Chapter 12
Model-Based Morphology: The Opening Spectrum

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12.1 Introduction

Filtering by morphological operations is particularly suited for removal of clutter and noise objects which have been introduced into noiseless binary images. The morphological filtering is designed to exploit differences in the spatial nature (shape, size, orientation) of the objects (connected components) in the ideal noiseless images as compared to the noise/clutter objects.

Since the typical noise models (union, intersection set difference, etc.) for binary images are not additive, and the morphological processing is strongly nonlinear, optimal filtering results conventionally available for linear processing in the presence of additive noise are not directly applicable to morphological filtering of binary images.

In this paper we describe a morphological filtering analog to the classic Wiener filter, a preliminary account having been given in [1]. The discussion begins in Section 2 with a review of the Wiener filter and its extension to a Binary Wiener filter; in these the underlying model entails decomposing the signal and additive noise into spectral elements in terms of an orthogonal basis set. Classic Wiener optimal estimation weights the respective spectral elements in the noisy signal according to the expected values of signal and noise energy across the spectrum. Section 3 extracts the essence of the algebraic structure underlying the derivation of the Wiener filter, doing so in a way that retains the concepts of energy and spectral decomposition, but eliminates the assumptions of noise additivity, orthogonal bases, and even the concept of inner product. The stage is thus set for the subsequent morphological filtering results where those assumptions do not apply. Section 4 derives an optimal morphological filter for binary images composed of the union (not sum) of the signal and noise connected components. The spectral decomposition of signal and noise is in terms of an ordered basis of connected components where the ordering is based on the morphological opening operation. (Such a basis is, in a certain sense, a "nested" collection of sets.) Thus the underlying model is based upon that ordered basis (which provides prototypes of signal and noise objects scattered throughout the binary image) and upon a morphological spectrum derived from openings. Section 5 expands the results of Section 4 beyond allowing signal and noise objects to be taken from a single ordered basis (e.g. an ordered set of discs). In Section 5, the collection of prototypes can include any number of coordinated ordered bases (e.g. an ordered set of discs, as well as an ordered set of squares, as well as several ordered sets of lines each at different orientations.)

Whereas in the first five section we restrict ourselves to finite-component spectral representation, in Section 6 we treat the continuous case for a single ordered basis. Section 7 extends these results to multiple ordered bases. In

Section 8 we compare the opening-spectrum filter discussed herein to mean-square morphological-filter estimation.

12.2 The Wiener Filter

Regarding the discrete Wiener filter, let $b_1, \ldots, b_n$ be an orthonormal basis. The model for the ideal random signal f is that $f = \sum_{n=1}^{N} \alpha_n b_n$ where $E[\alpha_n] = 0, V[\alpha_n] = \sigma_{f_n}^2$, and $E[\alpha_m \alpha_n] = 0, m \neq n$. The variances $\sigma_{f_n}^2$ are taken to be known. The model for the random noise g is that $g = \sum_{n=1}^{N} \beta_n b_n$ where $E[\beta_n] = 0, V[\beta_n] = \sigma_{g_n}^2$, and $E[\beta_m \beta_n] = 0, m \neq n$. Noise and signal are uncorrelated so that $E[\alpha_n \beta_m] = 0$.

The observed noisy signal is $f + g = \sum_{n=1}^{N} (\alpha_n + \beta_n) b_n$. The Wiener filtering problem is to determine weights $w_1, \ldots, w_N$ to make the estimate $\hat{f}$ of f, $\hat{f} = \sum_{n=1}^{N} w_n(\alpha_n + \beta_n) b_n$ minimize $E[\rho(f, \hat{f})]$, where ρ is a metric. In the case of Euclidean distance for the metric ρ, $E[\rho(f, \hat{f})] = E[\|f - \hat{f}\|^2]$.

Now,

$$\begin{aligned} \|f - \hat{f}\|^2 &= \|\sum_{n=1}^{N} \alpha_n b_n - \sum_{n=1}^{N} w_n(\alpha_n + \beta_n) b_n\|^2 \\ &= \sum_{n=1}^{N} [w_n(\alpha_n + \beta_n) - \alpha_n]^2 \end{aligned} \quad (1)$$

And

$$\begin{aligned} E\left[\|f - \hat{f}\|^2\right] &= \sum_{n=1}^{N} E[(w_n(\alpha_n + \beta_n) - \alpha_n)^2] \\ &= \sum_{n=1}^{N} w_n^2(\sigma_{f_n}^2 + \sigma_{g_n}^2) - 2 w_n \sigma_{f_n}^2 + \sigma_{f_n}^2 \end{aligned} \quad (2)$$

Hence, the minimizing weights are given by

$$w_n = \frac{\sigma_{f_n}^2}{\sigma_{f_n}^2 + \sigma_{g_n}^2}. \quad (3)$$

One can also define a binary Wiener filter, with weights restricted to 0 or 1. To determine the minimizing weights, we need just examine

$$w_n^2(\sigma_{f_n}^2 + \sigma_{g_n}^2) - 2 w_n \sigma_{f_n}^2 + \sigma_{f_n}^2 = \begin{cases} \sigma_{f_n}^2 & \text{if } w_n = 0 \\ \sigma_{g_n}^2 & \text{if } w_n = 1 \end{cases} \quad (4)$$

Hence, under the constraint that the $w_n \in \{0, 1\}$, the minimizing weights are given by

$$w_n = \begin{cases} 0 & \text{if } \sigma_{f_n}^2 < \sigma_{g_n}^2 \\ 1 & \text{otherwise} \end{cases} \quad (5)$$

In this case the estimate $\hat{f} = \sum_{n \in S} (\alpha_n + \beta_n) b_n$, where $S = \{n | w_n = 1\}$. Thus the optimal binary Wiener filter retains that part of the spectrum where the expected signal energy exceeds the expected noise energy, and discards the rest.

12.3 Optimal Filtering in the Generalized Case

This section restates the binary Wiener filter results, retaining the classic algebraic structure under far less restrictive assumptions than those of Section 2. The new assumptions will in fact be consistent with the morphological filter we will develop in Section 4. Specifically we now relax the assumptions of additive noise, vector norms, inner products, and orthonormal bases, replacing them with more general assumptions on the nature of noise inclusion, distance, energy, and spectral decomposition, and the relationships between them.

