The Language of Sets In Mathematics

In order to say things precisely, the language of mathematics is used. Here we endeavor to illustrate both the language and symbolism of basic mathematical set concepts.

1 Sets

A pack of dogs, a family of pigs and a bunch of bananas are examples of sets of things. A *set* is any collection of objects called *elements* or *members.* In the above examples, dogs, pigs, and bananas are elements of their respective sets.

Suppose we define a set by listing its members. Let the set A be the set of four people: Bob, Joy, Tammy, and Ken. Then we write

$$A = \{\text{Bob, Joy, Tammy, Ken}\}.$$

We list the elements in the set separated by commas and then we enclose the entire list in curly brackets.

In a set only one instance of each element in the set is allowed. Thus

$$A = \{\text{Bob, Bob, Joy, Tammy, Ken}\}$$

is not a set because the element Bob, occurs twice.

Listing the elements in a set is not the only way to define a set. Let A be the set of all positive even numbers less than 10. Then we write

$$A = \{x|x \text{ is even, positive and less than } 10\}.$$

This is read: "A is the set of all numbers x such that x is even, positive and less than 10. We might note that there are alternative ways of representing the same set. For example,

$$\begin{aligned} A &= \{x|x = 2n \text{ for some integer } n \text{ satisfying } 1 < n < 4\} \\ &= \{2, 4, 6, 8\}. \end{aligned}$$

For notational consistency, sets are usually denoted by capital letters such as

$$A, B, C, \ldots, W, X, Y, Z,$$

while the elements in sets are usually denoted by small letters such as

$$a, b, c, \ldots, w, x, y, z.$$

Sometimes we wish to refer to sets whose elements themselves are sets. To avoid confusion we call such a set of sets as a collection or family of sets and denote the collection with script letters. For example $\mathcal{S}$ is a collection containing A, B, C. $\mathcal{S} = \{A, B, C\}$, where the sets A, B, C are $A = \{a, b\}$, $B = \{b, c, d\}$, $C = \{d, e\}$.

When an element x is a *member of* or *belongs to* to a set A we write

$$x \in A.$$

When an element x is *not a member* of or *does not belong to* a set A we write

$$x \notin A.$$

Two sets A and B are *equal* if and only if they have the same elements, that is, every element which belongs to A also belongs to B and every element which belongs to B also belongs to A. This condition for the equality of two sets is known as the axiom of extension. The *equality* of two sets is denoted by the familiar equal symbol $=$. If set A equals set B we write

$$A = B.$$

If A and B are sets and every element belonging to A also belongs to B, then we say that A is a *subset* of B, and we write

$$A \subseteq B.$$

For example, if $A = \{a, b, c\}$ and $B = \{a, b, c, d\}$, then $A \subseteq B$ since every element of A is also an element of B.

This definition of subset implies that any set A is a subset of itself.

$$A \subseteq A \text{ for any set } A$$

For example, if $A = \{a, b, c\}$, then $A \subseteq A$ since every element of A is also an element of A.

If a set A is a subset of a set B, and if there exists some element which belongs to B and not to A, then A is a *proper* subset of B, and we write

$$A \subset B.$$

For example, if $A = \{a, b, c\}$ and $B = \{a, b, c, d\}$, then $A \subset B$ since the set B has the element d which is not in the set A.

If B is a subset of A, then A is called a *superset* of B, and we write

$$A \supseteq B.$$

In other words, B is a subset of A, if A contains B and another way of saying that A contains B is to say that A is a superset of B.

If B is a proper subset of A, then A is called a *proper superset* of B and we write

$$A \supset B.$$

We may construct a particular subset B of A by letting B be all the elements of A satisfying a specified constraint. If A is the set of all women, for example, then a subset B of A may be defined as the set of all women in A who are married. The specified constraint is indicated in the clause "who are married."

The principle which allows one to construct a subset B from a set A by specifying a constraint is known as the axiom of specification.

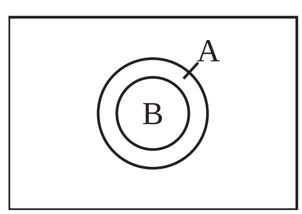
One immediate implication of the axiom of specification is that there exists a set $\emptyset$, called the *empty set,* which has no members. We can understand this the following way: Let A be a set with some members. Now use the axiom of specification with a constraint which is always false to define the set $\emptyset$. For example, let

$$\emptyset = \{x \in A | x \neq x\}.$$

Since there exists no element x not equal to itself, $\emptyset$ must be a set with no elements.

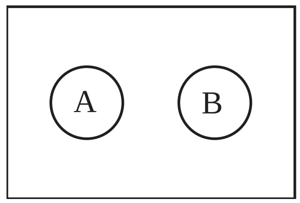
Any time sets are studied, all the sets studied are usually subsets of some fixed or given large set. Such a set is called the *universal set* and is denoted by U.

