giving a brief introduction to the Whitehead minimization problem. Let X be a finite alphabet, $X^{-1} = \{x^{-1} \mid x \in X\}$ be the set of formal inverses of letters from X, and $X^{\pm 1} = X \cup X^{-1}$. For a word w in the alphabet $X^{\pm 1}$ by $|w|$ we denote the length of w. A word w is called *reduced* if it does not contain subwords of the type xx^{-1} or $x^{-1}x$ for $x \in X$. Applying reduction rules $xx^{-1} \to \varepsilon, x^{-1}x \to \varepsilon$ (where ε is the empty word) one can reduce each word w in the alphabet $X^{\pm 1}$ to a reduced word $\overline{w}$. The word $\overline{w}$ is uniquely defined and does not depend on the order in a particular sequence of reductions. The set $F = F(X)$ of all reduced words over $X^{\pm 1}$ forms a group with respect to multiplication defined by $u \cdot v = \overline{uv}$ (i.e., to compute the product of words $u, v \in F$ one has to concatenate them and then reduce). The group F with the multiplication defined as above is called a *free* group with *basis* X. The cardinality $|X|$ is called the *rank* of $F(X)$. Free groups play a central role in modern algebra and topology.

A bijection $\phi : F \to F$ is called an *automorphism* of F if $\phi(uv) = \phi(u)\phi(v)$ for every $u, v \in F$. The set $Aut(F)$ of all automorphisms of F forms a group with respect to composition of automorphisms. Every automorphism $\phi \in Aut(F)$ is completely determined by its images on elements from the basis X since $\phi(x_1 \dots x_n) = \phi(x_1) \dots \phi(x_n)$ and $\phi(x^{-1}) = \phi(x)^{-1}$ for any letters $x_i, x_i \in X^{\pm 1}$. An automorphism $t \in Aut(F(X))$ is called a *Whitehead's automorphism* if t satisfies one of the two conditions below:
1) t permutes elements in $X^{\pm 1}$;
2) t fixes a given element $a \in X^{\pm 1}$ and maps each element $x \in X^{\pm 1}, x \neq a^{\pm 1}$ to one of the elements x, xa, $a^{-1}x$, or $a^{-1}xa$.

By $\Omega(X)$ we denote the set of all Whitehead's automorphisms of $F(X)$. It is known [5] that every automorphism from $Aut(F)$ is a product of finitely many Whitehead's automorphisms.

The automorphic orbit $Orb(w)$ of a word $w \in F$ is the set of all automorphic images of w in F:

$$Orb(w) = \{v \in F \mid \exists \varphi \in Aut(F) \text{ such that } \varphi(w) = v\}.$$

A word $w \in F$ is called *minimal* (or *automorphically minimal*) if $|w| \leq |\varphi(w)|$ for any $\varphi \in Aut(F)$. By w_{min} we denote a word of minimal length in $Orb(w)$. Notice that w_{min} is not unique. By $WC(w)$ (the *Whitehead's complex-*

ity of w) we denote a minimal number of automorphisms $t_1, \ldots, t_m \in \Omega(X)$ such that $t_m \ldots t_1(w) = w_{min}$. The algorithmic problem which requires finding w_{min} for a given $w \in F$ is called the *Minimization Problem* for F, it is one of the principal problems in combinatorial group theory and topology. There is a famous Whitehead's decision algorithm for the Minimization Problem, it is based on the following result due to Whitehead ([7]): if a word $w \in F(X)$ is not minimal then there exists an automorphism $t \in \Omega(X)$ such that $|t(w)| < |w|$. Unfortunately, its complexity depends on cardinality of $\Omega(X)$ which is exponential in the rank of $F(X)$. We refer to [4] for a detailed discussion on complexity of Whitehead's algorithms.

In this paper we focus on the *Recognition Problem* for minimal elements in F. It follows immediately from the Whitehead's result that $w \in F$ is minimal if and only if $|t(w)| \geq |w|$ for every $t \in \Omega(X)$ (such elements sometimes are called *Whitehead's minimal*). This gives one a simple deterministic decision algorithm for the Recognition Problem, which is of exponential time complexity in the rank of F. Below we construct a probabilistic classifier which is based on linear regression, it has real time complexity and gives correct answers with a sufficiently high probability.