Let f be any binary image in a set B of binary images and ψ be a mapping (a spectral decomposition) taking f into the N-tuple $(f_1, \ldots, f_N)$; that is $\psi : B \to B^N$. (In the case of the Wiener filter, the N-tuple $(f_1, \ldots, f_N)$ is $(\alpha_1 b_1, \ldots, \alpha_N b_N)$. Here, we incorporate into each f_n both the scalar and the basis elements.) Let ψ^{-1} be the inverse mapping re-assembling $(f_1, \ldots, f_N)$ back into f; that is $\psi^{-1} : B^N \to B$. The identity operator can be expressed as $\psi\psi^{-1}$ and $\psi^{-1}\psi$. For any two binary images f and g in B let there be defined a binary operation $<>$ such that $f <> g$ is also a binary image in B. When g is the noise, $f <> g$ corresponds to the observed noisy binary image. We require that $<>$ and ψ satisfy the relationship

$$\psi(f <> g) = (f_1 <> g_1, \ldots, f_N <> g_N). \tag{6}$$

Let ρ be the function evaluating the closeness of one image to another. Hence $\rho : B \times B \to [0, \infty)$. The function ρ must satisfy $\rho(f, h) = \sum_{n=1}^{N} \rho(f_n, h_n)$ where $\psi(f) = (f_1, \ldots, f_N)$ and $\psi(h) = (h_1, \ldots, h_N)$.

For any binary image g, we let $\#$ represent the operator which quantifies the energy in the binary image g; $\# : B \to [0, \infty)$. The operator $\#$ must satisfy $\#g = \sum_{n=1}^{N} \#g_n$, for spectral decomposition $\psi(g) = (g_1, \ldots, g_N)$. Finally, there is a relationship between ρ and $\#$: The distance between the binary image and the ideal image is just the energy in the noise image; $\rho(f <> g, f) = \#g$.

Let $w_n \in \{0, 1\}, n = 1, \ldots, N$ be binary weights and let the filtered binary image have a representation $(w_1(f_1 <> g_1), \ldots, w_N(f_N <> g_N))$ where

$$w_n(f_n <> g_n) = \begin{cases} f_n <> g_n & \text{if } w_n = 1 \\ \phi & \text{if } w_n = 0 \end{cases} \tag{7}$$

and ϕ is the binary image satisfying $f <> \phi = f$. The filtered binary image $\hat{f}$ itself can then be written as

$$\hat{f} = \psi^{-1}\left(w_1(f_1 <> g_1), \ldots, w_n(f_N, <> g_N)\right). \tag{8}$$

In essence the effect of the filtering is obtained by nulling spectral content of the observed noisy binary image.

The optimal filter parameters w_n are chosen to minimize

$$\begin{aligned} E[\rho(\hat{f}, f)] &= E\left[\sum_{n=1}^{N} \rho(\hat{f}_n, f_n)\right] = \sum_{n=1}^{N} E[\rho(w_n(f_n <> g_n), f_n)] \\ &= \sum_{n=1}^{N} E\left[\begin{cases} \#g_n & \text{if } w_n = 1 \\ \#f_n & \text{if } w_n = 0 \end{cases}\right]. \end{aligned} \tag{9}$$

Hence, the best value for w_n is given by

$$w_n = \begin{cases} 0 & \text{if } E[\#f_n] < E[\#g_n] \\ 1 & \text{otherwise.} \end{cases} \tag{10}$$

Then the index set S corresponding to the spectral content that will be included in the optimal filter output can be defined by $S = \{n | E[\#f_n] \geq E[\#g_n]\}$.

12.4 Optimal Binary Morphological Filter

To apply the foregoing algebraic filtering paradigm to mathematical morphology, we need to define the ideal random image model, the random noise model, the relationship of the observed image to the ideal random image and random noise, the formulation of representation operator ψ from morphological operators, the energy measure $\#$, and the closeness measure ρ. We begin with the representation operator ψ, which will be formulated relative to morphological opening, where the opening of binary image A by structuring element K is defined by

$$A \circ K = \bigcup \{K_x : K_x \subseteq A\} \tag{11}$$

where subscripts having names like x or y designate a translation of the set subscripted and where we assume all images are compact subsets of k-dimensional Euclidean space R^K. (See Haralick, Sternberg, and Zhuang [3], Dougherty and Giardina [4,5], or Serra [2] for the fundamental properties of the morphological opening.)

The representation operator ψ will be defined in a manner akin to the morphological granulometric pattern spectrum. To set up our definition for ψ in a way which relates to the ideal random image and noise models, we note that the opening operator has the following property: If $A = \bigcup_{i=1}^{I} A_i$, where each A_i is a connected component of A, and K is a connected structuring element, then

$$A \circ K = (\bigcup_{i=1}^{I} A_i) \circ K = \bigcup_{i=1}^{I} (A_i \circ K). \tag{12}$$

This property, that the opening of a union of connected components is the union of the openings, will be essential throughout our development. It is this property which motivates the following definition: Two sets A and B are said to not interfere with one another if and only if X, a connected component of $A \cup B$, implies that X is a connected component of A or of B but not both. It immediately follows that if A and B do not interfere with one another and K is a connected structuring element, then

$$(A \cup B) \circ K = (A \circ K) \cup (B \circ K). \tag{13}$$

The *opening-spectrum* operator ψ will be defined in terms of a set of openings. This set of openings will be based on the structuring elements in a naturally ordered morphological basis. We define a collection $\mathcal{K}$ of structuring elements to be an *opening spectrum basis* if and only if $K \in \mathcal{K}$ implies K is connected and $K, L \in \mathcal{K}$ implies $K \circ L = K$ or $K \circ L = \phi$. A opening-spectrum basis $\mathcal{K} = \{K(1), \ldots, K(M)\}$ is *naturally ordered* if and only if $K(1) = \{0\}$ and

$$K(i) \circ K(j) = \begin{cases} K(i) & j \leq i \\ \phi & j > i. \end{cases} \tag{14}$$

A simple example of an ordered opening-spectrum basis is a set of squares of increasing size, beginning with a square of one pixel.