Many times the relationship between sets can more easily be seen by drawing pictures of them. One such picture is the *Venn diagram.* Here the Universal set is represented by a large box and various subsets are represented by circles, squares or other closed figures lying within the large box. Areas which overlap are interpreted as elements which are common to the sets. If B is a subset of A, then the figure representing B is drawn so that it entirely lies within the figure drawn for A as illustrated below.

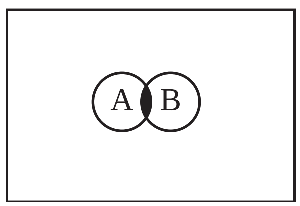


Sets A and B are *mutually exclusive* or *disjoint* if and only if they have no elements in common. Hence no element of A is an element of B and no element of B is an element of A.

If A and B are *disjoint* then the figures representing A and B are drawn so that there is no overlap between them as illustrated in the picture below.



If A is not a subset of B and A and B are not disjoint then the figures representing A and B are drawn so that there is some but not complete overlap between them, as illustrated in the picture below.



Many times it is useful to combine the sets we work with in various ways. For example, sometimes we might wish to combine the elements of two sets into one comprehensive set. The set union operation allows us to do precisely that. The union of two sets A and B is denoted by $A \cup B$ and is defined by

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

The union operation can be extended to any collection of sets, whether the collection is finite, countable, or uncountable. The axiom of unions states for any collection of sets, there exists a comprehensive set which contains as members all the elements which belong to each set from the given collection.

Let $\mathcal{E}$ be a collection of sets. Then we write the *union* of all the sets in $\mathcal{E}$ in the following way:

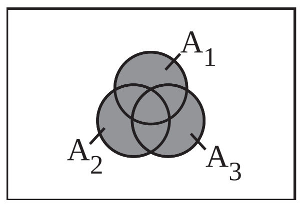
$$\bigcup_{A \in \mathcal{E}} A$$

The expression $A \in \mathcal{E}$ under the union symbol means that we are operating on all the sets in $\mathcal{E}$. The symbol A beside the union symbol means that we are taking the union of all sets A where A is one of the sets in $\mathcal{E}$, the set we are operating on.

From the definition of the union operation it should be clear that

$$\bigcup_{A \in \mathcal{E}} A = \{a | a \in A \text{ for some } A \in \mathcal{E}\}$$

If $\mathcal{E}$ consists of the sets A_1, A_2, A_3, $\mathcal{E} = \{A_1, A_2, A_3\}$, then the union of the sets A_1, A_2, A_3 can be represented on a Venn diagram by shading in all areas which lie in A_1 or A_2 or A_3. In the picture below, the shaded area represents $\bigcup_{A \in \mathcal{E}} A$, which may for this example be written as $A_1 \cup A_2 \cup A_3$.



Sometimes we might wish to form a set which has for its members only those elements which are common to two sets. The set intersection operation allows us to do that. The intersection of two sets A and B is denoted by $A \cap B$ and is defined by

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

The intersection operation can be extended to any collection of sets. Let $\mathcal{E}$ be a collection of sets. We write the intersection of all the sets in the collection as

$$\bigcap_{A \in \mathcal{E}} A$$

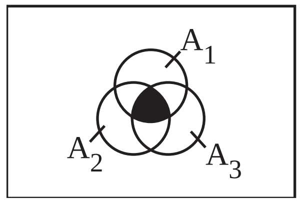
The expression $A \in \mathcal{E}$ under the intersection symbol means that the operation is for all sets A in $\mathcal{E}$. The symbol A beside the intersection symbol means that we are taking the intersection of all sets A where A is one of the sets in $\mathcal{E}$.

From the definition of the intersection operation it should be clear that

$$\bigcap_{A \in \mathcal{E}} A = \{a | a \in A \text{ for every } A \in \mathcal{E}\}$$

We should note that the axiom of specification allows us to construct the intersection.

If $\mathcal{E}$ consists of the sets A_1, A_2, A_3, $\mathcal{E} = \{A_1, A_2, A_3,\}$, then the intersection of the sets A_1, A_2, A_3 can be represented on a Venn diagram by shading all areas which are common to A_1, A_2, and A_3. In the picture below, the shaded area represents $\bigcap_{A \in \mathcal{E}} A$, which may for this example be written as $A_1 \cap A_2 \cap A_3$.



