Unimodal Search

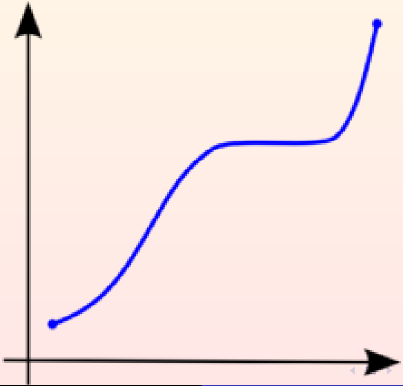
Robert M. Haralick

Computer Science, Graduate Center
City University of New York

Monotonic Increasing Functions

Definition

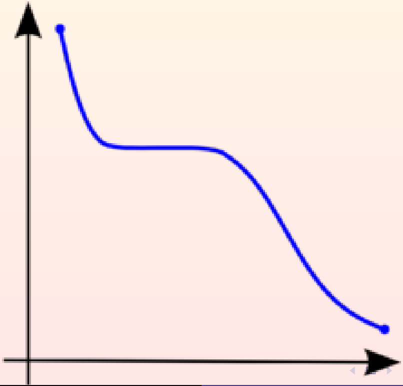
Let I be any interval of the real numbers R. A function $f : I \to R$ is Monotonically Increasing if and only if for every $(x, y) \in I \times I$, if $x \geq y$, then $f(x) \geq f(y)$.



Monotonic Decreasing Functions

Definition

Let I be any interval of the real numbers in R. A function $f : I \to R$ is Monotonically Decreasing if and only if for every $(x, y) \in I \times I$, if $x \geq y$, then $f(x) \leq f(y)$.



Strictly Increasing Functions

Definition

Let I be any interval of the real numbers in R. A function $f : I \to R$ is Strictly Increasing if and only if for every $(x, y) \in I \times I$, if $x > y$, then $f(x) > f(y)$.

Strictly Decreasing Functions

Definition

Let I be any interval of the real numbers in R. A function $f : I \to R$ is Strictly Decreasing if and only if for every $(x, y) \in I \times I$, if $x > y$, then $f(x) < f(y)$.

Unimodal Functions

Definition

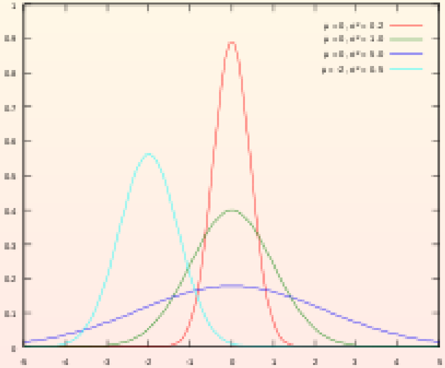
Let $[a, b]$ be any interval of the real numbers in R. A function $f : [a, b] \rightarrow R$ is Unimodal if and only if there exists $x^* \in [a, b]$ such that

- $f(x^*) \geq f(x), x \in [a, b]$
- f is strictly increasing in $[a, x^*]$
- f is strictly decreasing in $[x^*, b]$

Or

- $f(x^*) \leq f(x), x \in [a, b]$
- f is strictly decreasing in $[a, x^*]$
- f is strictly increasing in $[x^*, b]$

Unimodal Functions



Information in Unimodality

Suppose f is a unimodal function on $[0, L]$ with a maximum at x^*. Suppose $x_1 > x_2$ and $x_1, x_2 \in [0, L]$. Consider $f(x_1)$ and $f(x_2)$. There are 3 cases:

- $f(x_1) < f(x_2)$
- $f(x_1) > f(x_2)$
- $f(x_1) = f(x_2)$

$f(x_1) < f(x_2)$

If $f(x_1) < f(x_2)$, then it is impossible for the maximum to be in the interval $[x_1, L]$. The search need only continue in the interval $[0, x_1]$, an interval of length x_1.

$f(x_1) > f(x_2)$

If $f(x_1) > f(x_2)$, then it is impossible for the maximum to be in the interval $[0, x_2,]$. The search need only continue in the interval $[x_2, L]$, an interval of length $L - x_2$.

$f(x_1) = f(x_2)$

If $f(x_1) = f(x_2)$, then it is impossible for the maximum to be in the interval $[0, x_1]$ or $[x_2, L]$. The search need only continue in the interval $[x_1, x_2]$.
Without loss of generality, this case can be included either in case 1 or case 2.