Now we can define the operator ψ which produces a opening-spectrum with respect to a naturally ordered opening-spectrum basis $\mathcal{K} = \{K(1), \ldots, K(M)\}$. ψ is defined by $\psi(A) = (A_1, \ldots, A_M)$ where

$$A_m = A \circ K(m) - A \circ K(m+1) \tag{15}$$

for $m = 1, \ldots, M-1, A_M = A \circ K(M)$, and $K(1) = \{0\}$. A_m is that part of A which is open under $K(m)$ but not open under $K(m+1)$, except for A_M which is A opened by $K(M)$. A_M is the remainder. $K(1) = \{0\}$ assures that A_1 contains everything in A that does not fit any of the larger $K(M)$'s. It follows from this definition that for $i \neq j$, $A_i \cap A_j = \phi$. This happens because

$$\begin{aligned} A_i \cap A_j &= [A \circ K(i) - A \circ K(i+1)] \cap [A \circ K(j) - A \circ K(j+1)] \\ &= [A \circ K(i) \cap A \circ K(j)] \cap [A \circ K(i+1) \cup A \circ K(j+1)]^c \\ &= [A \circ K(\max\{i,j\})] \cap [A \circ K(\min\{i+1, j+1\})] \\ &= \phi \text{ since } \max\{i,j\} \geq \min\{i+1, j+1\} \text{ for any } i \neq j,\ i,j < M \end{aligned} \tag{16}$$

For the special case $j = M$ and for $i < M$, the derivation is

$$\begin{aligned} A_i \cap A_M &= [A \circ K(i) - A \circ K(i+1)] \cap A \circ K(M) \\ &= ([A \circ K(i) \cap [A \circ K(M)]) \cap [A \circ K(i+1)]^c \\ &= A \circ K(M) \cap [A \circ K(i+1)]^c \quad \text{since } A \circ K(i) \supseteq A \circ K(M) \\ &= \phi \quad \text{since } A \circ K(i+1) \supseteq A \circ K(M) \end{aligned} \tag{17}$$

It is easy to see that from the opening spectrum, $(A_1, \ldots, A_M)$, the original shape A can be exactly reconstructed since $\bigcup_{m=1}^{M} A_m = A$. This can be seen directly. Consider

$$\begin{aligned} \bigcup_{m=1}^{M} A_m = [A \circ K(1) - A \circ K(2)] \cup \ldots \\ \cup [A \circ K(M-1) - A \circ K(M)] \cup A \circ K(M) \end{aligned} \tag{18}$$

Since $K(i) \circ K(j) = K(i)$ for $i \geq j, A \circ K(j) \supseteq A \circ K(i)$ for $i \geq j$. (19)

Hence the sets $A \circ K(1), A \circ K(2), \ldots, A \circ K(M)$ are ordered in the sense that

$$A \circ K(1) \supseteq A \circ K(2) \supseteq \ldots \supseteq A \circ K(M) \tag{20}$$

From this it follows that for any $m \geq 2$,

$$[A \circ K(m-1) - A \circ K(m)] \cup A \circ K(m) = A \circ K(m-1) \tag{21}$$

Now by working from the right end of the union representation, taking two terms at a time, the entire union is seen to collapse to $A \circ K(1) = A$.

ψ^{-1} can then be defined by $\psi^{-1}(A_1, \ldots, A_M) = \bigcup_{m=1}^{M} A_m$. The existence of ψ^{-1} implies that the representation is unique in the sense that two different opening spectra must be associated with two different shapes and two different shapes must be associated with two different opening spectra. It implies, as well, that the representation is complete.

Next we discuss the spatial random process generation mechanism which produces binary image realizations. A spatial random process producing a set A is a non-interfering spatial Poisson process with respect to an ordered opening-spectrum basis $\mathcal{K}$ if and only if:

- For some Z, a Poisson distributed random number (with Poisson density parameter λ_A), which is the total connected component count of a binary image realization A;
- For some multinomial distributed numbers $L_1, \ldots, L_M$ with $\sum_{m=1}^{M} L_m = Z$ (with respective multinomial probabilities $p_1, \ldots, p_M$), which split the Z connected components into M subsets containing objects of the same type;
- For some randomly chosen translations $x_{mj}, m = 1, \ldots, M;\ j = 1, \ldots, L_m$;
- $A = \bigcup_{m=1}^{M} \bigcup_{j=1}^{L_m} K(m)_{x_{mj}}$, where the translated structuring elements do not interfere, i.e.,

$$K(i)_{x_{ij}} \bigcap K(m)_{x_{mn}} = \begin{cases} K(i)_{x_{ij}} & \text{if } i = m \text{ and } j = n \\ \phi & \text{otherwise.} \end{cases} \tag{22}$$

From this definition of a non-interfering random process, it follows that

$$\begin{aligned} A \circ K(\lambda) &= \left(\bigcup_{m=1}^{M} \bigcup_{j=1}^{L_m} K(m)_{x_{mj}} \right) \circ K(\lambda) \\ &= \bigcup_{m=1}^{M} \bigcup_{j=1}^{L_m} [K(m)_{x_{mj}} \circ K(\lambda)] \\ &= \bigcup_{m=\lambda}^{M} \bigcup_{j=1}^{L_m} K(m)_{x_{mj}} \end{aligned} \tag{23}$$

Moreover, if $\psi(A) = (A_1, \ldots, A_M)$, then

$$A_m = \bigcup_{j=1}^{L_m} K(m)_{x_{mj}} \tag{24}$$

for $m = 1, \ldots, M$. We interpret these results in the following manner: If A is opened by the λth basis structuring element, all components originating from "smaller" (lower-numbered) basis structuring elements are removed; the opening spectrum of A (with respect to the basis from which it was built) sorts A according to the index number of the underlying basis structuring elements, and leaves nothing out.

We consider both the ideal random image and the noise image to be generated by non-interfering random spatial processes. The observed noisy image is the union of the ideal image with a noise/clutter image. This motivates a definition of non-interfering spatial processes which here plays the role of the zero correlation between the coefficients of the image process and the coefficients of the noise process in the Wiener filter case. A random process generating realization D and a random process generating realization E are said to be non-interfering random processes if and only if D and E are always non-interfering sets for each realization.