Sometimes we may wish to refer to a set which has all the elements of the universal set U except for those elements in the set A. In this case we denote such a set by A^c and we call A^c the *complement* of A.

$$A^c = \{x \in U | x \notin A\}.$$

Other times we may wish to refer to a set which has all the elements of a set B except for those elements in the set A. Such a set may be expressed $B \cap A^c$. Very often this set is expressed by $B - A$, the minus sign being used to express the idea that the resultant set has all the elements of the set B minus or except those elements of the set A.

For finite sets, the number of elements in the set is finite. We count the number of elements in a finite set A using the notation $|A|$.

There is an abbreviated notation for $1, 2, \ldots, N$. It can be written as $[1, N]$

There are many types of collections of sets. Some of them are so important that they have specific names. If $\mathcal{E}$ is a collection of sets, each set in the collection being a subset of some set B, then $\mathcal{E}$ is called a *cover* of B or is said to cover B if and only if the union of all the elements in the subsets in the collection equals the set B; that is,

$$\bigcup_{A \in \mathcal{E}} A = B$$

Another important collection of sets is called a *partition.* If $\mathcal{E}$ is a collection of sets, each set in the collection being a subset of some set B, then $\mathcal{E}$ is called a partition of B if and only if each pair of subsets in $\mathcal{E}$ is disjoint (have no elements in common) and the union of all the elements in the subsets of the collection $\mathcal{E}$ equals the set B. A set A in a partition Π is called a *cell* or *block* of the partition.

In other words, a partition Π of a set B is a collection of mutually exclusive sets which cover B.

A third type of collection of sets is called a *power set.* A collection of sets $\mathcal{P}$ is called a power set of a set A if and only if each element of $\mathcal{P}$ is a subset of the set A and all possible subsets of the set A belong to $\mathcal{P}$.

A fourth type of collection of sets is called a *hierarchy*. Hierarchy is a means of ranking or organizing things. Different fields use the word in slightly different ways, but the essence of hierarchy is the following definition. A collection of sets $\mathcal{H}$ is called a hierarchy of a set A if and only if

1. $A \in \mathcal{H}$
2. $A = \bigcup_{X \in H} X$
3. $A, B \in \mathcal{H}$ and $A \cap B \neq \emptyset$ implies $A \subset B$ or $B \subset A$.

Hierarchies can be represented by a rooted tree. The set A is the root of the tree. Set $X \in \mathcal{H}$ is immediately below set $Y \in \mathcal{H}$ of the tree if and only if $X \subset Y$ and there is no set $Z \in \mathcal{H}$ such that $X \subset Z \subset Y$.

In general, a set X in a hierarchy $\mathcal{H}$ can be said to be of lower rank than a set Y if and only if $X \subset Y$.

Examples of hierarchies can be found in most organizational structures. Computer files in a file system are stored in a hierarchy of directories. Large electronic devices such as computers are usually composed of modules, which are themselves created out of smaller components (integrated circuits), which in turn are internally organized using hierarchial methods (e.g. using standard cells).

Biological taxonomic classifications form a hierarchy: kingdom, phylum, class, order, family, genus, species. Multiple organisms may be different species of the same genus. In the hierarchy, the organism is a singleton set. The species is the set of all organisms belonging to the species. The genus can be thought of as consisting of all organisms belonging to all species of the genus and so on.

Elements of a set may themselves have an internal structure. One of the more common structures is the ordered pair. The ordered pair has two components. The component which is written on the left is called the first component and the component which is written on the right is called the second component. The components are separated by a comma and the pair of components is enclosed in parentheses: if $d = (a, b)$ then we say that the element d is an ordered pair whose first component is a and whose second component is b. A common example of ordered pairs is married couples, the first component being the husband (wife) and the second component being the wife (husband).

Let A and B be sets. The *Cartesian product* of A with B is a set whose members are ordered pairs, the first component of each ordered pair being some element belonging to A and the second component of each ordered pair being some element belonging to B. The Cartesian product of A with B is denoted by $A \times B$. For example, if $A = \{a, b, c\}$ and $B = \{c, d, e\}$, then

$$A \times B = \{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, c), (c, d), (c, e)\}.$$

2 Binary Relations

Consider the statements "x is a student at y" or "x is greater than y." The first statement relates members of the set of all people to members of the set of schools. The second relates members of the set of real numbers to other members of that same set. A *binary relation* from set A to set B is the set of ordered pairs such that the first element (from A) is related, by a statement similar to those above, to the second element (from B). Hence the relation defined by the first statement above is the set of all ordered pairs whose first element is the name of a student and whose second element is the name of the school that the student attends.

The *domain* of a binary relation R where $R \subseteq A \times B$ is denoted by Dom (R) and is a subset of A containing elements which are the first element of at least one of the ordered pairs in R. For example the domain of the relation defined by "x is a student at y" is the set of all students which is clearly a subset of the set of all people.