Where To Place A Trial

If $f(x_1) < f(x_2)$, the maximum must be in the interval $[0, x_1]$.
If $f(x_1) > f(x_2)$ the maximum must be in the interval $[x_2, L]$.
If either of these intervals were larger than the other, the search could lose efficiency. Therefore

$$x_1 = L - x_2$$

The ratio of the length of the new interval of uncertainty to the length of the old interval of uncertainty is

$$r = \frac{x_1}{L}$$

Where To Place A Trial

If $f(x_1) < f(x_2)$, the interval of uncertainty is $[0, x_1]$ and the interior completed trial is x_2. We must place the next trial x_3 so that

$$x_2 = x_1 - x_3$$

The ratio of the length of the new interval of uncertainty to the length of the old interval of uncertainty is

$$r = \frac{x_2}{x_1}$$

System of Equations

$$
\begin{aligned}
x_1 &= L - x_2 \\
r &= \frac{x_1}{L} \\
r &= \frac{x_2}{x_1}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{x_1}{L} &= \frac{x_2}{x_1} \\
x_1^2 - x_2 L &= 0
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
x_1 + x_2 &= L \\
x_1^2 - x_2 L &= 0
\end{aligned}
$$

System of Equations

$$\begin{aligned} x_2 &= L - x_1 \\ x_1^2 - x_2 L &= 0 \end{aligned}$$

Substituting x_2 into the second equation,

$$\begin{aligned} x_1^2 - (L - x_1)L &= 0 \\ x_1^2 + x_1 L - L^2 &= 0 \\ \left(\frac{x_1}{L}\right)^2 + \left(\frac{x_1}{L}\right) - 1 &= 0 \\ \frac{x_1}{L} &= \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} \\ &= \frac{-1 \pm \sqrt{5}}{2} \end{aligned}$$

Golden Search

$$\frac{x_1}{L} = \frac{-1 \pm \sqrt{5}}{2}$$

Since, $\frac{x_1}{L} > 0$ and $\sqrt{5} > 1$

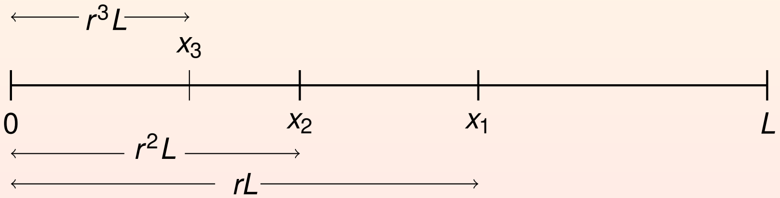
$$\begin{aligned} r &= \frac{x_1}{L} \\ &= \frac{-1 + \sqrt{5}}{2} \\ &\approx .618 \end{aligned}$$

Golden Search

$$
\begin{aligned}
r &= \frac{-1+\sqrt{5}}{2} \\
r &= \frac{x_1}{L} \\
x_1^2 - x_2 L &= 0 \\
\left(\frac{x_1}{L}\right)^2 &= \frac{x_2}{L} \\
r^2 &= \frac{x_2}{L}
\end{aligned}
$$

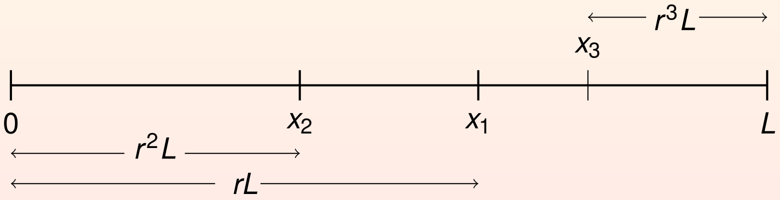
Golden Search

If the continued interval of uncertainty is the left interval, then



Golden Search

If the continued interval of uncertainty is the right interval, then



Golden Section Search For Maximum

```
float golden_section_max(float *f,float a,float b,float eps)
{
r=(-1.+sqrt(5))/2;
x1=a+r*(b-a);
x2=b-r*(b-a);
while abs(x1-x2) > eps
  {
  if(f(x1) < f(x2)) /left interval
    b=x1;
  else
    a=x2; /right interval
  x1=a+(b-a)*r;
  x2=b-r(b-a);
  }
return (a+b)/2;
}
```