We can now define an observed noisy image. Let A be a realization of a non-interfering spatial process (with respect to an ordered opening-spectrum basis $\mathcal{K}$) producing images of interest and let N be a realization of a non-interfering spatial process (with respect to the same $\mathcal{K}$) producing noise/clutter. We suppose that these processes do not interfere with one another. The observed noisy realization is defined as $A \cup N$. Let $\psi(A) = (A_1, \ldots, A_M), \psi(N) = (N_1, \ldots, N_M)$, and $\psi(A \cup N) = (B_1, \ldots, B_M)$. Then $B_m = A_m \cup N_m$. We reason as follows.

$$B_m = (A \cup N) \circ K(m) - (A \cup N) \circ K(m+1), \tag{25}$$

for $m = 1, \ldots, M-1$, and

$$B_M = (A \cup N) \circ K(M). \tag{26}$$

Because the processes do not interfere with one another,

$$\begin{aligned} B_m &= [A \circ K(m) \cup N \circ K(m)] - [A \circ K(m+1) \cup N \circ K(m+1)] \\ &= [A \circ K(m) - A \circ K(m+1)] \cup [N \circ K(m) - N \circ K(m+1)] \\ &= A_m \cup N_m \\ \text{and} \quad B_M &= A \circ K(m) \bigcup B \circ K(m) \\ &= A_M \bigcup N_M \end{aligned} \tag{27}$$

Thus we have just seen that

$$\psi(A \cup N) = (A_1 \cup N_1, \ldots, A_M \cup N_M). \tag{28}$$

The filtered image $\hat{A}$ will be based on selecting the most appropriate components from the opening-spectrum of $A \cup N$. Letting S be the set of components selected, we estimate A by $\hat{A}$ where

$$\hat{A} = \bigcup_{m \in S} (A_m \cup N_m) \text{ or } \hat{A} = \bigcup_{m \in S} B_m. \tag{29}$$

Thus by choosing the form of the estimation analogously to that of the binary Wiener filter, the estimation problem becomes one of choosing an appropriate index set S.

To determine S, we must first state our error criterion. For any two sets A and $\hat{A}$, we define the closeness (non-overlap) of A to $\hat{A}$ by $\rho(A, \hat{A}) = \#[(A-\hat{A}) \cup (\hat{A}-A)]$ where $\#$ is the set counting measure (pixel count, area). Our error criterion is then

$$E[\rho(A, \hat{A})] = E\left\{\#[(A - \hat{A}) \cup (\hat{A} - A)]\right\}. \tag{30}$$

To see how to choose S to minimize $E\left\{\#[(A - \hat{A}) \cup (\hat{A} - A)]\right\}$, first note that

$$\begin{aligned} A - \hat{A} &= \bigcup_{m=1}^{M} A_m - \bigcup_{m \in S} (A_m \cup N_m) = \bigcup_{\substack{m=1 \\ m \notin S}}^{M} A_m \\ \hat{A} - A &= \bigcup_{m \in S}^{M} A_m \cup N_m - \bigcup_{m=1}^{M} A_m = \bigcup_{m \in S} N_m. \end{aligned} \tag{31}$$

Hence,

$$\begin{aligned} \rho(A, \hat{A}) &= \#[(A - \hat{A}) \cup (\hat{A} - A)] \\ &= \#(A - \hat{A}) + \#(\hat{A} - A) \\ &= \# \bigcup_{\substack{m=1 \\ m \notin S}}^{M} A_m + \# \bigcup_{m \in S} N_m \\ &= \sum_{\substack{m=1 \\ m \notin S}}^{M} \#A_m + \sum_{m \in S} \#N_m \end{aligned} \tag{32}$$

The two summations above are respectively the area of the ideal image left out, plus the noise and clutter area left in. The individual terms decompose that area by spectral content.

Now, since each spectral component is built of translates of the same basis structuring elements, and since non-interference implies mutual exclusivity,

$$\begin{aligned} \#A_m &= \# \bigcup_{j=1}^{L_m} K(m)_{x_{mj}} \\ &= \sum_{j=1}^{L_m} \#K(m)_{x_{mj}} = L_m \#K(m) \end{aligned} \tag{33}$$

so that

$$E[\#A_m] = \#K(m) p_m \lambda_A \mathcal{A} \tag{34}$$

where p_m is the multinomial probability for the ideal image process, λ_A is the Poisson density parameter of the ideal image process, and $\mathcal{A}$ is the area of the image spatial domain. Likewise, $E[\#N_m] = \#K(m) q_m \lambda_N \mathcal{A}$, where q_m is the multinomial probability for the noise process and λ_N is the Poisson density parameter of the noise process.

To determine the index set S, we then have

$$E\left\{\#[(A-\hat{A})\cup(\hat{A}-A)]\right\} = E\left[\sum_{m=1}^{M}\begin{cases}\#A_m & m\notin S\\ \#N_m & m\in S\end{cases}\right] = \sum_{m=1}^{M}\begin{cases}E[\#A_m] & m\notin S\\ E[\#N_m] & m\in S\end{cases} \quad (35)$$

Hence, the best S is defined by

$$S = \{m|E[\#N_m] < E[\#A_m]\}, \quad (36)$$

or equivalently for the statistical assumptions made,

$$S = \{m|q_m\lambda_N < p_m\lambda_A\}. \quad (37)$$

A spectral component is retained according to the relative expectations of that component's "leave-out" of ideal image vs. "leave-in" of noise and clutter.

Figure 1 illustrates the concept of the filter. A is the ideal binary image; B is the observed noisy image. There are four structuring elements K(1), K(2), K(3), and K(4) which constitute an ordered basis. The four component images are given by

$$B1 = B\circ K(1) - B\circ K(2)$$
$$B2 = B\circ K(2) - B\circ K(3)$$
$$B3 = B\circ K(3) - B\circ K(4)$$
$$B4 = B\circ K(4)$$

Notice that all the binary-one pixels in B1 are noise. So the index set S, which selects which components constitute the filtered image, will not contain the index 1. The component images B2 and B3 contain more ideal image than noise so indices 2 and 3 are in S. Finally, the component image B4 has more noise than ideal image. Hence index 4 is not in S. The filtered image $\hat{A}$ is then defined by $\hat{A} = B2\cup B3$.