Similarly, the *range of* a binary relation R where $R \subseteq A \times B$ is denoted by Ran (R) and is a subset of B containing elements which are the second element of at least one of the ordered pairs in R. For example the range of the relation defined by "x is a student at y" is the set of all schools which is clearly a subset of the set of all educational institutions.

A binary relation R which relates members of a set A to members of the same set is said to be a binary relation *on* A. For example the statement "x is greater than y" relates members of the set of real numbers to other members of the set of real numbers. Thus the set of real numbers together with the specification "greater than" defines a relation *on* the set of real numbers.

Let R be a binary relation. Then, $R \subseteq A \times B$. Let $a \in A$. The subset of all elements of B which relate to a through R is denoted by $R(a)$. That is, $R(a) = \{b \in B \mid (a, b) \in B\}$. Let $b \in B$. The set of all elements of A which relation to b through R is denoted by $R^{-1}(b)$. $R^{-1}(b) = \{a \in A \mid (a, b) \in R\}$.

Let R be a binary relation on a set A: $R \subseteq A \times A$. The relation R is *reflexive* if and only if $a \in A$ implies $(a, a) \in R$.

If the statement that defines R is "x is greater than or equal to y" then R is *reflexive* because for each element a in A, the ordered pair (a, a) is an element of R. That is to say, 3 is related to 3 by the statement above, hence the ordered pair $(3, 3)$ is an element of R. Clearly the relation defined by the statement "x is greater than y" is not reflexive.

A binary relation R on A is the *identity* relation on A if R relates each element of A to only itself. For example the relation on the set of real numbers which is defined by the statement "x is equal to y" is the identity relation on the real numbers. This is so because the only ordered pairs in R are of the form (x, x).

A binary relation of R on A is *symmetric* if $(x, y) \in R$ implies $(y, x) \in R$. For example, consider the binary relation R on $A = \{1, 2, 3, 4\}$ of the form

$$R = \{(1, 2), (4, 1), (1, 1), (1, 4), (2, 1)\}.$$

This is symmetric. Clearly the relation on the real numbers which is defined by the statement "x is less than y" is not symmetric.

A binary relation of R on A is *transitive* if for $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$. For example, the binary relation on the set of real numbers defined by the statement "x is less than y" is transitive.

A binary relation R on a set A which is reflexive, symmetric, and transitive is called an *equivalence relation*.

An equivalence relation E on a set A determines a partition Π of A in the following way: $\Pi = \{E(a) \mid a \in A\}$. Thus two elements $a, b \in A$ that satisfy $(a, b) \in E$ are in the same cell or block of Π. It is easy to see that Π is indeed a partition of A. Since E is an equivalence relation, it is reflexive. Hence for any $a \in A$, $(a, a) \in E$. Thus $\cup_{a \in A} E(a) = A$. Let $a, b \in A$. Consider $E(a)$ and $E(b)$. Either $E(a) \cap E(b) = \emptyset$ or $E(a) = E(b)$. Suppose $E(a) \cap E(b) \neq \emptyset$. Let $c \in E(a) \cap E(b)$. Then $(a, c) \in E$ and $(b, c) \in E$. Since E is symmetric, $(b, c) \in E$ implies $(c, b) \in E$. Since E is transitive, $(a, c) \in E$ and $(c, b) \in E$ implies $(a, b) \in E$. Now suppose $x \in E(a)$. Then $(a, x) \in E$. Since E is symmetric, $(a, x) \in E$ implies $(x, a) \in E$. Since E is transitive, $(x, a) \in E$ and $(a, b) \in E$ imply $(x, b) \in E$. Since E is symmetric, $(x, b) \in E$ implies $(b, x) \in E$. Therefore, $x \in E(b)$. A similar argument shows that if $x \in E(b)$, then $x \in E(a)$. Hence if $E(a) \cap E(b) \neq \emptyset$, then $E(a) = E(b)$.

A binary relation R on a set A is *asymmetric* if and only if $(x, y) \in R$ implies $(y, x) \notin R$. The binary relation on the set of numbers defined by "x is less than y" is asymmetric.

A binary relation R on a set A is *antisymmetric* if and only if $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$. The relation on the set of numbers defined by "x less than or equal to y" is antisymmetric.

A binary relation R on a set A is a *partial order* on A if and only if R is reflexive, antisymmetric, and transitive. An example of a relation that induces a partial order is the subset relation. Let A be any collection of sets. Let $R \subseteq \mathcal{A} \times \mathcal{A}$ defined by

$$R = \{(X, Y) \in \mathcal{A} \mid X \subseteq Y\}$$

Since $X \subseteq X$ is true for every set X, $X \in \mathcal{A}$ implies $(X, X) \in R$. If $X, Y \in \mathcal{A}$, with $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$. Hence R is antisymmetric. Finally, if $(X, Y) \in R$ and $(Y, Z) \in R$, then $X \subseteq Y$ and $Y \subseteq Z$. Hence $X \subseteq Z$ which implies that $(X, Z) \in R$, making R transitive.