In fact, it is convenient to consider the Minimization Problem only for cyclically reduced words in F. A word $w = x_1 \ldots x_n \in F(X)$ ($x_i \in X^{\pm 1}$) is *cyclically reduced* if $x_1 \neq x_n^{-1}$. Clearly, every $w \in F$ can be presented in the form $w = u^{-1} \widetilde{w} u$ for some $u \in F(X)$ and a cyclically reduced element $\widetilde{w} \in F(X)$ such that $|w| = |\widetilde{w}| + 2|u|$. This $\widetilde{w}$ is unique and it is called a *cyclically reduced form* of w. Every minimal word in F is cyclically reduced, therefore, it suffices to construct a classifier only for cyclically reduced words in F.

2. Recognition of minimal words in F_2

In this section we describe a particular pattern recognition system for recognizing minimal elements in free groups of rank 2. The corresponding classifier is a supervised-learning classifier based on linear regression model with a decision rule of the Bayes' type.

2.1. Data generation: training datasets

A pseudo-random element w of $F = F_2(X)$ can be generated as a pseudo-random sequence $y_1, \ldots, y_l$ of elements $y_i \in X^{\pm 1}$ such that $y_i \neq y_{i+1}^{-1}$, where the length l is also chosen pseudo-randomly. However, it has been shown in [4] and in [3] that randomly taken cyclic reduced words in F are already minimal with asymptotic probability 1. Therefore, a set of randomly generated cyclically words in F would be highly biased toward the class of minimal elements. To obtain fair training datasets we use the following procedure.

For each positive integer $l = 1, \ldots, 1000$ we generate pseudo-randomly and uniformly 10 cyclically reduced words from $F(X)$ of length l. Denote the resulting set by W. Then using the deterministic Whitehead algorithm we construct the corresponding set of minimal elements

$$W_{min} = \{w_{min} \mid w \in W\}.$$

With probability 0.5 we substitute each $v \in W_{min}$ with the word $\widetilde{t(v)}$, where t is a randomly and uniformly chosen automorphism from $\Omega(X)$ such that $|\widetilde{t(v)}| > |v|$ (if $|\widetilde{t(v)}| = |v|$ we chose another $t \in \Omega(X)$, and so on). Now, the resulting set L is a set of pseudo-randomly generated cyclically reduced words representing the classes of minimal and non-minimal elements in approximately equal proportions. It follows from the construction that our choice of non-minimal elements w is not quite representative, since all these elements have Whitehead's complexity one (which is not the case in general). One may try to replace the automorphism t above by a random finite sequence of automorphisms from Ω to get a more representative training set. However, we will see in Section 3 that the training dataset L is sufficiently good already, so we elected to keep it as it is.

From the construction we know for each element $v \in L$ whether it is minimal or not. Finally, we create a training set

$$D = \{< v, P(v) > \mid v \in L\},$$

where

$$P(v) = \begin{cases} 1, & v \ is \ minimal; \\ 0, & otherwise. \end{cases}$$

2.2. Features

Let w be a reduced word in the alphabet $X^{\pm 1}$. In this section we describe the features of w which characterize a certain placement of specific words from $F(X)$ in w.

Let U_2 be the set of all words in F_2 that are length 2. Denote by $C(w, u)$ the number of subwords $u \in U_2$ occurring in w. The normalized value

$$C(w, u) \ / \ |w|$$

is a feature of w and feature vector

$$f(w) = \frac{1}{|w|} < C(w, u) \mid \forall u \in U_2 >$$

gives the numbers of occurrences of words of length two in w relative to the length of w.

This is the basic feature vector in all our considerations, it corresponds to the so-called *Whitehead graph* of w ([5]).

2.3. Decision Rule

The classification algorithm has to predict the value $P(w)$ of the predicate P for a given word w. We use the regression classifier as the basis of the decision rule, (see [1], [6]). For any word w having feature vector $f(w)$ we compute

$$\hat{P}(w) = \beta' f(w),$$

where $\hat{P}(w)$ is the value of $P(w)$ predicted by the regression model and β is the vector of regression coefficients.