12.5 Extension to Generalized (Tau-)Openings

The results we have just obtained can be extended to where the opening operation is changed to a generalized opening operation. Recall that in the previous section, each basic structuring element was just a set K. In the generalized opening operation, each basic structuring element is a collection Q of sets. The generalized opening of an image I with Q is then defined by:

$$I\circ Q = \bigcup_{L'\in Q} I\circ L'. \quad (38)$$

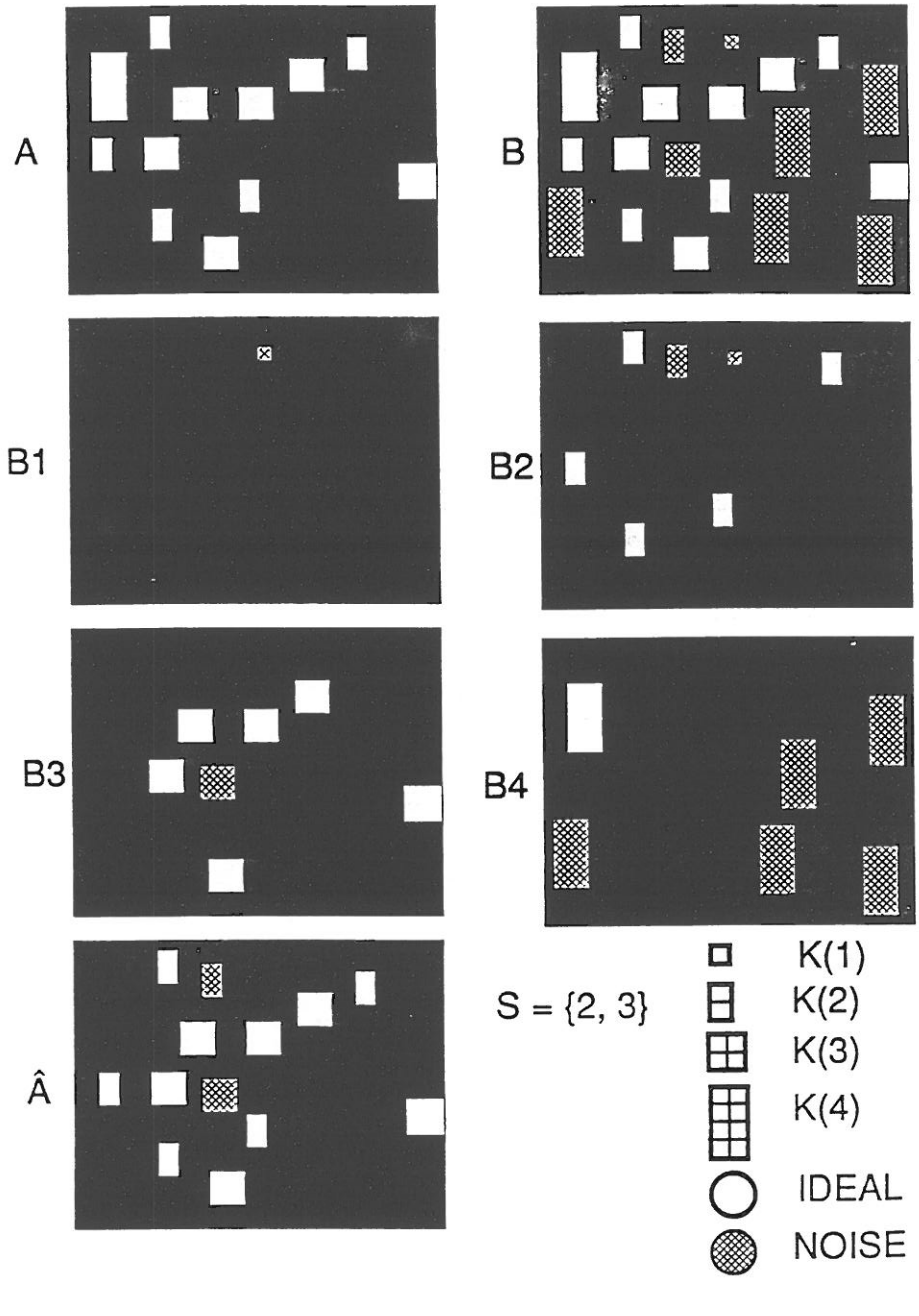

Figure 1 Figure 1 illustrates the filtering process. A is the ideal image; B is the observed noisy image. Using structuring elements K(1), K(2), K(3), and K(4) as the ordered basis produces component images B1, B2, B3, and B4. Component images B2 and B3 have more ideal image than noise, so the filtered image $\hat{A}$ is $B2 \cup B3$.

Regarding such generalized openings, Matheron [6] calls a filter Ψ a tau-opening if it satisfies four properties: it must be (1) anti-extensive, $\Psi(A) \subseteq A$; (2) translation invariant, $\Psi(A_x) = [\Psi(A)]_x$; (3) increasing, $A \subseteq B$ implies $\Psi(A) \subseteq \Psi(B)$; and (4) idempotent, $\Psi\Psi = \Psi$. The basic Matheron representation for tau-openings is that Ψ is a tau-opening if and only if there exists a collection Q such that Ψ is defined by eq. (38). Moreover, Q is a *base* for Inv[Ψ], the invariant class of Ψ; that is, the invariants for Ψ are unions of translations of elements in Q. For an elementary opening $A \circ K$, (K) is the base. The Matheron representation is discussed by Dougherty and Giardina [4,5], the gray-scale exension is given in [5], and both Serra [7] and Ronse and Heijmans [8] give lattice extensions.

The generalization is important because of the way it extends the underlying signal and noise spatial random process generation mechanism. For example, if the structuring elements were all line segments, the structuring element collection Q could consist of multiple orientation of line segments of the same length. The corresponding spatial random process would place non-interfering line segments at different orientations on the image. Or, the spatial random process could place non-interfering line segments, disks, or squares, on the image. For each size, the corresponding structuring element collection could be: line segments of the given size at a variety of orientations, a disk of the given size, and a square of the given size.

To see how the generalized opening can be used, we illustrate the case for which each structuring element collection contains exactly two structuring elements. Let $\mathcal{K} = \{K(1), \ldots, K(M)\}$ and $\mathcal{J} = \{J(1), \ldots, J(M)\}$ be naturally ordered opening bases. Define the collection $\mathcal{Q}$ by $\mathcal{Q} = \{Q(1), \ldots, Q(M)\}$ where $Q(m) = \{K(m), J(m)\}, m = 1, \ldots, M$. To make the ordering of the collection $\mathcal{K}$ and the collection $\mathcal{J}$ compatible, we require that

$$K(i) \circ J(j) = J(i) \circ K(j) = \phi \tag{39}$$

for $j > i$. Q is called a *generalized opening basis.*

Now, using the generalized opening operator, consider

$$\begin{aligned} K(i) \circ Q(j) &= K(i) \circ K(j) \cup K(i) \circ J(j) \\ &= \begin{cases} K(i) & i \geq j \\ \phi & \text{otherwise} \end{cases} \end{aligned} \tag{40}$$