Let A be a set and R be a partial order on A. If for every $(x, y) \in A$, either $(x, y) \in R$ or $(y, x) \in R$, then R is said to be a *chain.*

Let $A = \{a_1, \ldots, a_N\}$ be a finite set and R be chain. Then the elements of A can be ordered $< a_{i_1}, \ldots, a_{i_N} >$ such that $(a_{i_n}, a_{i_{n+1}}) \in R, n = 1, \ldots, N-1$.

Let A be a set and $B \subseteq A$. If R is a partial order on A and $(m, x) \in R$ for every $x \in B$, then m is said to be a *lower bound* of B.

Let A be a set and $B \subseteq A$. If R is a partial order on A and $(m, m^*) \in R$ for every m satisfying $(m, x) \in R$ for every $x \in B$, then m^* is said to be a *greatest lower bound* of B.

Let A be a set and $B \subseteq A$. If R is a partial order on A and $(x, m) \in R$ for every $x \in B$, then m is said to be an *upper bound* of B.

Let A be a set and $B \subseteq A$. If R is a partial order on A and $(m^*, m) \in R$ or every m satisfying $(x, m) \in R$ for $x \in B$ then m^* is said to be a *lowest upper bound* of B.

Let A be a set and R be a partial order on A. If for every $a, b \in A$, there exists a greatest lower bound, called the meet, m_* of a and b, $(m_*, a) \in R$ and $(m_*, b) \in R$ and there exists a lowest upper bound, called the join, m^* of a and b, $(a, m^*) \in R$ and $(b, m^*) \in R$, then R is said to be a *lattice.*

A binary relation R from A to B is said to be *defined everywhere* if Dom $R = A$. For example, if R is a binary relation from $A = \{1, 2, 3, 4\}$ to $B = \{a, b, c, d, e\}$ of the form

$$R = \{(1, d), (2, a), (3, d), (4, c), (1, c)\}$$

then R is defined everywhere since every element in A is paired to at least one element of R.

A binary relation R from A to B is said to be *single valued* if $(x, y) \in R$ and $(x, z) \in R$ implies $y = z$. That is to say, no element in A will appear in more than one of the ordered pairs of R. For example if R is a binary relation from $A = \{1, 2, 3, 4\}$ to $B = \{a, b, c, d, e\}$ of the form

$$R = \{(1, c), (2, e), (4, b)\},$$

Then R is single valued since no element in A appears in more than one ordered pair of R.

Let $H \subseteq A \times B$. If H is defined for every A and is single valued, then H is called a *function* That is, a function H assignes to each element of A some element of B. If H is a function, we write $H : A \to B$.

A binary relation R from A to B is said to be *onto* B if Ran $R = B$. For example, if R is a binary relation from $A = \{1, 2, 3, 4\}$ to $B = \{a, b, c, d, e\}$ of the form $R = \{(1, a), (1, b), (2, c), (3, e), (4, d)\}$, then R is onto B since every element in B is paired to at least one element of A.

3 N-Ary Relations

Let $A_1, \ldots, A_N$ be sets. The Cartesian product $A_1 \times A_2 \times \ldots \times A_N$ consists of a set of all N-tuples whose first component is an element from A_1, whose second component is an element from A_2, ..., and whose N^{th} component is an element from A_N. The Cartesian product $A_1 \times A_2 \times \ldots \times A_N$ can be more compactly written as $\bigtimes_{n=1}^{N} A_N$. The Cartesian product of A with itself N times, $\underbrace{A \times A \times \ldots A}_{N \text{ times}}$ can be more compactly written A^N.

Definition 1. *A relation R is said to be a N-ary relation if and only if $R \subseteq A_1 \times A_2 \times \ldots A_N$. If $R \subseteq A^N$, then R is said to be a N-ary relation on A.*

Definition 2. *Let $R \subseteq A^N$ and $H \subseteq A \times B$. The relation composition of R with H is denoted by $R \circ H$ and is defined by*

$$\begin{aligned} R \circ H = \{(b_1, \ldots, b_N) \in B^N \quad | \\ & \textit{for some } (a_1, \ldots a_n) \in R, \\ & (a_n, b_n) \in H, n = 1, \ldots, N\} \end{aligned}$$

For example, if $A = \{a, b, c\}$, $B = \{1, 2\}$, $R = \{(a, b, b), (c, b, a), (b, a, a), (a, a, c), (b, c, c)\}$ and $H = \{(a, 1), (b, 1), (c, 2)\}$, then

$$R \circ H = \{(1, 1, 1), (2, 1, 1), (1, 1, 2), (1, 2, 2)\}$$

Definition 3. *Let $R \subseteq A^N$ and $S \subseteq B^N$. If $h : A \to B$ and $R \circ h \subseteq B$, then h is said to be a homomorphism of R into S.*

If $h : A \to B$ and $R \circ h = S$, then h is said to be a homomorphism of R onto S.