Unfortunately, we cannot guarantee that $P(w)$ is, indeed, a linear function of $f(w)$. We explore non-linear dependencies by using a general quadratic mapping. Let $f_\varphi(w) = \varphi(f(w))$ be a vector consisting of components of $f(w)$ and all their pair-wise products written in some order. The corresponding prediction value

$$\hat{P}(w) = \beta'_\varphi f_\varphi(w).$$

The decision rule, $\mathcal{R}(w)$, of minimal or not is made according to the following formula:

$$\mathcal{R}(w) = \begin{cases} 1, & if\ \hat{P}(w) > \Theta; \\ 0, & otherwise. \end{cases} \tag{1}$$

where Θ is a given threshold. However, there is an ambiguity in selection of the parameter Θ in the decision rule (1).

Here we elected to use the following Bayesian type of the decision rule. Suppose an event $\hat{P}(w) = \alpha$, where $\alpha \in \mathbb{R}$, is observed. We are going to make a prediction on whether $P(w) = 1$ or $P(w) = 0$ based on estimations of conditional probabilities

$$\Pr(P(w) = 1 | \hat{P}(w) = \alpha) \ \text{ and } \ \Pr(P(w) = 0 | \hat{P}(w) = \alpha).$$

Let $P_1(w)$ and $P_0(w)$ denote the events $P(w) = 1$ and $P(w) = 0$ respectively. Similarly, by $\hat{P}_\alpha(w)$ we denote event $\hat{P}(w) = \alpha$. Theoretically, the decision rule is:

$$\mathcal{R}(w) = \begin{cases} 1, & \Pr(P_1(w)|\hat{P}_\alpha(w)) > \Pr(P_0(w)|\hat{P}_\alpha(w)); \\ 0, & otherwise. \end{cases} \tag{2}$$

Since we cannot compute the conditional probabilities above precisely, we estimate them as follows. We partition the set $\mathbb{R}$ into intervals Δ of equal length. Now, let $\hat{P}_\Delta(w)$ denote event $\hat{P}(w) \in \Delta$. We estimate the conditional probabilities:

$$\Pr(P_1(w)|\hat{P}_\Delta(w)) \ \text{ and } \ \Pr(P_0(w) = |\hat{P}_\Delta(w))$$

Using Bayes' formula one can rewrite the probabilities above ($i = 0, 1$):

$$\Pr(P_i(w)|\hat{P}_\Delta(w)) = \frac{\Pr(\hat{P}_\Delta(w)|P_i(w))\Pr(P_i(w))}{\Pr(\hat{P}_\Delta(w))}$$

Therefore

$$\Pr(P_1(w)|\hat{P}_\Delta(w)) > \Pr(P_0(w)|\hat{P}_\Delta(w))$$

if and only if

$$\Pr(\hat{P}_\Delta(w)|P_1(w))\mathrm{Pr}_1 > \Pr(\hat{P}_\Delta(w)|P_0(w))\mathrm{Pr}_0$$

The probabilities $\mathrm{Pr}_1 = \Pr(P_1(w))$ and $\mathrm{Pr}_0 = \Pr(P_0(w))$ are prior probabilities corresponding to the distribution of minimal and non-minimal elements among the inputs given to the classifier. It is safe to assume that the prior probabilities are equal. Thus the inequality above takes the form

$$\Pr(\hat{P}_\Delta(w)|P_1(w)) > \Pr(\hat{P}_\Delta(w)|P_0(w))$$

The conditional probabilities above can be estimated from the given training dataset D. For $i = 0, 1$ put

$$d_i(\Delta) = |\{w \mid \hat{P}(w) \in \Delta, < w, i >\in D\}| \ / \ |D|$$

Then

$$\Pr(\hat{P}(w) \in \Delta | P(w) = i) \approx d_i(\Delta), \quad i = 0, 1.$$

Finally we can define the following decision rule, which is a variation of the Bayes' decision rule above:

$$\mathcal{R}(w) = \begin{cases} 1, & \hat{P}(w) \in \Delta \ and\ d_1(\Delta) > d_0(\Delta); \\ 0, & \hat{P}(w) \in \Delta \ and\ d_0(\Delta) > d_1(\Delta). \end{cases} \tag{3}$$

2.4. Test datasets

To test and evaluate our pattern recognition system we generate several test datasets of different types:

- A test set S_e which is generated by the same procedure as for the training set D, but independently of D.
- A test set S_R of pseudo-randomly generated cyclically reduced elements of $F(X)$, as described in Section 2.1.
- A test set S_P of pseudo-randomly generated cyclically reduced *primitive* elements in $F(X)$. Recall that $w \in F(X)$ is primitive if and only if there exists a sequence of Whitehead automorphisms $t_1 \dots t_m \in \Omega(X)$ such that $t_m \dots t_1(x) = w$ for some $x \in X^{\pm 1}$. Elements in S_P are generated by the procedure described in [4], which, roughly speaking, amounts to a random choice of $x \in X^{\pm 1}$ and a random choice of a sequence of automorphisms $t_1 \dots t_m \in \Omega(X)$.
- A test set S_{10} which is generated in a way similar to the procedure used to generate the training set D. The only difference is that the non-minimal elements are obtained by applying not one, but several randomly chosen automorphisms from $\Omega(X)$. The number of such automorphisms is chosen uniformly randomly from the set $\{1, \dots, 10\}$, hence the name.

Some comparative characteristics of the generated datasets are given in Table 1.

	size	% min	% non-min	avg($\|w\|$)	max($\|w\|$)
D	10000	51.9	48.1	541	1202
S_e	5000	49.5	50.5	542	1200
S_{10}	5000	48.6	51.4	691	10629
S_R	5000	98.8	1.2	499	998
S_P	6000	0	100	30	3443

Table 1. Description of the datasets.

3. Results of experiments

Let $f(w)$ be the feature mapping discussed in Section 2.2, and $f_\varphi(w)$ be the image of $f(w)$ under the quadratic mapping φ as was discussed in Section 2.3.

We run experiments with two different classifiers $\mathbf{P}$ and $\mathbf{P}_\varphi$ which are based on the linear regression model applied to vectors $f(w)$ and $f_\varphi(w)$.

The results of evaluation of the accuracy of the classifier $\mathbf{P}$ are given in Table 2a. This data shows that the accuracy of $\mathbf{P}$ decreases when the Whitehead's complexity of inputs grows.

However, the classifier $\mathbf{P}_\varphi$ achieves almost perfect classification accuracy (see Table 2b).

	S_R	S_e	S_{10}	S_P
$\|w\| > 0$	0.960	0.954	0.828	0.567
$\|w\| > 4$	0.962	0.957	0.828	0.532
$\|w\| > 100$	0.984	0.975	0.824	0.494

(a)

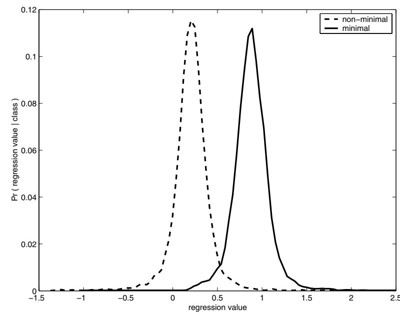
	S_R	S_e	S_{10}	S_P
$\|w\| > 0$	0.991	0.995	0.996	0.945
$\|w\| > 4$	0.993	0.996	0.996	1.000
$\|w\| > 100$	1.000	1.000	1.000	1.000

(b)

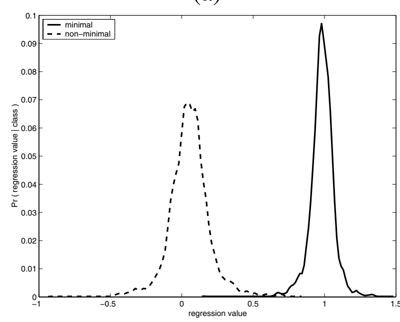
Table 2. Classification accuracy of the classifiers a) P; b) P_φ.

Conclusions:

- The classifier $\mathbf{P}_\varphi$ is remarkably reliable;
- Very short words are more difficult to classify (perhaps, because they do not provide sufficient information for the classifiers);
- The estimated conditional probabilities for $\mathbf{P}_\varphi$ (which come from the Bayes' decision rule, see Section 2.3) are presented in Figure 1b. Clearly, the classes of minimal and non-minimal elements are separated around 0.5 with a small overlap. So the regression works perfectly with the threshold $\Theta \approx 0.6$. From the figure we can see that the probability of misclassification of classifier $\mathbf{P}$ is much higher then the one for $\mathbf{P}_\varphi$.



(a)



(b)

Figure 1. Conditional probabilities obtained with a) P; b) P_φ.

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