Likewise,

$$\begin{aligned} J(i) \circ Q(j) &= J(i) \circ K(j) \cup J(i) \circ J(j) \\ &= \begin{cases} J(i) & i \geq j \\ \phi & \text{otherwise} \end{cases} \end{aligned} \tag{41}$$

Suppose that a realization A for a non-interfering process can be written as

$$A = \left[\bigcup_{m=1}^{M} \bigcup_{j=1}^{L_j^K} K(m)_{x_{mj}} \right] \bigcup \left[\bigcup_{m=1}^{M} \bigcup_{j=1}^{L_j^J} J(m)_{y_{mj}} \right] \tag{42}$$

where the sets in the collection

$$\{K(m)_{x_{mj}}, J(m)_{y_{mj}} : i = 1, \ldots, L_m^K, j = 1, \ldots, L_m^J\}_{m=1}^M \tag{43}$$

are naturally non-interfering. Then

$$\begin{aligned} A \circ Q(\lambda) &= \left[\bigcup_{m=1}^{M} \bigcup_{j=1}^{L_j^K} K(m)_{x_{mj}} \bigcup \bigcup_{m=1}^{M} \bigcup_{j=1}^{L_j^J} J(m)_{y_{mj}}\right] \circ Q(\lambda) \\ &= \bigcup_{m=1}^{M} \bigcup_{j=1}^{L_j^K} [K(m)_{x_{mj}} \circ Q(\lambda)] \bigcup \bigcup_{m=1}^{M} \bigcup_{j=1}^{L_j^J} [J(m)_{x_{mj}} \circ Q(\lambda)] \\ &= \bigcup_{m=\lambda}^{M} \bigcup_{j=1}^{L_j^K} K(m)_{x_{mj}} \bigcup \bigcup_{m=\lambda}^{M} \bigcup_{j=1}^{L_j^J} J(m)_{y_{mj}} \end{aligned} \tag{44}$$

Moreover, applying the spectrum definition of eq. (15) to the generalized opening Q yields

$$\begin{aligned} A_m &= A \circ Q(m) - A \circ Q(m+1) \\ &= \bigcup_{n=m}^{M} \bigcup_{j=1}^{L_j^K} K(n)_{x_{nj}} \bigcup \bigcup_{n=m}^{M} \bigcup_{j=1}^{L_j^J} J(n)_{y_{nj}} \\ &\qquad - \bigcup_{n=m+1}^{M} \bigcup_{j=1}^{L_j^K} K(n)_{x_{nj}} \bigcup \bigcup_{n=m+1}^{M} \bigcup_{j=1}^{L_j^J} J(n)_{y_{nj}} \\ &= \bigcup_{j=1}^{L_j^K} K(m)_{x_{nj}} \bigcup \bigcup_{j=1}^{L_j^J} J(m)_{y_{nj}} \end{aligned} \tag{45}$$

From this it is clear that the representation operator ψ based on $\mathcal{Q}$ has an inverse and $A = \bigcup_{m=1}^{M} A_m$. Furthermore, $A_i \bigcup A_j = \phi$ and $\#A = \sum_{m=1}^{M} \#A_m$. This fulfills the required conditions described in Section 3. Furthermore, results for $\mathcal{Q}$ containing collections of pairs of structuring elements are immediately generalizable to collections having any number of structuring elements.

To extend the optimal index set S given by eq. (28) to the situation where Q contains pairs, $Q(m) = \{K(m), J(m)\}$, we need only recognize that there are now four noninterferring processes to consider: (1) a signal process involving $\{K(m)\}$ with Poisson parameter λ_{AK} and multinomial probabilities p_{Km}, (2) a signal process involving $\{J(m)\}$ with Poisson parameter λ_{AJ} and multinomial probabilities p_{Jm}, (3) a noise process involving $\{K(m)\}$ with Poisson parameter λ_{NK} and multinomial probabilities q_{Km}, and (4) a noise process involving $\{J(m)\}$ with Poisson parameter λ_{NJ} and multinomial probabilities q_{Jm}. Since eq. (35) still applies, eq. (45) applied to both signal and noise yields

$$\begin{aligned} E[p(A, \hat{A})] = &\sum_{m \notin S} A\#K(m)[\lambda_{AK} p_{Km} + \lambda_{AJ} p_{Jm}] + \\ &\sum_{m \in S} A\#K(m)[\lambda_{NK} q_{Km} + \lambda_{NJ} q_{Jm}] \end{aligned} \tag{46}$$

Thus, the best S is defined by

$$S = \{m : \lambda_{NK} q_{Km} + \lambda_{NJ} q_{Jm} < \lambda_{AK} p_{Km} + \lambda_{AJ} p_{Jm} \tag{47}$$

Extension to more than two-structuring-element opening bases is straightforward.

12.6 Continuous Opening Spectra

In the present section we extend the preceding notions to the case of continuously parameterized openings, and in doing so relate the preset spectral theory to the granulometric theory of Matheron [6]. Because Euclidean granulometric theory does not apply to discrete space, it is at once recognized that the theory of the preceding sections is not rendered superfluous by the Euclidean approach: specifically, the theory of discrete opening spectra applies to both discrete and Euclidean space, whereas the continuous-spectra approach only applies to Euclidean space.

Matheron [6] defines a granulometry to be a family of binary-image operators $(\Psi_t), t \geq 0$, for which Ψ_t is antiextensive and monotonically increasing, $\Psi_t\Psi_r = \Psi_r\Psi_t = \Psi_{\max(t,r)}$ for all $t, r > 0$, and Ψ_0 is the identity. Here t is a generalized scale parameter. He further defines a Euclidean granulometry to be a granulometry for which Ψ_t is translation invariant and $\Psi_t(A) = t\Psi_t(A/t)$ for $t > 0$. If K is a convex, compact set, then the parameterized opening $\Psi_t(A) = A \circ tK$ is an Euclidean granulometry. Moreover, a deep theorem of Matheron [6] states that, for compact K, $A \circ tK$ is a granulometry if and only if K is convex. In particular, $tK \circ rK = tK$ whenever $t \geq r$ if and only if K is convex [clearly $tK \circ rK = \emptyset$ for $t < r$].