If h is one-one, and $R \circ h \subseteq S$, then h is said to be a monomorphism of R into S. In this case, a copy of R is in S.

If $h : A \to B$, h is one-one, and $R \circ h = S$, then h is an isomorphism of R onto S.

Definition 4. *Let P be a partial order on A and Q be a partial order on B. A relation $m \subseteq A \times B$ is said to be monotonically increasing if and only if m is a homomorphism from P into Q.*

Definition 5. (Subrelation) *Let $R \subseteq \bigtimes_{n=1}^N L_n$. A subset $S \subseteq R$ is a subrelation of R if and only if for some $A_n \subseteq L_n$, $n = 1, \ldots, N$, $S = \{(a_1, \ldots, a_N) \in \bigtimes_{n=1}^N A_n \mid (a_1, \ldots, a_N) \in R\}$.*

Definition 6. (Projection) *Let $R \subseteq \bigtimes_{n=1}^N L_n$ and $I = \{i_1, \ldots, i_M\} \subseteq \{1, \ldots, N\}$. The projection operator π, projecting with respect to the index set I is defined by*

$$\pi_I(R) \quad = \quad \{(x_1, \ldots, x_M) \in \times_{m=1}^M L_{i_m} \mid \textit{for some } (a_1, \ldots, a_N) \in R,\ a_{i_m} = x_m,\ m = 1, \ldots, M\}$$

We can also take projections of tuples.

$$\pi_I(a_1, \ldots, a_N) \quad = \quad (a_{i_1}, \ldots, a_{i_M})$$

Definition 7. (Component Wise Mapping) *A mapping $f : \bigtimes_{n=1}^N L_n \to \bigtimes_{n=1}^N M_n$, is called a component wise mapping if and only if the mapping f works component by component. That is $f = (f_1, \ldots, f_N)$ and $f_n : L_n \to M_n$, $n = 1, \ldots, N$ so that*

$$f(a_1, \ldots, a_N) = (f_1(a_1), \ldots, f_N(a_N))$$

If $I = \{i_1, \ldots, i_M\} \subset \{1, \ldots, N\}$, we use the notation f_I to mean $f_I = (f_{i_1}, \ldots, f_{i_M})$.

Having a component mapping immediately leads to a composition of a relation with a component mapping.

Definition 8. (Composition) *Let $R \subseteq \bigtimes_{n=1}^N L_n$ and f a component wise mapping $f : \bigtimes_{n=1}^N L_n \to \bigtimes_{n=1}^N M_n$. The composition of R with f, denoted by $R \circ f$ is defined by*

$$R \circ f = \{(b_1, \ldots, b_N) \in \bigtimes_{n=1}^{N} M_n \mid b_n = f_n(a_n) \ \textit{for some} \ (a_1, \ldots, a_N) \in R\}$$

Also if $u = (u_1, \ldots, u_N) \in \bigtimes_{n=1}^N L_n$, then $u \circ f = (f_1(u_1), \ldots, f_N(u_N))$.

Proposition 1. *Let $f : \bigtimes_{n=1}^N \to \bigtimes_{n=1}^N M_n$ be a component wise mapping. Let $I \subset \{1, \dots, N\}$. Then for any $u \in \bigtimes_{n=1}^N L_n$, $\pi_J(u) \circ f_J = \pi_J(u \circ f)$*

Proposition 2. (Preservation of Subset Relation) *Let $R \subseteq \bigtimes_{n=1}^N L_n$ and $S \subseteq \bigtimes_{n=1}^N L_n$ satisfy $R \subset S$. Let f be a component wise mapping on $\bigtimes_{n=1}^N L_n$. Then $R \circ f \subset S \circ f$.*

Proof. Let $(b_1, \dots, b_N) \in R \circ f$. Then for some $(a_1, \dots, a_N) \in R$, $f_n(a_n) = b_n,\ n = 1, \dots, N$. But $R \subset S$, so that $(a_1, \dots, a_N) \in R$ implies $(a_1, \dots, a_N) \in S$. Now $f_n(a_n) = b_n,\ n = 1, \dots, N$ and $(a_1, \dots, a_N) \in S$ implies by definition of composition of a relation with a function that $(b_1, \dots, b_N) \in S$. □

Proposition 3. (Preservation of Everywhere Defined Subspaces) *Let $R \subseteq \bigtimes_{n=1}^N L_n$ and f be a component wise mapping. $f : \bigtimes_{n=1}^N L_n \to \bigtimes_{n=1}^N M_n$. Let $I \subseteq \{1, \dots, N\}$. Then*

$$\pi_I(R) = \bigtimes_{i \in I} \pi_i(R) \text{ implies } \pi_I(R \circ f) = \bigtimes_{i = \in I} \pi_i(R \circ f)$$

Proposition 4. (Component Wise Mappings: Intersections and Unions) *Let $R \subseteq \bigtimes_{n=1}^N L_n$, $S \subseteq \bigtimes_{n=1}^N L_n$ and f be a component wise mapping on $\bigtimes_{n=1}^N L_n$. Then*

- $(R \cap S) \circ f \subseteq (R \circ f) \cap (S \circ f)$
- $(R \cup S) \circ f = (R \circ f) \cup (S \circ f)$

Proof. Let $(b_1, \dots, b_N) \in (R \cap S) \circ f$. Then for some $(a_1, \dots, a_N) \in R \cap S$, $f_n(a_n) = b_n,\ n = 1, \dots, N$. $(a_1, \dots, a_N) \in R \cap S$ implies $(a_1, \dots, a_N) \in R$ and $(a_1, \dots, a_N) \in S$. Now $(a_1, \dots, a_N) \in R$ and $f_n(a_n) = b_n,\ n = 1, \dots, N$ imply $(b_1, \dots, b_N) \in R \circ f$. And $(a_1, \dots, a_N) \in S$ and $f_n(a_n) = b_n,\ n = 1, \dots, N$ imply $(b_1, \dots, b_N) \in S \circ f$. $(b_1, \dots, b_N) \in R \circ f$ and $(b_1, \dots, b_N) \in S \circ f$ imply $(b_1, \dots, b_N) \in (R \circ f) \cap (S \circ f)$.

Let $(b_1, \dots, b_N) \in (R \cup S) \circ f$. Then for some $(a_1, \dots, a_N) \in R \cup S$, $f_n(a_n) = b_n,\ n = 1, \dots, N$. $(a_1, \dots, a_N) \in R \cup S$ implies $(a_1, \dots, a_N) \in R$ or $(a_1, \dots, a_N) \in S$. Now $(a_1, \dots, a_N) \in R$ and $f_n(a_n) = b_n,\ n = 1, \dots, N$ imply $(b_1, \dots, b_N) \in R \circ f$. And $(a_1, \dots, a_N) \in S$ and $f_n(a_n) = b_n,\ n = 1, \dots, N$ imply $(b_1, \dots, b_N) \in S \circ f$. $(b_1, \dots, b_N) \in R \circ f$ or $(b_1, \dots, b_N) \in S \circ f$ imply $(b_1, \dots, b_N) \in (R \circ f) \cup (S \circ f)$.

Let $(b_1, \dots, b_N) \in (R \circ f) \cup (S \circ f)$. Then $(b_1, \dots, b_N) \in R \circ f$ or $(b_1, \dots, b_N) \in S \circ f$ $(b_1, \dots, b_N) \in R \circ f$ implies that for some $(a_1, \dots, a_N) \in R$, $f_n(a_n) = b_n,\ n = 1, \dots, N$. $(b_1, \dots, b_N) \in S \circ f$ implies that for some $(a'_1, \dots, a'_N) \in S$, $f_n(a'_n) = b_n,\ n = 1, \dots, N$. If for some $(a_1, \dots, a_N) \in R$, $f_n(a_n) = b_n,\ n = 1, \dots, N$ or if for some $(a'_1, \ldots, a'_N) \in S$, $f_n(a'_n) = b_n$, $n = 1, \ldots, N$, then for some $(a^*_1, \ldots, a^*_N) \in R \cup S$, $f_n(a^*_n) = b_n$, $n = 1, \ldots, N$. Now by definition of composition of a relation with a component wise function, $(b_1, \ldots, b_N) \in (R \cup S) \circ f$.

□

Proposition 5. *Let $R \subseteq \bigtimes_{n=1}^{N} L_n$. Let $I, J \subseteq \{1, \ldots, N\}$. Then,*

$$\pi_I(\pi_{I \cup J}(R)) = \pi_I(R)$$

Abstract groups define a relation. First we give the definition of a group.