For continuous parameter $t \geq 0$, $\mathcal{K} = \{K(t)\}$ will be called for an *ordered opening basis* if and only if $K(t) \circ K(r) = K(t)$ for $r \leq t$ and $K(t) \circ K(r) = \emptyset$ for $r > t$. One way, but certainly not the only way, to generate such a class $\mathcal{K}$ is to consider a compact, conex set K, and define $K(t) = tK$.

The spectrum operator ψ can be adapted to the continuous setting by defining $\psi(A) = [A(t)]_{t \geq 0}$, where

$$A(t) = A \circ K(t) - \bigcup_{r>t} A \circ K(r) \tag{48}$$

For $t \neq t'$, $A(t) \cap A(t') = \emptyset$. To see this, suppose without loss of generality that $t' > t$. Then

$$\begin{aligned} A(t) \cap A(t') &= [A \circ K(t) \cap A \circ K(t')] \cap \\ &\quad \left\{ \left[\bigcup_{r>t} A \circ K(r) \right] \cup \left[\bigcup_{r>t'} A \circ K(r) \right] \right\}^c \\ &= A \circ K(t') \cap \left[\bigcup_{r>t} A \circ K(r) \right]^c \end{aligned} \tag{49}$$

which is null since the latter union includes $A \circ K(t')$.

In the present continuous setting we must adopt a more general view of the non-interfering spatial process. To do so we generalize the random grain model employed by Sand and Dougherty [9] in their analysis of the statistical distributions for granulometric pattern-spectrum moments. Specifically, we assume that to form a realization A, a component number Z is selected from a Poisson distribution with mean μ_A, parameters $t_1, t_2, \ldots, t_Z$ are independently selected from some distribution Π_A possessing density $f_A(t)$, translations $x_1, x_2, \ldots, x_Z$ are randomly chosen, and

$$A = \bigcup_{m=1}^{Z} K(t_m)_{x_m} \tag{50}$$

where the components are non-interfering. There are several salient points regarding this more general model:

1. It reduces to the former discrete model if the parameter class is finite.
2. Equation (22) holds.
3. Equation (23) holds, its new form being

$$A \circ K(\lambda) = \bigcup_{m=\lambda}^{Z} K(t_m)_{x_m} \tag{51}$$

4. Equation (24) holds, its new form being

$$A(t) = \bigcup_{m} \{K(t_m)_{x_m} : t_m = t\} \tag{52}$$

If we assume that noise realization N derives from a similar non-interfering process with Poisson mean μ_N and t_k selected from a distribution Π_N possessing density $f_N(t)$, then $\psi(A \cup N) = [A(t) \cup N(t)]$. The estimate $\hat{A}$ for A is given by eq. (29) with t_m in place of m; however, in the present context S is a subset of $[0, \infty)$ and is not a discrete set. The estimation problem is to find S for which $E[\rho(A, \hat{A})]$ is minimzed, with # now denoting area.

Similarly to eq. (32), it can be shown that

$$\rho(A, \hat{A}) = \sum_{t_m \notin S} \#A(t_m) + \sum_{t_k \in S} \#N(t_k) \tag{53}$$

Because the component counts for both signal and noise are random, $E[p(A, \hat{A})]$ does not easily reduce; however, if we make the simplifying assumption that the component counts are fixed, say at the respective means μ_A and μ_N, we then obtain

$$E[\rho(A, \hat{A})] = \mu_A \int_{S^c} \#K(t) f_A(t) dt + \mu_N \int_S \#K(t) f_N dt \tag{54}$$

To see the manner in which we arrive at eq. (54), let Λ denote the first summand in eq. (54) and let

$$\Lambda_m = \begin{cases} \#A(t_m), & \text{if } t_m \notin S \\ 0, & \text{otherwise} \end{cases} \tag{55}$$

Then

$$E[\Lambda] = \sum_{m=1}^{\mu_A} E[\Lambda_m] = \mu_A E[\Lambda_1] = \mu_A \int_{S^c} \#K(t)] f_A(t) dt \tag{56}$$

The second summand in eq. (54) is handled similarly. The best S is given by

$$S = \{t : \mu_N f_N(t) < \mu_A f_A(t)\} \tag{57}$$

Note the similarity to the discrete solution given in eq. (37).

12.7 Extension of Continuous Spectra to Tau-Openings

Generalization of the continuous theory to tau-openings with $Q(t) = \{K(t), J(t)\}$ proceeds along similar lines to the generalization in the discrete case, under the assumption that $K(t) \circ J(t') = J(t) \circ K(t') = \emptyset$ for $t' > t$. For instance, eq. (44) and (45) become

$$A \circ Q(\lambda) = \bigcup_{m=\lambda}^{Z} K(t_m)_{x_m} \bigcup \bigcup_{k=\lambda}^{W} J(t_k)_{x_k} \tag{58}$$

$$A(t) = \bigcup_m \{K(t_m)_{x_m} : t_m = t\} \bigcup \bigcup_k \{J(t_k)_{x_k} : t_k = t\} \tag{59}$$

where Z and W are the respective Poisson variables for $\{K(t)\}$ and $\{J(t)\}$, possessing respective means μ_{A_K} and μ_{A_J}, and it is assumed that the corresponding parameter sequences derive from the respective densities f_{A_K} and f_{A_J}. Generalization to more than two structuring-element sequences is immediate.

Like the signal, the noise too can be generated by both $\{K(t)\}$ and $\{J(t)\}$ with the $K(t)$ and $J(t)$ Poisson variables possessing means μ_{N_K} and μ_{N_J}, respectively, and the corresponding parameter sequences possessing probability densities f_{N_K} and f_{N_J}, respectively. Then the error equation takes the form

$$\begin{aligned} E[p(A, \hat{A})] = \mu_{A_K} \int_{S^c} \#K(t) f_{A_K}(t) dt + \mu_{A_J} \int_{S^c} \#J(t) f_{A_J}(t) dt \\ + \mu_{N_J} \int_S \#K(t) f_{N_J}(t) dt + \mu_{N_J} \int_S \#J(t) f_{N_J}(t) dt \end{aligned} \tag{60}$$

The preceding equation extends to any finite number of structuring-element sequences. In particular, if any of the Poisson means are zero, then the equation reduces to one in which the signal and noise are generated by different primitive shapes, which shows that our model allows signal and noise to be generated by different primitives.