Definition 9. **(Group)** *The pair $(G, *)$ is a group if and only if G is a set and $*$ is a binary operation and they satisfy*

1. *$g, h \in G$ imply $g * h \in G$*
2. *$g, h, k \in G$ imply $g * (h * k) = (g * h) * k$*
3. *There exists an element $e \in G$ such that $g * e = e * g = g$ for every $g \in G$*
4. *$g \in G$ implies there exists $h \in G$ such that $g * h = h * g = e$*

The relation R defined by a group $(G, *)$ is a relation of triples. $R \subset G \times G \times G$ defined by

$$R = \{(g, h, k) \in G \times G \times G \mid g * h = k\}$$

The subgroups of a group correspond to sets which are subrelations of the relation defined by the group.

Definition 10. **(Subgroup)** *Let $(G, *)$ be a finite group and $H \subset G$ satisfy $h_1, h_2 \in H$ imply $h_1 * h_2 \in H$. Then $(H, *)$ is called a subgroup of $(G, *)$.*

Proposition 6. *Let $(G, *)$ be a group and let R be the relation defined by the group $(G, *)$. Let $(H, *)$ be a subgroup of $(G, *)$ and let S be the relation defined by subgroup $(H, *)$. Then $S \subseteq R$*

We can see how subrelations correspond to the subgroups of a group.

Proposition 7. **(Subrelation Subgroup Correspondence)** *Let $(G, *)$ be a group. Define $R = \{(g_1, g_2, g_3) \in G \times G \times G \mid g_3 = g_1 * g_2\}$. Then $(H, *)$ is a subgroup of $(G, *)$ if and only if subrelation*

$$S = \{(h_1, h_2, h_3) \in R \mid (h_1, h_2, h_3) \in H \times H \times H\}$$

satisfies

$$\pi_{\{1,2\}} S = H \times H$$

Definition 11. (Simple Automaton) *The triple* (S, Σ, δ) *is a simple automaton if and only if* $\delta : S \times \Sigma \to S$. *The set* S *is called the set of states and the set* Σ *is called the set of inputs.*

In a simple automaton, every input can be applied to every state and it will produce a transition to a new state, which is possible the same as the starting state. The subautomatons of a simple automaton correspond to sets which are subrelations of the relation defined by the simple automaton.

Definition 12. (Subautomaton) *The triple* $(T, \Sigma, \delta|_T)$, $T \subseteq S$, *is a subautomaton of the simple automaton* (S, Σ, δ) *if and only if for every* $t \in T$ *and* $\sigma \in \Sigma$, $\delta|_T(t, \sigma) \in T$

Definition 13. Simple Incomplete Automaton *A simple incomplete automaton is a triple* (S, Σ, δ) *where* $\delta \subseteq S \times \Sigma \times S$ *and* δ *is single-valued on* $S \times \Sigma$. *In a simple incomplete automaton,* δ *is not required to be defined everywhere on* $S \times \Sigma$.

An example incomplete automaton $(S\Sigma, \delta)$ is shown below. $S = \{a, b, c\}$, $\Sigma = \{\sigma_1, \sigma_2\}$

$$\delta = \{(a, \sigma_1, b), (b, \sigma_1, c), (c, \sigma_2, a)$$

Notice that input σ_2 cannot be applied to state a or b and σ_1 cannot be applied to state c.

Relations may also be defined on collections of sets. A binary relation R defined on a collection of sets is said to be:

- **symmetric** if and only if $(X, Y) \in R$ implies $(Y, X) \in R$.
- **exclusive** if and only if $(X, Y) \in R$ implies $X \cap Y = \emptyset$.
- **hereditary** if and only if for any subset $A \subseteq X$ and for any subset $B \subseteq Y$, if $(X, Y) \in R$, then $(A, B) \in R$.
- **extensive** if and only if $(\{a\}, \{b\}) \in R$ for every $a \in A, b \in B$ implies $(A, B) \in R$.

A binary relation S defined on a collection of sets is said to be a **separation** relation if and only if S is symmetric, exclusive, hereditary, and extensive. If $(X, Y) \in S$, then the set X is said to be separated from Y with respect to the separation relation S. If $(X, Y) \notin S$, then the set X is said to be connected to Y with respect to the separation relation S.

Let S be a separation relation. A set A is said to be disconnected with respect to the separation relation S is and only if for some two celled partition $\Pi = \{\Pi_1, \Pi_2\}$ on $A, \Pi_1 \neq \emptyset$ and $\Pi_2 \neq \emptyset$, $(\Pi_1, \Pi_2) \in S$.

Let S be a separation relation. A set A is said to be connected with respect to the separation relation S if and only if it is not disconnected with respect to the separation relation S. That is, if and only if for every two celled partition $\Pi = \{\Pi_1, \Pi_2\}$ on A, $\Pi_1 \neq \emptyset, \Pi_2 \neq \emptyset, (\Pi_1, \Pi_2) \notin S$.

Let S be a separation relation. An element x is said to be *connected* to an element y in a set A if and only if there exists some connected subset B of A containing both x and y.