As in the single-opening situation, there is a close connection between the present theory and the Matheron theory for Euclidean granulometries. Matheron

[6] calls a class of images **K** a generator of a Euclidean granulometry $\{\Psi_t\}$ if the invariant class of Ψ_1 consists of unions of translations of scalar multiples $tK, t \geq 1$, of elements $K \in \mathbf{K}$. If it happens that the images of **K** are convex, then

$$\Psi_t(A) = \bigcup\{A \circ tK : K \in \mathbf{K}\} \tag{61}$$

is a granulometry with generator **K**. Now suppose $\mathbf{K} = \{K_1, K_2, \ldots, K_p\}$ is finite and $tK_1 \circ t'K_j = \emptyset$ for $t' > t$ and $i \neq j$. If for any K_i in **K** we define $K_i(t) = tK_i$, then $\{K_i(t)\}$ is a basis in our present spectral sense. Assuming each realization A of the spatial process is formed in the usual way from this basis, the spectral component $A(t)$ takes the form

$$A(t) = \Psi_t(A) - \bigcup_{r>t} \Psi_r(A) \tag{62}$$

Note that it has been demonstrated in [6] that $\Psi_r(A) < \Psi_t(A)$ for $r > t$, and that the invariant class of Ψ_r is a subset of the invariant class of Ψ_t.

12.8 Interpretation of the Optimal Estimator $\hat{A}$ in the context of Optimal Morphological Estimation

Morphological dilation and erosion operations are translation invariant and increasing. This motivates calling mappings which are increasing and translation invariant morphological filters. A general framework for the characterization of statistically optimal morphological filters has been developed by Dougherty [10, 11, 12, 13, 14]. An interesting question is how do we treat the problem of optimally estimating one random variable by a morphological funciton of a finite number of observations? Included in the discussions [10, 11] is the manner in which we apply constraints to the filter, so that the optimal estimator is a particular type of morphological filter, say tau-opening or linear operator. A key class of increasing, translation invariant mappings are the alternating sequential filters of Sternberg [15] and Lougheed [16] (see Serra [17]), and an optimization criterion for these has been developed by Schonfeld and Goutsias [18, 19]. In the present section, we wish to briefly investigate the relationship between the optimal filter based on the opening spectrum and the general problem of morphological estimation.

Returning again to the Wiener filter, the weights w_n of eq (3) provide the estimate $\hat{f}$ of f relative to an orthonormal basis $b_1, b_2, \ldots, b_n$, with the summation over this basis serving as the inversion back to the spatial domain. In the general algebriac paradigm of Section 3, f is found from the weighted representation by applying ψ^{-1}. When applying optimal estimation relative to the morphological representation in terms of $\mathcal{K}$, equation (29) provides the required inversion. An interesting and important question can be posed: Does the estimator $\hat{A}$ possess a morphological representation? That is, can we write $\hat{A} = \Omega(A \cup N)$, where Ω is a "morphological operation?" If by "morphological operation" we mean an increasing, translation invariant mapping, then $\hat{A}$ possesses no such representation. Indeed from the manner in which S is chosen, it can be seen that if $A \cup N'$ is obtained from $A \cup N$ by replacing a noise component $K(m)_x$ of N by a noise component $K(m')_y$ where $m' > m, m \in S, m' \notin S$, and $K(m')_y$ properly contains $K(m)_x$, then $A \cup N$ is a proper subset of $A \cup N'$, but, according to eq. (25), the filtered version of $A \cup N'$ is a proper subset of the filtered version of $A \cup N$. Thus, the optimal fitler determined by eq. (29) is not necessarily increasing (although it might be).

Whether we take the weak definition of a morphological filter adopted in [5, 10], that of being increasing and translation invariant, or the strong definition adopted by Serra [7], which includes idempotence (without assuming translation invariance because the definition lies in the context of lattice theory), the mapping $\Omega(A)$ defined by eq. (29) is not necessarily a morphological filter. Consequently, even though the measure $E[p(A, \hat{A})]$ can be interpreted as mean-square error in the binary setting, the operator Ω is not necessarily expressable in terms of the Matheron expansion as an union of erosions, and it is precisely this expansion in which the mean-square optimization theory of [10] is framed.

Nevertheless, the estimation operator is translation invariant and can be expressed "morphologically," where here we mean that it can be expressed using ordinary morphological operations in conjunction with set-theoretic operations. The desired expression is immediate from the definition of the spectrum operator ψ, and is imply eq. (25) applied to S. Rewritten, eq. (29) takes the form

$$A = \bigcup_{m \in S} (A \cup N) \circ K(m-1) - (A \cup N) \circ K(m) \tag{63}$$

12.9 Conclusion

For the problem of filtering corrupted binary images of the form $A \cup N$, we have chosen an appropriate morphological opening spectral decomposition, as well as distance and energy measures resulting in an appropriate measure of estimation error. Based upon these choices (which are quite different from the analogous choices for the additive noise/linear filter problem, and which eliminate the requirement for orthogonality or an inner product space) we have derived optimal filtering results analogous to conventional Weiner filtering results based on image and noise energy contents in each spectral bin.

The assumptions on the image and noise models in order for the results to be valid are presently fairly strong. The image and noise connected components are modeled as translated copies of objects from a single ordered opening basis set (Sections 4 and 6) or a collection of such basis sets (Sections 5 and 7). In addition there is a non-interference (non-overlap) condition so that all objects remain distinct and no objects are created that fail to arise directly from basis sets.

These conditions guarantee sufficiency. However, they are actually stronger than need be. They were sufficient to guarantee that $(A \cup B) \circ K = A \cup B$ and $\#(A \cup B) = \#A + \#B$. It is easy to create instances in which $(A \cup B) \circ K = A \cup B$ and A and B are not non-interfering sets. If A and B are not exclusive then $\#(A \cup B) \leq \#A + \#B$. So if the sets overlap, the quantities we have been computing will be strict upper bounds. However, in this case, the overlapping can be regarded as a random process and instead of computing $\#(A \cup B)$ a composition of $E[\#(A \cup B)] = k(\#A + \#B)$ for an appropriate $0 < k < 1$ can be made. Along these lines, the possibility of generalizing the results is quite strong.

In addition, in order to better handle irregular or ill-defined noise sets, as well as ideal (noise free) images comprised of families of objects for which no simple ordered opening basis is obvious, we are working on extending our results to instances where the assumptions on image and noise objects are relaxed. In particular, extension to the case where the objects are in some sense well-sorted by one or more bases is being sought in derivations and experiments.

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